Construction of SDE-based wind speed models with exponential autocorrelation

Rafael Zárate Miñano

Escuela de Ingeniería Minera e Industrial de Almadén

Universidad de Castilla-La Mancha
Introduction

Motivation

- Use of wind speed models in the analysis of many aspects of power systems
  - power system economics and operation
  - generation capacity reliability evaluation
  - dynamic studies and control of wind turbines

- Different types of models
  - time series models
  - four-component composite models
  - models based on Kalman filters
  - based on stochastic differential equations
Introduction
Motivation

- The wind speed must be properly characterized since the reliability of the different studies depends on it.

- Statistical characterization:
  - Probability distribution: Weibull, Gamma, etc ...
  - Autocorrelation: Exponential or Power-law

- Models proposed in the literature fail in reproducing one of the above characteristics.
Introduction
Contribution

We relay on basic stochastic calculus concepts and tools...

- Stationary Markov processes
- Regression theorem
- Itô formula
- Fokker-Planck equation

... to derive a method to construct SDEs that exactly reproduce both the probability distribution and the exponential autocorrelation of the wind speed
Outlines of Stochastic Calculus
Stationary Markov processes

- Markov process: Stochastic process without memory
  - The future of the process only depends on the present but it is independent on the past

- Stationary process: The probability distribution is time-invariant
  - $E[x(t)] = \mu(t) = \mu$
  - $\sqrt{E[(x(t) - \mu)^2]} = \sigma(t) = \sigma$
Outlines of Stochastic Calculus
Stationary Markov processes

- For autocovariance and autocorrelation, stationarity implies:

\[
    c(s, t) = E \left[ (x(s) - \mu(s)) \cdot (x(t) - \mu(t)) \right]
\]

\[
    r(s, t) = \frac{E \left[ (x(s) - \mu(s)) \cdot (x(t) - \mu(t)) \right]}{\sigma(s) \cdot \sigma(t)}
\]

depend only on the time lag \( \tau = t - s \), that is

\[
    c(s, t) = c(\tau) = E \left[ (x(t - \tau) - \mu) \cdot (x(t) - \mu) \right]
\]

\[
    r(s, t) = r(\tau) = \frac{E \left[ (x(t - \tau) - \mu) \cdot (x(t) - \mu) \right]}{\sigma^2}
\]
Outlines of Stochastic Calculus
Regression theorem

\[
\frac{dE[x(t)]}{dt} = -\alpha \cdot E[x(t)]
\]

\[\downarrow\]

\[
\frac{dc(\tau)}{d\tau} = -\alpha \cdot c(\tau)
\]

\[\downarrow\]

Autocovariance \quad \rightarrow \quad c(\tau) = \sigma^2 \cdot e^{-\alpha \cdot \tau}

Autocorrelation \quad \rightarrow \quad r(\tau) = e^{-\alpha \cdot \tau}
Outlines of Stochastic Calculus

Stochastic differential equations

- Differential form:

\[ dx(t) = a(x(t), t) \cdot dt + b(x(t), t) \cdot dW(t) \]

- Integral form:

\[ x(t) - x_0 = \int_0^t a(x(u), u) \cdot du + \int_0^t b(x(u), s) \cdot dW(u) \]
Outlines of Stochastic Calculus
The Itô formula

\[ dg(x(t), t) = \]

\[ \left[ \frac{\partial g(x(t), t)}{\partial t} + a(x(t), t) \cdot \frac{\partial g(x(t), t)}{\partial x(t)} \right. \]

\[ + \left. \frac{1}{2} \cdot b^2(x(t), t) \cdot \frac{\partial^2 g(x(t), t)}{\partial x^2(t)} \right] \cdot dt \]

\[ + b(x(t), t) \cdot \frac{\partial g(x(t), t)}{\partial x(t)} \cdot dW(t) \]
Outlines of Stochastic Calculus
Fokker-Planck equation

\[ \frac{\partial p(x(t), t)}{\partial t} = \]

\[ - \frac{\partial}{\partial x(t)} \left[ a(x(t), t) \cdot p(x(t), t) \right] \]

\[ + \frac{1}{2} \cdot \frac{\partial^2}{\partial x^2(t)} \left[ b^2(x(t), t) \cdot p(x(t), t) \right] \]
Proposed Building Method of the SDE Model
Use of the Fokker-Planck equation

- For stationary process:

\[ a(x(t), t) = a(x(t)) \]
\[ b(x(t), t) = b(x(t)) \]
\[ p(x(t), t) = p(x(t)) \]

- Fokker-Planck equation reduces to:

\[ 0 = -a(x(t)) \cdot p(x(t)) + \frac{1}{2} \cdot \frac{\partial}{\partial x(t)} \left[ b^2(x(t)) \cdot p(x(t)) \right] \]
Proposed Building Method of the SDE Model
Use of the Fokker-Planck equation

- Solving the Fokker-Planck equation for $a(x(t))$:

$$a(x(t)) = b(x(t)) \cdot \frac{\partial b(x(t))}{\partial x(t)} + \frac{1}{2} \cdot b^2(x(t)) \cdot \frac{\partial \ln p(x(t))}{\partial x(t)}$$

- Solving the Fokker-Planck equation for $b^2(x(t))$:

$$b^2(x(t)) = \frac{2}{p(x(t))} \cdot \int_{-\infty}^{x(t)} a(z(t)) \cdot p(z(t)) \cdot dz(t)$$
Proposed Building Method of the SDE Model
Use of the Itô formula

- We look for a differential equation for the autocovariance:

\[ g(x(t)) = (x(t) - \mu) \cdot (x(s) - \mu), \quad \text{where } s < t \]

- Applying the Itô formula:

\[
\frac{\partial g(x(t))}{\partial t} = 0 \\
\frac{\partial g(x(t))}{\partial x(t)} = x(s) - \mu \\
\frac{\partial^2 g(x(t))}{\partial x^2(t)} = 0
\]
Proposed Building Method of the SDE Model
Use of the Itô formula

- Result:

\[ d[(x(t) - \mu) \cdot (x(s) - \mu)] = \]
\[ a(x(t)) \cdot (x(s) - \mu) \cdot dt + b(x(t)) \cdot (x(s) - \mu) \cdot dW(t) \]

with initial condition \((x(s) - \mu)^2\).

- Integral form:

\[ (x(t) - \mu) \cdot (x(s) - \mu) - (x(s) - \mu)^2 = \]
\[ \int_s^t a(x(u)) \cdot (x(s) - \mu) \cdot du + \int_s^t b(x(u)) \cdot (x(s) - \mu) \cdot dW(u) \]
Proposed Building Method of the SDE Model
Use of the Itô formula

- Applying the expectation operator:

\[
E[(x(t) - \mu) \cdot (x(s) - \mu)] - E[(x(s) - \mu)^2] = \\
E \left[ \int_s^t a(x(u)) \cdot (x(s) - \mu) \cdot du \right] \\
+ E \left[ \int_s^t b(x(u)) \cdot (x(s) - \mu) \cdot dW(u) \right]
\]

and taking into account that:

\[
E \left[ \int f(x(t)) \cdot dW(t) \right] = 0
\]
Proposed Building Method of the SDE Model
Use of the Itô formula

\[
E [(x(t) - \mu) \cdot (x(s) - \mu)] - E [(x(s) - \mu)^2] = \\
\int_s^t E [a(x(u)) \cdot (x(s) - \mu)] \cdot du
\]

- Coming back to the differential form:

\[
\frac{dE [(x(t) - \mu) \cdot (x(s) - \mu)]}{dt} = E [a(x(t)) \cdot (x(s) - \mu)]
\]
Proposed Building Method of the SDE Model
Use of the Itô formula

- By comparing

\[
\frac{dE[(x(t) - \mu) \cdot (x(s) - \mu)]}{dt} = E[a(x(t)) \cdot (x(s) - \mu)]
\]

with

\[
\frac{dc(s, t)}{dt} = -\alpha \cdot c(s, t)
\]

where

\[
c(s, t) = E[(x(t) - \mu) \cdot (x(s) - \mu)]
\]

it is clear that

\[
a(x(t)) = -\alpha \cdot (x(t) - \mu)
\]
Proposed Building Method of the SDE Model

Summary

- To have a wind speed model with exponential autocorrelation:

1. Perform a statistical analysis of the wind speed data
   - Identify the probability distribution \( p(x(t)) \)
   - Identify the autocorrelation coefficient \( \alpha \)
Proposed Building Method of the SDE Model

Summary

2. Use a SDE of the form:

\[ dx(t) = a(x(t)) \cdot dt + b(x(t)) \cdot dW(t) \]

- The drift term is:

\[ a(x(t)) = -\alpha \cdot (x(t) - \mu) \]

- The diffusion term is computed from:

\[ b^2(x(t)) = \frac{2}{p(x(t))} \cdot \int_{-\infty}^{x(t)} -\alpha \cdot (z(t) - \mu) \cdot p(z(t)) \cdot dz(t) \]
Examples
Three-parameter Beta distribution

\[ p_B(x) = \begin{cases} 
\frac{1}{\lambda_3 \cdot B(\lambda_1, \lambda_2)} \cdot \left( \frac{x}{\lambda_3} \right)^{\lambda_1-1} \cdot \left( \frac{\lambda_3 - x}{\lambda_3} \right)^{\lambda_2-1} & \text{if } x > 0 \\
0 & \text{if } x \leq 0 
\end{cases} \]

\[ a(x) = -\alpha \cdot \left( x - \frac{\lambda_1 \cdot \lambda_3}{\lambda_1 + \lambda_2} \right) \]

\[ b(x) = \sqrt{\frac{2 \cdot \alpha \cdot (\lambda_3 - x) \cdot x}{\lambda_1 + \lambda_2}} \]
Examples
Two-parameter Gamma distribution

\[ p_G(x) = \begin{cases} 
\frac{1}{\lambda_2^{\lambda_1} \cdot \Gamma(\lambda_1)} \cdot x^{\lambda_1-1} \cdot \exp \left( -\frac{x}{\lambda_2} \right) & \text{if } x > 0 \\
0 & \text{if } x \leq 0
\end{cases} \]

\[ a(x) = -\alpha \cdot (x - \lambda_1 \cdot \lambda_2) \]

\[ b(x) = \sqrt{2 \cdot \alpha \cdot \lambda_2 \cdot x} \]
Examples
Two-parameter Weibull distribution

\[ p_W(x) = \begin{cases} 
\frac{\lambda_1}{\lambda_2} \cdot \left( \frac{x}{\lambda_2} \right)^{\lambda_1 - 1} \cdot \exp \left( - \left( \frac{x}{\lambda_2} \right)^{\lambda_1} \right) & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases} \]

\[ a(x) = -\alpha \cdot \left( x - \lambda_2 \cdot \Gamma \left( 1 + \frac{1}{\lambda_1} \right) \right) \]

\[ b(x) = \sqrt{b_1(x) \cdot b_2(x)} \]
Examples
Two-parameter Weibull distribution

\[ b_1(x) = 2 \cdot \alpha \cdot \frac{\lambda_2}{\lambda_1^2} \cdot x \cdot \left( \frac{\lambda_2}{x} \right)^{\lambda_1} \]

\[ b_2(x) = \lambda_1 \cdot \exp\left( \left( \frac{x}{\lambda_2} \right)^{\lambda_1} \right) \cdot \Gamma \left( 1 + \frac{1}{\lambda_1}, \left( \frac{x}{\lambda_2} \right)^{\lambda_1} \right) - \Gamma \left( \frac{1}{\lambda_1} \right) \]
Numerical Simulations
Three-parameter Beta distribution

Figure: Three-parameter Beta distribution model.
Numerical Simulations
Two-parameter Gamma distribution

Figure: Two-parameter Gamma distribution model.
Numerical Simulations
Two-parameter Weibull distribution

Figure: Two-parameter Weibull distribution model.
Case Study

- Modeling the wind speed in Wellington, New Zealand

- Data set: hourly-mean values for whole year 2014 (8760 values)
**Case Study**

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(a)

**Figure:** (a) Negative log likelihood value of the PDFs parameter estimation; (b) Generalized Gamma PDF fit to the data histogram and histogram of the simulated process.
Case Study

Figure: (a) Autocorrelation analysis of data and autocorrelation of the simulated process; (b) Power spectral density of data and of the simulated process.