Small-Signal Stability Analysis of Delay System

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Outline

• Delay System Modelling
• Small-Signal Stability
• Eigenvalue-based Approach
• Analytical Approach
• General Cases
Definitions in Delay System

• Delay-independent System
  The system remains stable for any positive value of delay.

• Delay-dependent System
  The system is stable without delay, but it will become unstable due to some specific delays.

• Delay Margin
  Delay margin is a constant value. In a delay-dependent system, if the magnitude of the delay is larger than the delay margin, the system collapse.
Small-Signal Stability

- **Index 1 Hessenberg Form System**

\[ \dot{x} = f(x, y, y_d, u) \]
\[ 0 = g(x, y, u) \]
\[ x_d = x(t - \tau) \]
\[ y_d = y(t - \tau) \]

- **Stationary Point:**

\[ 0 = f(x_0, y_0, x_0, y_0, u_0) \]
\[ 0 = g(x_0, y_0, u_0) \]
Small-Signal Stability

• Linearizing the system at the stationary solution yields:

\[ \Delta \dot{x} = f_x \Delta x + f_y \Delta y + f_{yd} \Delta y_d \]

\[ 0 = g_x \Delta x + g_y \Delta y \]

Eliminating \( \Delta y \):

\[ \Delta \dot{x} = A_0 \Delta x + A_1 \Delta x(t - \tau) \]

\[ A_0 = f_x - f_y g_y^{-1} g_x \]

\[ A_1 = -f_{yd} g_y^{-1} g_x \]

• Characteristic equation:

\[ \Delta(\lambda) = \lambda I - A_0 - A_1 e^{-\lambda \tau} \quad \text{Transcendental!} \]
Eigenvalue-based Approach

Characteristic Equation:

\[ \Delta(\lambda) = \lambda I - A_0 - A_1 e^{-\lambda \tau} \]

Discretization:


Transform the differential algebraic equations in a continuous boundary value problem with partial derivatives and using Chebyshev’s discretization scheme to calculate the eigenvalues.
Eigenvalue-based Method

• Two useful properties:

➢ The characteristic equation only has a finite number of characteristic roots in any vertical strip of the complex plane:

\[ \lambda \in \mathbb{C} : \alpha < \Re(\lambda) < \beta \]

➢ There exists a number such that all characteristic roots are confined to the half-plane:

\[ \gamma \in \mathbb{R}, \lambda \in \mathbb{C} : \Re(\lambda) < \gamma \]
Do we really have to calculate the approximate eigenvalues of the delay system to find the delay margin?

\[ \Delta(\lambda) = \lambda I - A_0 - A_1 e^{-\lambda \tau} \]
Analytical Approach

• Example

\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x(t - \tau) \]

At the delay margin the system is supposed to have two eigenvalues on the imaginary axis, let’s assuming:

\[ \lambda_1 = j \omega, \lambda_2 = -j \omega \]

Then we have:

\[ \Delta(\lambda_1) = \lambda_1 I - A_0 - A_1 e^{-\lambda_1 \tau} = 0 \]
\[ \Delta(\lambda_2) = \lambda_2 I - A_0 - A_1 e^{-\lambda_2 \tau} = 0 \]

According to the above equations, we have:

\[ \lambda = \pm j, \quad \bar{\tau} = \pi \]
Eigenvalue-based Approach

- Real part of the dominate eigenvalue as a function of delay:
Analytical Approach

- Decision function of the crossing direction:

\[
\text{sign}\{\Re\left[\frac{d \lambda}{d \tau}\right]_{\lambda=j\omega}\}
\]

If positive (+), crossing towards instability

If negative (-), crossing towards stability
Advantages of Eigenvalue-based Approach

- Comparing with analytical method:
  
  Easy and less trick!

- Comparing with time-domain method:
  
  Lyapunove second method, linear matrix inequality method and etc.

  Easy and fast.

  Less conservative!
Delay System Modelling

➢ Retarded Delay Differential Equation (RDDE):

\[ \dot{x} = f(x, x_d, y, y_d, u) \]
\[ 0 = g(x, x_d, y, y_d, u) \]

➢ Neutral Delay Differential Equation (NDDE):

\[ \dot{x} = f(x, x_d, y, y_d, \dot{x}_d, u) \]
\[ 0 = g(x, x_d, y, y_d, u) \]
Characteristic Equations

- **Index-1 Hessenerg Form System**
  \[ \Delta(\lambda) = \lambda I - A_0 - A_1 e^{-\lambda \tau} \]

- **Non Index-1 Hessenerg Form System**
  \[ \Delta(\lambda) = \lambda I - A_0 - \sum_k A_k e^{-\lambda_k \tau} \]

- **Neutral Delay System**
  \[ \Delta(\lambda) = \lambda I - A_0 - \sum_k A_k e^{-\lambda_k \tau} - B e^{-\lambda \tau} \]
Implements in Dome

- Retarded Delay System

\[
\begin{align*}
\dot{x} &= f(x, y, x_d, y_d, u) \\
0 &= g(x, y, x_d, y_d, u)
\end{align*}
\]

- Neutral Delay System

\[
\begin{align*}
\dot{x} &= f(x, y, x_d, \dot{x}_d, y_d, u) \\
0 &= g(x, y, x_d, y_d, u) \\
\dot{x} &= y \\
0 &= f(x, y, x_d, y_d, u) - y
\end{align*}
\]
Delay-Independent Neutral System

- Food-limited Dynamic Population Model

\[ \dot{S}(t) = rS(t) \left[ 1 - \frac{S(t-\tau)}{K} + c \frac{\dot{S}(t-\tau)}{K} \right] \]

\[ r = \frac{\pi}{\sqrt{3}} + \frac{1}{20} \]

\[ c = \frac{\sqrt{3}}{2\pi} - \frac{1}{25} \]

\[ K = 1 \]
Delay-Independent Neutral System

- Linear Partial Element Equivalent Circuit (PEEC)

\[ \dot{x} = L x(t) + M x(t - \tau) + N \dot{x}(t - \tau) \]

\[
\frac{L}{100} = \begin{bmatrix}
-7 & 1 & 2 \\
3 & -9 & 0 \\
1 & 2 & -6
\end{bmatrix}
\]

\[
\frac{M}{100} = \begin{bmatrix}
1 & 0 & -3 \\
-0.5 & -0.5 & -1 \\
-0.5 & -1.5 & 0
\end{bmatrix}
\]

\[
72N = \begin{bmatrix}
-7 & 1 & 2 \\
3 & -9 & 0 \\
1 & 2 & -6
\end{bmatrix}
\]
Delay Modelling

- Constant Delay
- Distributed Delay

\[ \tau = \int_b^a w(\theta) x(t-\theta) \, d\theta \]

- Periodic Time-varying Delay

\[ \tau(t) = \tau_0 + \delta f(\Omega t) \]

- Stochastic Delay
References


Thanks for your attention!

Any Questions?