Methods in optimization, system theory and networks for mathematical modelling

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2nd WORKSHOP on DYNAMIC SYSTEM MODELLING

23-24 March 2017

Overview

1. Optimization for the analysis of large scale data sets

2. Fractional Operators

3. A discrete calculus formulation for elasticity, plasticity and damage

4. Modelling a network with Partial Differential Equations

Optimization for the analysis of large scale data sets

- Applications from the fields of signal processing, machine learning and statistics;
- Large-scale problems impose restrictions on methods that have been so far employed;
- The new methods have to be:
  (a) memory efficient;
  (b) ideally, within seconds they should offer noticeable progress towards a solution.

Second Order Methods in Optimisation

- With Dr. K. Fountoulakis (University of Berkeley, USA) & Professor J. Gondzio (University of Edinburgh, UK) we are concerned with the following family of optimisation problems:

$$\text{minimize } f_c(x) := \sum_{i=1}^{p} c_i \|W_i^\intercal x\|_1 + \frac{1}{2}\|Ax - b\|_2^2. \qquad (1)$$

- Where $x \in \mathbb{R}^n$, $c_i \in \mathbb{R}^p$ are vectors of weights for the regularizers;
- $W_i : \mathbb{R}^n \to E^l$, where $E = \mathbb{R}$ or $\mathbb{C}$;
- $A : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$, $b \in \mathbb{R}^m$ are the sampled data;
- Very large matrices !!!

Four major applications

Type	**p**	**W**	$\boldsymbol{\varphi(x)}$
(I) Least-Squares	1	Identity	$\|Ax-b\|_2^2$
(II) ℓ_1-analysis	1	Arbitrary matrix $W:\mathbb{R}^m\to\mathbb{R}^l$	$\|Ax-b\|_2^2$
(III) Total Variation (TV)	1	$W:\mathbb{R}^m\to\mathbb{C}^{m-1}$ (Tridiagonal matrix)	$\|Ax-b\|_2^2$
(IV) TV & ℓ_1-analysis	2	$W_1:\mathbb{R}^m\to\mathbb{C}^{m-1}$ $W_2:\mathbb{R}^m\to\mathbb{R}^l$	$\|Ax-b\|_2^2$

Note: Total variation is a process, most often used in digital image processing, that removes noise;

The first order method

- minimize $f(x)$, $x \in \mathbb{R}^n$;
- A first order approximation is: $Q := f(y) + \nabla f(y)^T(x - y)$;
- The good:
  (a) They avoid matrix factorizations;
  (b) They have low memory requirements;
  (c) They sometimes offer fast progress in the initial stages of optimization.
- The bad (as demonstrated by numerical experiments):
  (a) They miss **essential** information;
  (b) Slow practical convergence.

A simple description of Second order methods

- minimize $f(x)$, $x \in \mathbb{R}^n$;
- A second order approximation is:
  $Q := f(y) + \nabla f(y)^T(x-y) + \frac{1}{2}(x-y)^T \nabla^2 f(y)(x-y)$;
- By setting $\nabla Q = 0_{n,1}$ we get:

$$\nabla^2 f(y)(x-y) = -\nabla f(y);$$

- Where $\nabla^2 f(y)$ is a matrix $n \times n$, $\nabla f(y)$ is a vector $n \times 1$;
- Find optimal x.

Basic Steps

1. Appropriate **smoothing** of the problem; 1st and 2nd order derivatives of norm 1 are not defined !!
2. Compute **derivatives of complex valued matrix functions**;
3. Solve the system that appears via the second order method (Newton conjugate gradient method). Efficient **preconditioning techniques** are for fast iterative solution;
4. **Perturbation analysis** for optimal solutions;
5. Numerical results & Compare our method with others in the literature.

Second order methods

Smoothing

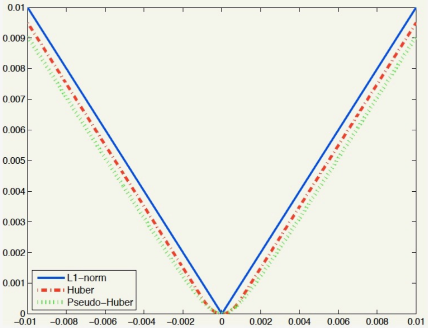
Nesterov's smoothing for the ℓ_1-norm: Huber function (only first-order differentiable),

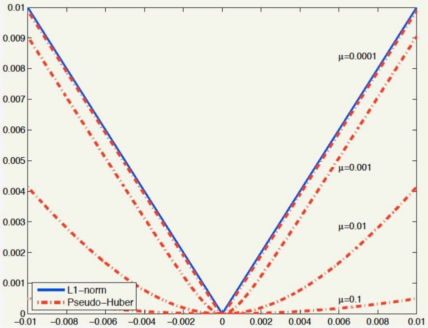
- used in NestA by S. R. Becker and J. Bobin and E. J. Candés,
- and in TFOCS by S. R. Becker and E. J. Candés and M. C. Grant, for the dual.

Second-order derivative???
Replace ℓ_1-norm with pseudo-Huber function

$$\psi_\mu(x) = \mu \sum_{i=1}^{m} \left(\sqrt{1 + \frac{x_i^2}{\mu^2}} - 1 \right).$$

Smoothing





Empirical observation: the error in the reconstructed components of the signal is $\mathcal{O}(\mu)$.

Second order methods

Control of spectrum and warm-start

Control of spectrum

- At any x, $0 \prec \nabla^2 f_c^\mu(x) \preceq \left(\frac{1}{\mu}\sum_{i=1}^{p} c_i C(W_i) + \lambda_1\right) I_m$,

  where λ_1 is the largest eigenvalue of $\nabla^2\varphi(x)$

Warm-start of **minimize** $\mathbf{f}_c(\mathbf{x})$

- Zero optimal solution if for an index i

$$c_i \geq \|W_i(W_i^\intercal W_i)^{-1}\nabla\varphi(0)\|_\infty \text{ for } l_i > m$$

  while the rest c_j, $j \neq i$, regularization parameters are zero

Warm-start of **minimize** $\mathbf{f}_c^\mu(\mathbf{x})$ (empirical observation)

- Few Newton-type iterations such that $\|\nabla f_c^\mu(x)\| \leq \epsilon$, when

$$c_i \geq \|W_i(W_i^\intercal W_i)^{-1}\nabla\varphi(0)\|_\infty$$

  while the rest c_j, $j \neq i$, regularization parameters are zero

References

- Dassios I., Fountoulakis K., Gondzio J., *A Preconditioner for a Primal-Dual Newton Conjugate Gradients Method for Compressed Sensing Problems.* **SIAM (Society for Industrial and Applied Mathematics) Journal on Scientific Computing**. Accepted (2016).
- Dassios I., Fountoulakis K., Gondzio J., *A Second-order Method for Compressed Sensing Problems with Coherent and Redundant Dictionaries.* **Edinburgh Research Group in Optimization.** Technical Report ERGO-14-007 (2014).

Software

- The software "pdNCG: primal-dual Newton Conjugate Gradients" for the above mentioned problem is available for free online !!!
- The solver is memoryless, it requires only matrix-vector product operations, hence it is appropriate for large-scale instances.

Numerical results on TV – problems & Comparisons

Signal Processing Problems

We reproduce experiments as given by papers/demos in existing state-of-the-art solver packages.

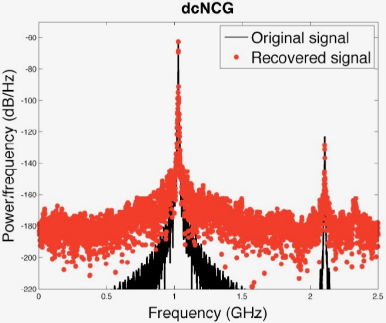
Solvers

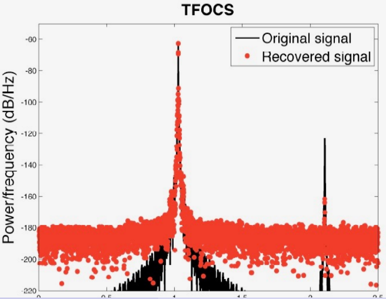
- TFOCS (Templates for First-Order Conic Solvers): Auslender and Teboulle's single-projection method., by S. R. Becker, E. J. Candés and M. C. Grant

ℓ_1-Analysis (recovery of radar pulses)

Info

- W is a Gabor frame
- A is a block diagonal matrix, with ± 1 for entries
- Sub-sampling is $\frac{1}{12}$
- Noise is added so that the small pulse has SNR 0.1 dB
- dcNCG ($\mu = 1.0e$-5): time=0.3 min., rel. err.=1.56e-3
- TFOCS: time=1.0 min., rel. err.=1.82e-3



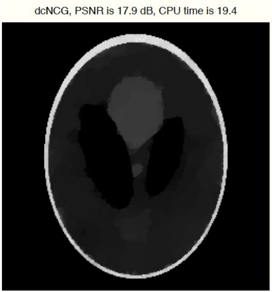


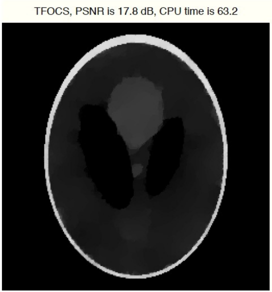
Isotropic Total-Variation

Info

- A is a partial Fourier matrix
- Sub-sampling is $\frac{1}{4}$
- SNR 10 dB
- dcNCG ($\mu = 1.0e\text{-}4$): time=19.4 sec., PSNR.=17.9 dB
- TFOCS: time=63.2 sec., PSNR=17.8 dB

$$PSNR = 20\log_{10}\left(\frac{\sqrt{n_1 n_2}}{\|x-\bar{x}\|_F}\right)$$



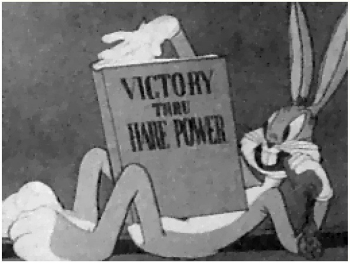


Isotropic Total-Variation and ℓ_1-Analysis (denoising)

Info

- Reg.: $\alpha\|Wx\|_1 + \beta\|Dx\|_1$
- W is 9/7 bi-orthogonal wavelet transform
- A is the identity
- SNR 10 dB
- dcNCG ($\mu = 1.0e\text{-}4$): time=6.6 sec., PSNR=27.6 dB
- TFOCS: time=12.7 sec., PSNR=27.5 dB

dcNCG, PSNR 27.6 dB, CPU time is 6.6

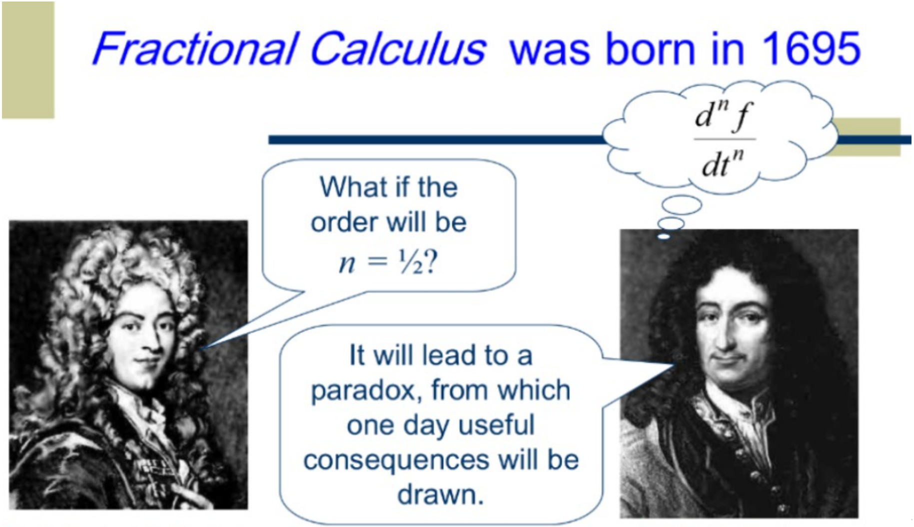


TFOCS, PSNR 27.5 dB, CPU time is 12.7

Notes – Future Work

- Interest in applying our results in Image Processing;
- TV – problems: Apply Fractional Operators that can detect the edge direction and enhance the texture details.

Fractional Operators



Open problems on Fractional Derivatives

- Total Variation problems – to handle better the texture details of image;
- The hyperbolic heat conduction equation (HHE),

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

breaks down at length and time scales of the order of 1 nm $(10^{-9} m)$ and $1 ps (= 10^{-12} s)$;

- Would a time-fractional heat conduction equation be a suitable physical model to describe sub-nanometric thermal transport?

$$\tau \frac{\partial^\gamma T}{\partial t^\gamma} + \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

Where $1 \leq \gamma \leq 2$.

Nabla Difference Operator

- $\nabla Y_k = Y_k - Y_{k-1}$;
- $\nabla^2 Y_k = \nabla(\nabla Y_k) = Y_k - 2Y_{k-1} + Y_{k-2}$;

  $\vdots$
- $\nabla^n Y_k = \frac{1}{\Gamma(n+1)} \sum_{j=0}^{n} (-1)^j \frac{1}{\Gamma(j+1)\Gamma(n-j+1)} Y_{k-j}$, $n \in \mathbb{N}$.

Nabla Fractional Difference Operator

- The nabla operator of n-th order, n **Natural**, applied to a vector of sequences $Y_k : \mathbb{N}_\alpha \to \mathbb{C}^m$ is defined by:

$$\nabla^n Y_k = \sum_{j=0}^{n} a_j Y_{k-j} = a_0 Y_k + a_1 Y_{k-1} + \ldots + a_n Y_{k-n},$$

  where $a_j = (-1)^j \frac{1}{\Gamma(n+1)\Gamma(j+1)\Gamma(n-j+1)}$.
- The nabla fractional operator of n-th order, n **Fractional**, applied to a vector of sequences $Y_k : \mathbb{N}_\alpha \to \mathbb{C}^m$ is defined by

$$\nabla^n_\alpha Y_k = \sum_{j=\alpha}^{k} b_j Y_j = b_k Y_k + b_{k-1} Y_{k-1} + \ldots + b_\alpha Y_\alpha,$$

  where $b_j = \frac{1}{\Gamma(-n)}(k-j+1)^{\overline{-n-1}}$;
- A tool for time scale analysis and storage.

Systems of Fractional Nabla Difference Equations

(a) Generalized non-autonomous system of fractional nabla difference equations:

$$F\nabla_{\alpha}^{n} Y_k = G_k^{(k)} Y_k + G_k^{(k-1)} Y_{k-1} + \ldots + G_k^{(\alpha)} Y_\alpha + V_k.$$

(b) Generalized non-autonomous system of fractional nabla difference equations of multiple orders:

$$F\nabla_{A}^{N} Y_k = \sum_{i=\alpha}^{k} G_i Y_i + V_k.$$

Where $Y_k = \begin{bmatrix} y_k^{(1)} & y_k^{(2)} & \cdots & y_k^{(m)} \end{bmatrix}^T$ &
$\nabla_A^N Y_k = \begin{bmatrix} \nabla_{\alpha_1}^{n_1} y_k^{(1)} & \nabla_{\alpha_2}^{n_2} y_k^{(2)} & \cdots & \nabla_{\alpha_m}^{n_m} y_k^{(m)} \end{bmatrix}^T$.

Matrix pencils

Good Knowledge of Matrix Pencil Theory is required !!

1. If the coefficients are square matrices and the leading coefficient is singular then:
   - The matrix pencil has **finite** and **infinite** eigenvalues, or,
   - The determinant of the pencil is identically zero;
2. Coefficients can be non-square matrices; Characteristic polynomial is not defined via the determinant of the pencil.

We need a good Matrix Decomposition & an appropriate Transformation to split the initial system into subsystems.

Singular linear system of fractional nabla difference equations

- Consider the singular fractional discrete time system of the form

$$F\nabla_0^n Y_k = GY_k + V_k, \quad k = 1, 2, ...$$

- with known initial conditions

$$Y_0$$

- where $F, G \in \mathbb{C}^{r\times m}$, $V_k \in \mathbb{C}^m$.
- The matrices F and G can be non square (when $r \neq m$) or
- square ($r = m$) and F singular (det$F$=0).

Matrix pencils

- For the pencil $sF - G$: when $F = I_m$, G is square, the zeros of $\det(sF - G)$ are the eigenvalues of G.
- Generalized eigenvalue problem

$$sFX = GX.$$

(a) If F, G square, $\det(sF - G) \equiv p(s)$ and F is singular, the pencil has infinite eigenvalues.

$$FX = s^{-1}GX.$$

If F is singular with a null vector X, then $FX = 0_{m,1}$, so that X is an eigenvector of the reciprocal problem corresponding to eigenvalue $s^{-1} = 0$; i.e., $s = \infty$.

(b) Moreover even with F, G square, it is possible $\det(sF - G) \equiv 0$, independent of s;

(c) F, G can be non-square matrices.

Matrix pencils

Definition

The matrix pencil $sF - G$ is called:
A. Regular when $r = m$ and $\det(sF - G) \equiv p(s)$
B. Singular when $r \neq m$ or $r = m$ and $\det(sF - G) \equiv 0$

Regular Pencil

A singular pencil has invariants of the following type:

- *finite elementary divisors* of the type $(s - a_j)^{p_j}$;
- *infinite elementary divisors* of the type $\hat{s}^q = \frac{1}{s^q}$.

Regular pencil

- From the regularity of $sF - G$, there exist nonsingular matrices $P \in \mathbb{R}^{m \times m}$, $Q \in \mathbb{R}^{m \times m}$ such that
- $$PFQ = F_W, \quad PGQ = G_W$$
- $$F_W = blockdiag\{I_p, H_q\},$$
  $$G_W = blockdiag\{J_p, I_q\}.$$

  H_q is a nilpotent matrix with index q_* and J_p a Jordan matrix, the rest of the matrices are sparse and except zeros they have only the number 1.

Regular pencil

By considering the transformation

$$Y_k = Q \begin{bmatrix} Z_k^p \\ Z_k^q. \end{bmatrix}$$

and

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix},$$

Regular pencil

We arrive easily at the subsystems

$$\nabla_0^n Z_k^p = J_p Z_k^p + P_1 V_k;$$

$$H_q \nabla_0^n Z_k^q = Z_k^q + P_2 V_k.$$

Singular Pencil

A regular pencil has invariants of the following type:

- *finite elementary divisors* of the type $(s - a_j)^{p_j}$;
- *infinite elementary divisors* of the type $\hat{s}^q = \frac{1}{s^q}$;
- *column minimal indices* of the type $\epsilon_1 = \epsilon_2 = ... = \epsilon_g = 0 < \epsilon_{g+1} \leq ... \leq \epsilon_d$;
- *row minimal indices* of the type $\zeta_1 = \zeta_2 = ... = \zeta_h = 0 < \zeta_{h+1} \leq ... \leq \zeta_t$.

Singular pencil

- From the regularity of $sF - G$, there exist nonsingular matrices $P \in \mathbb{R}^{r\times r}$, $Q \in \mathbb{R}^{m\times m}$ such that
- $$PFQ = F_K, \quad PGQ = G_K$$
- $$F_K = blockdiag\{I_p, H_q, F_\epsilon, F_\zeta, 0_{h,g}\},$$
$$G_K = blockdiag\{J_p, I_q, G_\epsilon, G_\zeta, 0_{h,g}\}.$$

  H_q is a nilpotent matrix with index q_* and J_p a Jordan matrix, the rest of the matrices are sparse and except zeros they have only the number 1.

Singular pencil

By considering the transformation

$$Y_k = Q \begin{bmatrix} Z_k^p \\ Z_k^q \\ Z_k^\epsilon \\ Z_k^\zeta \\ Z_k^g . \end{bmatrix}$$

and

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_p & Q_q & Q_\epsilon & Q_\zeta & Q_g \end{bmatrix},$$

We arrive easily at the subsystems

$$\nabla_0^n Z_k^p = J_p Z_k^p + P_1 V_k;$$

$$H_q \nabla_0^n Z_k^q = Z_k^q + P_2 V_k;$$

$$F_\epsilon \nabla_0^n Z_k^\epsilon = G_\epsilon Z_k^\epsilon + P_3 V_k;$$

$$F_\zeta \nabla_0^n Z_k^\zeta = G_\zeta Z_k^\zeta + P_4 V_k;$$

$$0_{h,g} \cdot \nabla_0^n Z_{k+1}^g = 0_{h,g} \cdot Z_k^g + P_5 V_k.$$

Results

1. Theorem for existence of solutions for both regular and singular case;
2. Theorem for uniqueness of solutions for given initial or boundary condition;
3. Formulas for the case of unique solutions.
4. Optimal Solutions for the case of non-consistent initial and boundary conditions;
5. Reformulated version of the Kalman filter, in order to produce an algorithm and gain an optimal solution;
6. Stability;
7. Robustness;
8. Duality: The analysis of the prime system provides all information we need for its Dual system and Transpose Dual system.

- Dassios I., *Optimal solutions for non-consistent singular linear systems of fractional nabla difference.* **Circuits, Systems and Signal Processing**, Springer, Volume 34, Issue 6, pp. 1769-1797 (2015).
- Dassios I., Baleanu D., Kalogeropoulos, G., *On non-homogeneous singular systems of fractional nabla difference equations*, **Applied Mathematics and Computation,** Elsevier, Volume 227, pp. 112-131 (2014).
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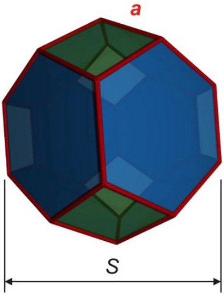
References

- Dassios I., *Stability and robustness of singular systems of fractional nabla difference equations.* **Circuits, Systems and Signal Processing**, Springer. Accepted (2016).
- Dassios I., *Geometric relation between two different types of initial conditions of singular nabla fractional discrete time systems.* **Mathematical Methods in the Applied Sciences**, Willey. Accepted (2016).
- Dassios I., Baleanu D., *Duality of singular linear systems of fractional nabla difference equations.* **Applied Mathematical Modeling**, Elsevier, Volume 39, Issue 14, pp. 4180-4195 (2015).

A mathematical model for elasticity, plasticity and damage

- Collaboration with AMEC Foster Wheeler, UK and Prof Andrey Jivkov (University of Manchester, UK);
- Dr. Peter James (AMEC Foster Wheeler, UK) provides specific isotropic materials;
- The task is: When Forces are applied to random parts of the material to simulate the
  (a) elastic;
  (b) plastic;
  (c) and failure behavior of the material.
- We propose a discrete lattice model.

Discrete lattice model



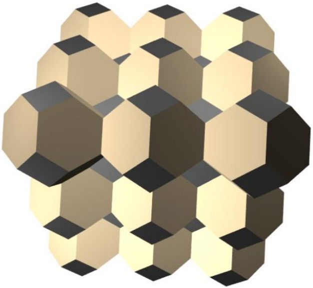


Figure: Truncated octahedron (left), representing an average single grain. Assembly of cells, filling compactly a 3D region (right), representing topologically averaged polycrystal.

Discrete lattice model

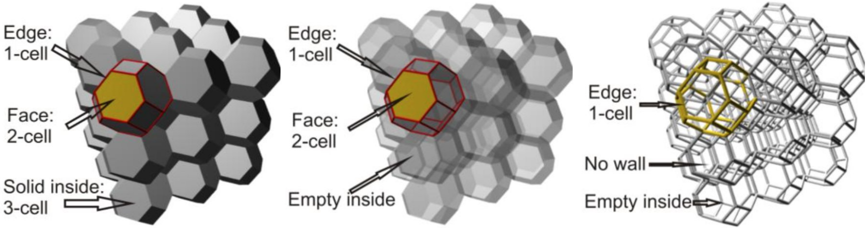
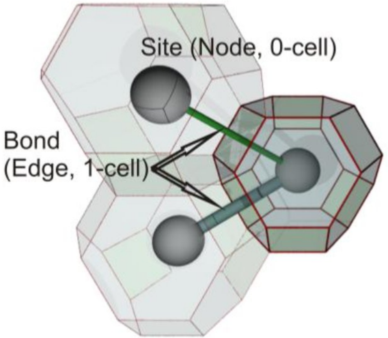


Figure: Illustration of 3-complex (left), 2-complex (centre), and 1-complex (right), embedded in $\mathbb{R}^3$. Illustration based on the topologically averaged polycrystal.

Discrete lattice model



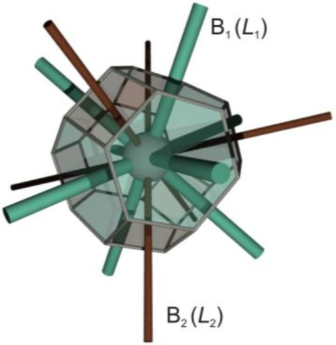
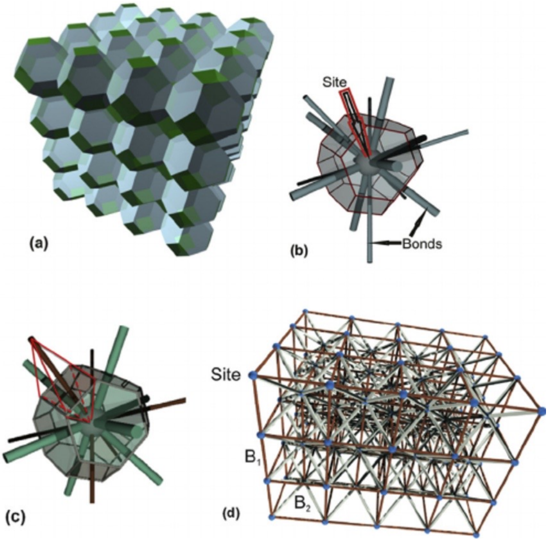


Figure: Illustration of site-bond model construction as a reduction of 3-complex to 1- complex (graph). Left figure shows a 3-cell with two neighboring 3-cells which goes into 3 sites (0-cells, graph nodes) connected by bonds (1-cells, graph edges). Right figure shows a single 3-cell with the 14 bonds (1-cells, graph edges) in the substituting site-bond topology.

Discrete lattice model



Steps – A simple description of the problem

- **Initial Positions** of n nodes and m edges

$$D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix} \in \mathbb{R}^{n\times 3}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^{m\times 3}$$

$D_j = (D_1^j, D_2^j, D_3^j),\ j = 1, 2, ..., n$
$b_i = (b_1^i, b_2^i, b_3^i),\ i = 1, 2, ..., m$

- Then:

$$AD = b,$$

A is the incidence matrix (Laplacian) of the graph, describes the connectivity;

Steps – A simple description of the problem

- **New Positions** after Forces are applied to the material

$$AX = y,$$

X, y new position of nodes and edges;

- **New Positions** of n nodes and m edges

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \in \mathbb{R}^{n\times 3}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \in \mathbb{R}^{m\times 3}$$

$X_j = (X_1^j, X_2^j, X_3^j)$, $j = 1, 2, ..., n$
$y_i = (y_1^i, y_2^i, y_3^i)$, $i = 1, 2, ..., m$.

- **Relation between Forces and new positions of edges**.

$$F = K(y)y;$$

where

$$F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix} \in \mathbb{R}^{m\times 3}, \quad K(y) = diag\left\{\frac{|F_1|}{|y_1|}, \frac{|F_2|}{|y_2|}, \dots, \frac{|F_m|}{|y_m|}\right\} \in \mathbb{R}^{m\times m}.$$

Steps – A simple description of the problem

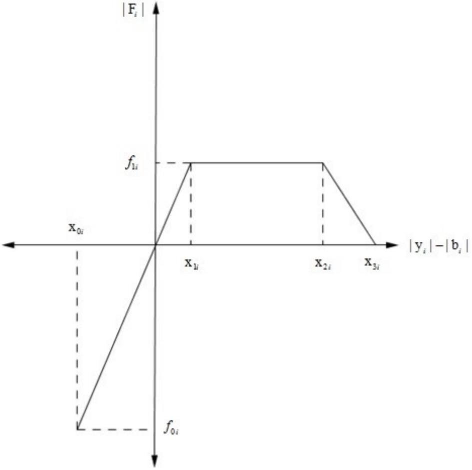


Figure: Example of Conceptual edge forces-displacement relation.

Steps – A simple description of the problem

- **Relation between Forces and reaction Forces on the edges**

$$A^T F = B;$$

- We conclude to the **Non-linear system**

$$A^T K(AX)AX = B.$$

Results

1. By using the facilities (laboratories) of the University of Manchester we can have graphs (plots) between the magnitude of the forces and the difference between initial & new lengths of the edges;
2. We solve numerically the non-linear system;
3. We built a Software which predicts the elasticity, plasticity and cracking for the material provided.

References

- Dassios I., Jivkov A., Abu-Muharib A., James P., *A mathematical model for plasticity and damage: A discrete calculus formulation.* **Journal of Computational and Applied Mathematics**, Elsevier. Accepted (2016).

- The software implementing the above results is in the use of "AMEC Wheeler Foster".

A Partial Differential Equation's geometric evolution problem

- Networks of curves in bounded planar domains formed by triple junctions:

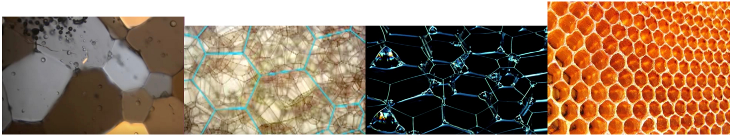


Figure: Polycrystalline camphor-ethanol mixture, soap bubbles, honeycomb

A Partial Differential Equation's geometric evolution problem

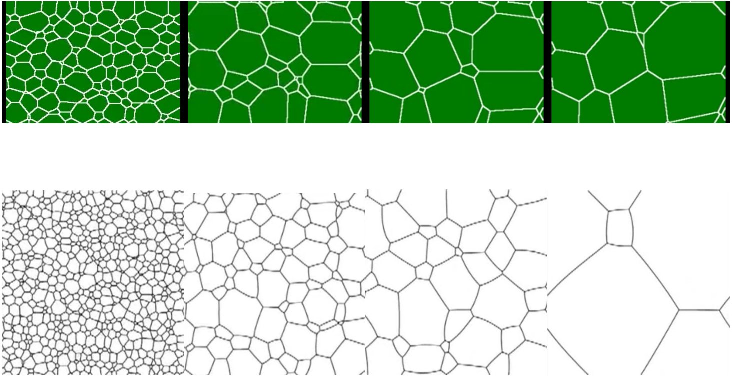


Figure: Evolution of a network

Problem formulation

- The evolution of $G_i(s,t)$ is described by

$$\frac{\partial G_i}{\partial t} = \frac{\partial^2 G_i}{\partial s^2}, \quad i = 1, 2, ..., m, \tag{2}$$

- Where

$$G_i(\cdot, t) : [0, L_i(t)] \to \Omega,$$

be smooth functions such that $G_i(\cdot, t)\mid_{(0,L_i(t))}$ is an embedding and $\|\partial_s G_i(s,t)\| = 1$;

$$L_i : [0, +\infty) \to [0, +\infty),$$

satisfying $L_i(0) = 0$. For each $t \geq 0$ $L_i(t)$ is the length of the curve G_i; $t \geq 0$ is time, s, $0 \leq s \leq L_i(t)$, arc length parameter and $G_i(s,t)$, contained in Ω.

Subject to four conditions

1. Incidence at the point at which the curves intersect:

$$G_i(L_i(t),t) = G_p(L_p(t),t) = G_r(L_r(t),t),$$

2. Angle conditions at the point at which the curves intersect:

$$G_{is}(L_i(t),t)\cdot G_{ps}(L_p(t),t) = cos\frac{2\pi}{3};$$

3. Incidence at $\partial\Omega$:

$$b(G_i(L_i(t),t) = 0;$$

4. Angle conditions at $\partial\Omega$:

$$< G_{is}(L_i(t),t), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla b(G_i) >= 0.$$

$G_{is} = \frac{\partial G_i}{\partial s}$, $G_{iss} = \frac{\partial^2 G_i}{\partial s^2}$, $G_{it} = \frac{\partial G_i}{\partial t}$, $\partial\Omega$ is the boundary of Ω, $b(\cdot,\cdot)$ is a C^1 real function of two variables that describes locally the boundary $\partial\Omega$ and $< \cdot,\cdot >$ is Euclidean inner product.

Steps

- We formulate the problem in a way that the arc length parameter s takes its values in a domain independent from time t (parametrisation);
- With an appropriate transformation $s = s(x)$ we arrive at the non-linear PDEs:

$$\frac{\partial \Gamma_i}{\partial t} = \frac{1}{|\partial \Gamma_i/\partial x|^2}\frac{\partial^2 \Gamma_i}{\partial x^2} - \frac{\partial \Gamma_i}{\partial x}\frac{1}{(ds/dx)^3}\frac{d^2 s}{dx^2}$$

  and will be defined in the set $\mathcal{D}$=$[0, l_i] \times [0, +\infty)$.
- Define the Linearized Operator;
- Find the Eigenvalues & Eigenfunctions of this operator;
- Results on the stability of the steady states in terms of the geometry of the boundary.

References

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- Boutarfa B., Dassios I., *A stability result for a network of two triple junctions on the plane.* **Mathematical Methods in the Applied Sciences**, Willey. Accepted (2016).
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Notes – Future Work

- Identifying the problem in $3D$;
- Research on Grain growth: The increase in size of grains (crystallites) in a material at high temperature. This occurs when recovery and recrystallisation are complete and further reduction in the internal energy can only be achieved by reducing the total area of grain boundary;

–THANK YOU FOR YOUR ATTENTION–

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