

Stochastic Cubature Rule for Probabilistic Power Flow with High-Dimensional Uncertainty

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Abstract This paper proposes the stochastic cubature rule (SCR)-based uncertainty quantification method for probabilistic power flow (PPF) analysis under high-dimensional uncertainties. By applying the Haar-distributed orthogonal transformation to the conventional cubature rule, the coordinate components of the cubature points are redistributed across dimensions. SCR requires only a small number of sample points with equal positive weights. Analytical results indicate that SCR can improve higher-order moment approximation and help mitigate the “non-local sampling effect”. Combined with the copula transformation, the proposed method further accommodates complex distributions and correlations. Tests on modified IEEE 300-and 3120-bus systems evaluate the proposed method against several representative methods in terms of statistical moment calculation, probability density function (PDF) fitting, and confidence interval (CI) estimation. The results verify that SCR achieves satisfactory accuracy, computational efficiency, and robustness under diverse uncertainty conditions.

Keywords: Correlation, cubature rule, orthogonal rotation, probabilistic power flow, uncertainty.

1 Notation and Abbreviations

Tables 1 and 2 provide a complete list of symbols and abbreviations, respectively, utilized in the paper.

2 Introduction

2.1 Motivations

Probabilistic Power Flow (PPF) is an essential tool for the analysis of power system performance under source-load uncertainty [1]. Typical PPF applications include optimal operation, expansion planning, and stability evaluation [2, 3]. In general, PPF is required to provide accurate solutions under stringent computation time requirements. However, high-dimensional uncertainty sources, nonlinear power-flow models, and strong dependencies among uncertain inputs pose substantial challenges to achieve these

Table 1: List of Symbols

Symbol	Description	Symbol	Description
N	Dimension of the random input vector	ξ_n	n -th SCR cubature point
$\mathbf{Y}$	System response output variables	$\mathbf{T}$	Gaussian random matrix used for QR decomposition
$\mathbf{f}(\cdot)$	Model of deterministic power flow	$\widehat{\mathbf{Q}}$	Orthogonal factor obtained from QR decomposition
$\mathbf{X}$	Random input vector with arbitrary distribution	$\mathbf{R}$	Upper-triangular factor obtained from QR decomposition
$h(\cdot)$	Response function for statistical quantity evaluation	$\mathbf{D}$	Diagonal sign-correction matrix
$\mathcal{P}(\cdot)$	PDF of input random variables	R_{ii}	i -th diagonal element of $\mathbf{R}$
$\mathcal{D}$	Support of input random variables	$\text{diag}(\cdot)$	Diagonal-matrix operator
$\mathbb{E}[\cdot]$	Expectation operator	$\text{sign}(\cdot)$	Sign function
$\bar{\mathbf{Y}}$	Mean of output variables	$\widetilde{M}_{4,j}$	SCR estimate of the fourth-order moment in the j -th dimension
k	Order of statistical moment	N_ℓ	Number of distinct dimension indices in (19)
$(\cdot)^{\circ k}$	Element-wise k -th power	b_ℓ	Selected dimension index in (19)
$\circ$	Hadamard product	a_ℓ	Nonnegative integer
P_l	Active power demand of load	A	Sum of exponent parameters in (19)
Q_l	Reactive power demand of load	$!!$	Double factorial operator
$G_N(\cdot)$	Gaussian PDF of load power	$\text{Var}[\cdot]$	Variance operator
$\bar{P}_l/\sigma_{P_l}$	Mean/STD of active load power	$\widetilde{M}_{2k,j}$	SCR estimate of the $2k$ -th moment in the j -th dimension
$\bar{Q}_l/\sigma_{Q_l}$	Mean/STD of reactive load power	N_m	Number of paired observations for Spearman correlation
$P_{WT}/P_{WT\max}$	Actual/rated active power of wind turbines	α, β	Random variables used in the rank-correlation definition
$v/v_r/v_{ci}/v_{co}$	Actual/rated/cut-in/cut-out wind speeds	$r_{-\alpha m}/r_{-\beta m}$	Ranks of the m -th observations of α and β
$G_W(v)$	Weibull PDF of wind speed	$\bar{r}_{-\alpha}/\bar{r}_{-\beta}$	Average ranks of α and β
α_w/β_w	Shape/scale parameter of Weibull distribution	ρ_s	Spearman correlation coefficient
P_{PVG}	Active power output of photovoltaic systems	$H(\cdot)$	Joint distribution function
$\gamma/\gamma_{\max}$	Actual/maximum solar irradiance	$\varphi_i(\cdot)$	Marginal distribution function of the i -th random variable
A_{pv}	Effective area of the photovoltaic cell array	$C(\cdot)$	copula function
η	Energy efficiency of PV power plants	s, t	Integration variables in the Gaussian copula
$G_B(\gamma)$	Beta PDF of solar irradiance	ρ	Pearson correlation coefficient
$\Gamma(\cdot)$	Gamma function	$\mathbf{U}$	Random vector following the uniform distribution
α_b, β_b	Shape parameters of Beta distribution	U_i	i -th element of $\mathbf{U}$
ξ	Independent standard Gaussian random vector	X_i	i -th element of $\mathbf{X}$
$\mathbf{I}_N$	N -dimensional identity matrix	Σ_Z	Covariance matrix of random variables $\mathbf{Z}$
$\mathcal{T}(\cdot)$	Transformation from Gaussian space to the original random-variable space	$\mathbf{L}$	Cholesky factor of Σ_Z
$\phi_N(\cdot)$	PDF of the standard Gaussian distribution	$\mathbf{Z}$	Correlated Gaussian random vector
ξ_n	n -th conventional cubature point	$\mathbf{Z}_n$	n -th correlated Gaussian cubature point
W_n	Weight of the n -th cubature point	$\mathbf{X}_n$	n -th cubature point in the original random-variable space
ζ_n	n -th deterministic cubature direction	$\mathbf{Y}_n$	DPF response corresponding to $\mathbf{X}_n$
$\xi_{n,i}$	i -th component of the n -th cubature point	ε_{CI}	Averaged relative error of the 95% CI width
$\mathbf{Q}$	Random orthogonal matrix	w_i/w_i^*	Estimated/reference 95% CI width at bus i
$\mathcal{O}_N$	Orthogonal group of order N	ε_{KLD}	KLD-based PDF estimation error
μ	Normalized Haar measure on $\mathcal{O}_N$	$\vartheta(x)/\vartheta^*(x)$	Estimated/reference PDF
$\mathbb{S}^{N-1}$	Unit hypersphere in the N -dimensional space	$\varpi(n)$	Index mapping of an antipodal cubature-point pair
$\mathbf{q}_n$	Haar-rotated cubature direction	$F(\cdot)$	CDF of the standard Gaussian distribution
$\mathbf{e}$	Unit vector in $\mathbb{R}^N$		

Table 2: List of Abbreviations

Abbreviation	Description	Abbreviation	Description
CI	Confidence interval	MC	Monte Carlo method
CP	Cubature point	PCE	Polynomial chaos expansion
CR	Cubature rule	PDF	Probability density function
DPF	Deterministic power flow	PEM	Point estimate method
EV	Electric vehicle	PPF	Probabilistic power flow
EVc	Charging-station electric vehicle	PV	Photovoltaic system
EVr	Residential electric vehicle	QR	QR decomposition
FT	Field-theoretic density estimation	REG	Renewable energy generator
GMM	Gaussian mixture model	SCR	Stochastic cubature rule
GP	Gaussian process	STD	Standard deviation
KDE	Kernel density estimation	UT	Unscented transformation
KLD	Kullback–Leibler divergence	WF	Wind farm
LRA	Low-rank approximation	CDF	Cumulative distribution function

targets [4]. These challenges motivate the development of efficient and scalable uncertainty quantification methods for PPF analysis.

2.2 Literature review

Mainstream methods for PPF include Monte Carlo-based methods, analytical methods, surrogate-model-based methods, and approximation methods. Monte Carlo (MC) simulation propagates a large number of random input samples through the deterministic power flow (DPF) model and estimates the probability distributions of the resulting system responses [5]. It can provide high numerical accuracy but generally requires a large number of DPF evaluations. To reduce the number of model evaluations, field-theoretic (FT) density estimation has been employed to reconstruct output probability distributions from a limited number of samples [6, 7]. Analytical methods [8, 9] often rely on model simplifications, series expansions, or specific assumptions regarding input distributions and dependence structures. Their accuracy may deteriorate for strongly nonlinear and high-dimensional PPF problems. Surrogate-model-based methods mainly include low-rank approximation (LRA) [10], polynomial chaos expansion (PCE) [11, 12], and Gaussian process (GP) [13–15] methods. They reduce model evaluation costs by constructing approximate mappings from uncertain inputs to output responses. Approximate methods mainly include the point estimate method [16] (PEM) and the unscented transformation (UT) [17] method, both of which characterize input uncertainties using a limited number of deterministic points and associated weights. However, matching higher-order statistical moments in PEM may require additional estimation points or more sophisticated point-selection schemes. In high-dimensional problems, UT may assign a large negative weight to the central point, potentially compromising its numerical stability. Moreover, UT has limited capability to capture higher-order statistical moments [18, 19].

In 2009, Arasaratnam [20] introduced the cubature Kalman filter based on the spherical-radial cubature rule (CR). Compared with UT, CR employs equal positive weights and therefore avoids numerical difficulties associated with large negative sigma-point weights. CR can provide efficient numerical ap-

proximation for the Gaussian-weighted integral. It has been widely applied in fields such as navigation, aircraft attitude estimation, fault detection and isolation, and power system state estimation. Nevertheless, its application to PPF analysis in power systems has been rarely explored. Reference [21] develops three CR-based schemes for estimating the output moments of probabilistic optimal power flow and employs the Nataf transformation to map correlated uncertainties into the independent standard Gaussian space. However, there are two inherent limitations of CR. First, each conventional CP contains only one nonzero coordinate with magnitude $\sqrt{N}$, where N is the dimension of the random input vector. Consequently, the coordinate magnitude is concentrated along a single axis, resulting in increasingly extreme marginal excursions and severe mismatch of higher-order moments as N increases. This behavior contributes to the non-local sampling effect in high-dimensional spaces [22]. Second, the CR point set is constructed for Gaussian-weighted integrals. Direct application of these symmetric points in an asymmetric or non-Gaussian input space may therefore fail to accurately represent the original probabilistic characteristics.

In summary, existing PPF methods still struggle to simultaneously achieve high computational accuracy, efficiency, and scalability to high-dimensional problems. Developing probabilistic methods that require fewer samples, ensure robust numerical stability, and accurately capture probabilistic characteristics remains an important research focus.

2.3 Contributions

To improve the accuracy and efficiency of PPF under high-dimensional uncertainties, the SCR-based PPF method is developed. The Haar-distributed orthogonal transformation is introduced into the conventional CR to redistribute the coordinate components of the CPs across dimensions without increasing the number of DPF evaluations. Analytical results indicate that SCR can improve higher-order moment approximation and help mitigate the “non-local sampling effect”. Furthermore, the copula transformation is incorporated to accommodate correlated and complex uncertainties. Numerical studies under varying uncertainty dimensions, renewable penetration levels, and correlation strengths demonstrate the accuracy, efficiency, and robustness of the proposed method.

2.4 Organization

The remainder of this paper is organized as follows. Section 3 formulates the PPF problem and introduces probabilistic models of typical uncertain sources. Section 4 first introduces the conventional CR and analyzes its high-dimensional limitations, then develops the Haar-rotation-based SCR and investigates its moment-matching properties. The copula-based extension and the complete SCR-based PPF procedure are also subsequently presented. Section 5 evaluates the proposed method through numerical case studies, and Section 6 concludes the paper and discusses future work.

3 Probabilistic power flow

3.1 Formulation of the PPF problem

PPF aims to quantify the statistical characteristics of power system responses resulting from uncertain power injections. Nonlinear mapping defined by the DPF can be written as

$$\mathbf{Y} = \mathbf{f}(\mathbf{X}), \quad (1)$$

where $\mathbf{X}$ and $\mathbf{Y}$ represent input uncertainties and corresponding system responses, respectively. For an arbitrary response function $\mathbf{h}(\mathbf{Y})$, the corresponding statistical quantity of $\mathbf{Y}$ can be expressed as the following multidimensional probabilistic integral

$$\mathbb{E}[\mathbf{h}(\mathbf{Y})] = \int_{\mathcal{D}} \mathbf{h}(\mathbf{f}(\mathbf{X})) \mathcal{P}(\mathbf{X}) d\mathbf{X}, \quad (2)$$

where $\mathcal{P}(\cdot)$ denotes the probability density function (PDF) of input random variables, $\mathcal{D} \subseteq \mathbb{R}^N$ denotes their support, and $\mathbb{E}[\cdot]$ denotes the expectation operator. $\mathbf{h}(\cdot)$ provides a unified representation for different statistical quantities. For example, setting $\mathbf{h}(\mathbf{Y}) = \mathbf{Y}$ yields the mean $\bar{\mathbf{Y}} = \mathbb{E}[\mathbf{h}(\mathbf{Y})]$; whereas setting $\mathbf{h}(\mathbf{Y}) = (\mathbf{Y} - \bar{\mathbf{Y}})^{\circ k}$ yields the k -th order moment. Here, $(\cdot)^{\circ k}$ denotes the element-wise power and $\circ$ denotes the Hadamard product. In practice, the estimated statistical moments characterize the probabilistic features of system responses and provide the basis for reconstructing the output PDFs and evaluating probabilistic indices.

3.2 Probabilistic characteristics of typical uncertain sources

Stochastic features of power systems primarily stem from time-varying loads and stochastic renewable energy generators (REGs). Models considered in this paper include conventional loads and EV charging loads on the demand side, and stochastic wind and photovoltaic (PV) generation on the generation side. Mathematical models for the probabilistic characteristics of the above uncertainty sources are provided below.

3.2.1 Conventional loads

Gaussian distributions are commonly adopted to characterize the active-and reactive-power demands, P_l and Q_l , respectively [23]:

$$G_N(P_l) = \frac{1}{\sqrt{2\pi}\sigma_{P_l}} \exp\left[-\frac{(P_l - \bar{P}_l)^2}{2\sigma_{P_l}^2}\right], \quad (3)$$

$$G_N(Q_l) = \frac{1}{\sqrt{2\pi}\sigma_{Q_l}} \exp\left[-\frac{(Q_l - \bar{Q}_l)^2}{2\sigma_{Q_l}^2}\right], \quad (4)$$

where $\bar{P}_l$ and σ_{P_l} are the mean and standard deviation (STD) of P_l , respectively. $\bar{Q}_l$ and σ_{Q_l} are the mean and STD of Q_l , respectively.

3.2.2 Electric vehicles

Charging behavior of a single EV is primarily characterized by its operational status, battery capacity, and daily driving mileage. A detailed charging-demand model for individual EVs and EV aggregates is provided in [24]. On this basis, this paper considers a cluster of EVs connected to the same bus and characterizes their aggregate charging behavior using queuing theory [25]. Based on the K-S test results reported in [24], the aggregate charging power of charging-station EVs (EVc) is modeled using the Weibull distribution, whereas that of residential EVs (EVr) is modeled using the Gaussian distribution.

3.2.3 Wind power plants

The active power output of the wind turbine is approximately as a piecewise function of the wind speed:

$$P_{WT} = \begin{cases} 0, & v \leq v_{ci}, \\ \frac{P_{WT\max}}{v_r - v_{ci}}v - \frac{P_{WT\max}}{v_r - v_{ci}}v_{ci}, & v_{ci} < v \leq v_r, \\ P_{WT\max}, & v_r < v \leq v_{co}, \\ 0, & v > v_{co}, \end{cases} \quad (5)$$

where $P_{WT\max}$ is the rated power of the wind turbine generator. v_r , v_{ci} , and v_{co} denote the rated, cut-in, and cut-out wind speeds, respectively. v represents the actual wind speed. The randomness of wind speed is characterized by the Weibull distribution, and the PDF is expressed as follows [26]:

$$G_W(v) = \frac{\alpha_w}{\beta_w} \left(\frac{v}{\beta_w}\right)^{\alpha_w-1} \exp\left[-\left(\frac{v}{\beta_w}\right)^{\alpha_w}\right], \quad v > 0, \quad (6)$$

where α_w and β_w denote the shape and scale parameters of the Weibull distribution, respectively.

3.2.4 PV power plants

The active power output of a PV power plant is linearly proportional to the solar irradiance [27]:

$$P_{\text{PVG}} = \gamma A_{\text{pv}} \eta, \quad (7)$$

where A_{pv} denotes the effective area of the photovoltaic cell array, η represents the energy conversion efficiency of the PV power station, and γ is the solar irradiance, which is generally assumed to follow the Beta distribution:

$$G_{\text{B}}(\gamma) = \frac{\Gamma(\alpha \cdot b + \beta \cdot b)}{\Gamma(\alpha \cdot b) \Gamma(\beta \cdot b)} \cdot \left(\frac{\gamma}{\gamma_{\max}} \right)^{\alpha \cdot b - 1} \cdot \left(1 - \frac{\gamma}{\gamma_{\max}} \right)^{\beta \cdot b - 1}, \quad (8)$$

where $\Gamma(\cdot)$ denotes the Gamma function. $\gamma_{\max}$ represents the maximum solar irradiance within a period. $\alpha \cdot b$ and $\beta \cdot b$ are the shape parameters of the Beta distribution.

4 Stochastic cubature rule

According to Section 3.1, the PPF problem can be interpreted as the numerical evaluation of multidimensional probabilistic integrals. Therefore, PPF results can be approximated by applying efficient numerical integration schemes, which motivates the development of the proposed method. Section 4.1 reviews the conventional CR and analyzes its high-dimensional limitations. Section 4.2.1 and 4.2.2 introduce the Haar-rotation-based SCR and theoretically characterize its moment-matching properties, followed by numerical validation in Section 4.2.3. Section 4.2.4 extends SCR to correlated and non-Gaussian uncertainties, and Section 4.2.5 summarizes the complete SCR-based PPF procedure.

4.1 Principle and limitation of conventional cubature rule

The CR is an efficient numerical integration scheme for Gaussian-weighted integrals. Let $\boldsymbol{\xi} \in \mathbb{R}^N$ denotes the independent standard Gaussian random vector, i.e., $\boldsymbol{\xi} \sim \mathcal{N}(0, \mathbf{I}_N)$. Through the transformation $\mathcal{T}(\boldsymbol{\xi}) = \mathbf{X}$, the probabilistic integral in (2) can be rewritten as

$$\mathbb{E}[\mathbf{h}(\mathbf{Y})] = \int_{\mathbb{R}^N} \mathbf{h}(\mathbf{f}(\mathcal{T}(\boldsymbol{\xi}))) \phi_N(\boldsymbol{\xi}) d\boldsymbol{\xi}, \quad (9)$$

where $\phi_N(\cdot)$ is the PDF of $\boldsymbol{\xi}$; $\mathcal{T}(\cdot)$ is the transformation function from $\boldsymbol{\xi}$ to the actual power-system random variables $\mathbf{X}$. As a non-product rule, CR uses a fixed set of axis-aligned CPs $\boldsymbol{\xi}_n$, $n = 1, \dots, 2N$, together with their corresponding weights to approximate the above integral, i.e.

$$\mathbb{E}[\mathbf{h}] \approx \sum_{n=1}^{2N} W_n \mathbf{h}(\mathbf{f}(\mathcal{T}(\boldsymbol{\xi}_n))), \quad (10)$$

where $W_n = 1/(2N)$ is the weight of ξ_n , $\sum_{n=1}^{2N} W_n = 1$; $\xi_n = (\sqrt{N}\zeta)_n$, where ζ is constructed from a set of basis vectors in the N -dimensional random variable space with corresponding sign variations. For example, $\zeta \in \mathbb{R}^2$ represents the following set of points:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}. \quad (11)$$

Accuracy of approximation methods can be characterized by their moment-matching conditions. CPs of CR method satisfy:

$$\sum_{n=1}^{2N} W_n \xi_{n,i} = 0, \quad (12)$$

$$\sum_{n=1}^{2N} W_n \xi_{n,i}^2 = 1, \quad (13)$$

$$\sum_{n=1}^{2N} W_n \xi_{n,i}^3 = 0, \quad (14)$$

where $\xi_{n,i}$ denotes the i -th element of ξ_n . These conditions exactly match the corresponding moments of the independent standard Gaussian distribution. CR is algebraically exact up to degree three using only $2N$ points. Moreover, CR is independent of the explicit form of the DPF model, allowing the power-flow solver to be treated as a black box. As each CP has an equal positive weight, ratio between the sum of absolute weights and their algebraic sum equals unity, avoiding numerical instability caused by negative weights.

However, CR exhibits limited capability in matching higher-order moments. As shown in (11), each CP has only one nonzero component, so its coordinate magnitude is concentrated along a single dimension. For any dimension i , only two CPs have nonzero components in that dimension, with values $\sqrt{N}$ and $-\sqrt{N}$, respectively. Consequently, the estimated higher-order even moments scale with increasing powers of N . Taking the fourth- and sixth-order moments as examples, their values estimated by CR are N and N^2 , respectively, whereas the corresponding exact moments of the standard Gaussian distribution are 3 and 15. This mismatch becomes increasingly pronounced as the dimension increases and propagates through the DPF mapping, reducing estimation accuracy of output statistics.

4.2 Stochastic cubature rule-based PPF method

4.2.1 Haar rotation and point set reconstruction

To improve higher-order moment approximation without increasing the number of CPs, the random orthogonal matrix $\mathbf{Q}$ drawn according to the Haar measure is introduced to rotate the original axis-aligned cubature-point set. The orthogonal transformation preserves the central symmetry while

redistributing their coordinate components across dimensions. Define the orthogonal group $\mathcal{O}_N$, where $\mathcal{O}_N = \{\mathbf{Q} \in \mathbb{R}^{N \times N} : \mathbf{Q}^T \mathbf{Q} = \mathbf{I}_N\}$. There is a unique normalized, left and right-translation invariant measure μ [28, 29]. For any measurable $\mathcal{X} \subset \mathcal{O}_N$, μ satisfies

$$\mu(\mathbf{Q}\mathcal{X}) = \mu(\mathcal{X}\mathbf{Q}) = \mu(\mathcal{X}). \quad (15)$$

For any unit vector $\mathbf{e} \in \mathbb{R}^N$, there is

$$\mathbf{Q}\mathbf{e} \sim \text{Uniform}(\mathbb{S}^{N-1}), \quad (16)$$

where $\mathbb{S}^{N-1}$ denotes the unit hypersphere in the N -dimensional space. Furthermore, according to Borel's lemma, for any fixed i and j , $\sqrt{N}Q_{ij}$ converges in distribution to the standard Gaussian distribution as $N \rightarrow \infty$.

Core operation of SCR involves the introduction of $\mathbf{Q}$ into the CPs generation process. CP set of SCR is:

$$\tilde{\boldsymbol{\xi}}_n = \sqrt{N}\mathbf{q}_n, \quad n = 1, 2, \dots, 2N, \quad (17)$$

where $\mathbf{q}_n = \mathbf{Q}\boldsymbol{\zeta}_n$, and the weight of each CP is still $1/(2N)$. In this paper, QR-based procedure is employed to generate matrix $\mathbf{Q}$ [30], with the detailed steps given as follows: First, an $N \times N$ matrix $\mathbf{T}$ is generated with independent standard normal entries, i.e., $T_{ij} \sim N(0, 1)$. Second, the QR decomposition is performed, $\mathbf{T} = \hat{\mathbf{Q}}\mathbf{R}$. Third, since the sign convention of $\mathbf{R}$ affects $\hat{\mathbf{Q}}$, the diagonal correction matrix $\mathbf{D}$ is introduced as $\mathbf{D} = \text{diag}(\text{sign}(R_{11}), \dots, \text{sign}(R_{NN}))$, where $\text{diag}(\cdot)$ denotes the operator that forms a diagonal matrix, R_{ii} is the i -th diagonal element of the matrix $\mathbf{R}$, and

$$\text{sign}(R_{ii}) = \begin{cases} 1, & R_{ii} \geq 0, \\ -1, & R_{ii} < 0. \end{cases} \quad (18)$$

Then, $\mathbf{Q}$ can be calculated as $\mathbf{Q} = \hat{\mathbf{Q}}\mathbf{D}$.

4.2.2 Theoretical analysis of moment-matching properties

Due to the orthogonality of $\mathbf{Q}$ and the symmetric positive-negative structure of the CPs, SCR exactly preserves the second-order moment matrix of the standard Gaussian distribution. Moreover, all odd-order moments vanish due to the symmetric structure of SCR points. The following analysis therefore focuses on even-order moments.

According to (16), for any fixed n , $\tilde{\boldsymbol{\xi}}_n$ is uniformly distributed on the $\sqrt{N}\mathbb{S}^{N-1}$. Based on Proposition

2.5 in Meckes' work [29], there is

$$\mathbb{E} \left[\prod_{\ell=1}^{N_\ell} \tilde{\xi}_{n, b_\ell}^{2a_\ell} \right] = \frac{N^A \prod_{\ell=1}^{N_\ell} (2a_\ell - 1)!!}{N(N+2) \cdots (N+2A-2)}, \quad A = \sum_{\ell=1}^{N_\ell} a_\ell, \quad (19)$$

where $\tilde{\xi}_{n, b_\ell}$ is the b_ℓ -th element of $\tilde{\xi}_n$; $b_\ell \in \{1, \dots, N\}$ denotes the selected dimension index, and $b_{N_1}, \dots, b_{N_\ell}$ are distinct; N_ℓ is the number of selected distinct dimensions; a_ℓ is nonnegative integer; and $!!$ denotes the odd double factorial. According to (19), expectation of the fourth-order moment estimated by SCR CPs is

$$\begin{aligned} \mathbb{E} [\widetilde{M}_{4,j}] &= \mathbb{E} \left[\frac{1}{2N} \sum_{n=1}^{2N} \tilde{\xi}_{n,j}^4 \right] \\ &= \frac{1}{2N} \sum_{n=1}^{2N} \mathbb{E} [\tilde{\xi}_{n,j}^4] \\ &= N^2 \cdot \frac{3!!}{N(N+2)} \\ &= \frac{3N}{N+2}, \end{aligned} \quad (20)$$

where $\widetilde{M}_{4,j}$ is the estimate of the fourth-order moment in the j -th dimension. The absolute bias of the fourth-order moment estimator is

$$\left| \mathbb{E} [\widetilde{M}_{4,j}] - 3 \right| = \frac{6}{N+2}. \quad (21)$$

This bias decreases with N increasing and approaches zero as $N \rightarrow \infty$, indicating that $\widetilde{M}_{4,j}$ is asymptotically exact in expectation. To further quantify the dispersion of $\widetilde{M}_{4,j}$ with respect to $\mathbf{Q}$, its variance is derived as follows:

$$\begin{aligned} \text{Var} [\widetilde{M}_{4,j}] &= \mathbb{E} \left[\left(\widetilde{M}_{4,j} \right)^2 \right] - \left[\mathbb{E} \left(\widetilde{M}_{4,j} \right) \right]^2 \\ &= \frac{24N^2(N-1)}{(N+2)^2(N+4)(N+6)}, \end{aligned} \quad (22)$$

and the detailed derivation of (22) is provided in Appendix A. For sufficiently large N , $\text{Var} [\widetilde{M}_{4,j}] \approx 24/N$, indicating that dispersion of $\widetilde{M}_{4,j}$ across different Haar rotations decreases as dimension increases. Combining (21) and (22), both the bias and variance of $\widetilde{M}_{4,j}$ vanish as $N \rightarrow \infty$. Specifically, the mean-square error satisfies

$$\mathbb{E} \left[\left(\widetilde{M}_{4,j} - 3 \right)^2 \right] = \text{Var} (\widetilde{M}_{4,j}) + \left(\mathbb{E} [\widetilde{M}_{4,j}] - 3 \right)^2 \rightarrow 0, \quad (23)$$

indicating that $\widetilde{M}_{4,j}$ converges in mean square to the exact fourth-order moment.

The above fourth-order analysis can be extended to higher even-order moments. Setting $N_\ell = 1$,

$a_1 = k$ in (19), for any fixed positive integer k , the $2k$ -th order moment estimated using the SCR CPs in the j -th dimension satisfies

$$\begin{aligned}\mathbb{E}[\widetilde{M}_{2k,j}] &= \frac{1}{2N} \sum_{n=1}^{2N} \mathbb{E}[\widetilde{\xi}_{n,j}^{2k}] \\ &= (2k-1)!! \prod_{r_a=1}^{k-1} \frac{N}{N+2r_a}.\end{aligned}\tag{24}$$

For any fixed k , each factor $N/(N+2r_a)$ approaches unity as $N \rightarrow \infty$, while $(2k-1)!!$ is the exact $2k$ -th order moment of the standard Gaussian distribution. Therefore, for any fixed even order, the SCR moment estimate is asymptotically exact in expectation over the Haar measure. The dispersion of higher even-order moment estimators can be analyzed following a similar procedure to the fourth-order moment.

4.2.3 Numerical validation of moment-matching properties

To numerically verify the theoretical results in Section 4.2.2, a 1000-dimensional standard Gaussian space is considered. SCR is compared with CR and the classical PEM ($2N+1$ points) [16] in terms of moment matching and sampling characteristics. Table 3 reports the moment estimates and corresponding relative errors. For SCR, 500 independent Haar realizations are performed to assess the influence of the randomness of $\mathbf{Q}$, and the average estimated values and average relative errors are reported. CR exhibits rapidly increasing errors for higher-order even moments. PEM accurately matches the low-order moments of the standard Gaussian distribution, but its errors increase markedly for higher even-order moments. In contrast, SCR provides consistently accurate estimates of higher-order even moments. The mean fourth-order moment is 2.99, close to the exact value of 3. Although the average relative error of SCR increases with moment order, it remains substantially lower than those of CR and PEM over the investigated higher-order moments. These results demonstrate that SCR maintains superior moment-estimation accuracy across different Haar realizations of $\mathbf{Q}$.

Figure 1 further shows the PDFs reconstructed from the sampling points using kernel density estimation (KDE), where $\text{SCR}(\mathbf{Q}_1), \dots, \text{SCR}(\mathbf{Q}_{10})$ denote ten independent Haar realizations. The SCR curves closely follow the exact Gaussian density with only minor variations across different $\mathbf{Q}$, whereas CR shows the largest deviation, followed by PEM. Overall, the results demonstrate that SCR effectively captures the statistical characteristics of high-dimensional Gaussian distributions using only $2N$ sampling points.

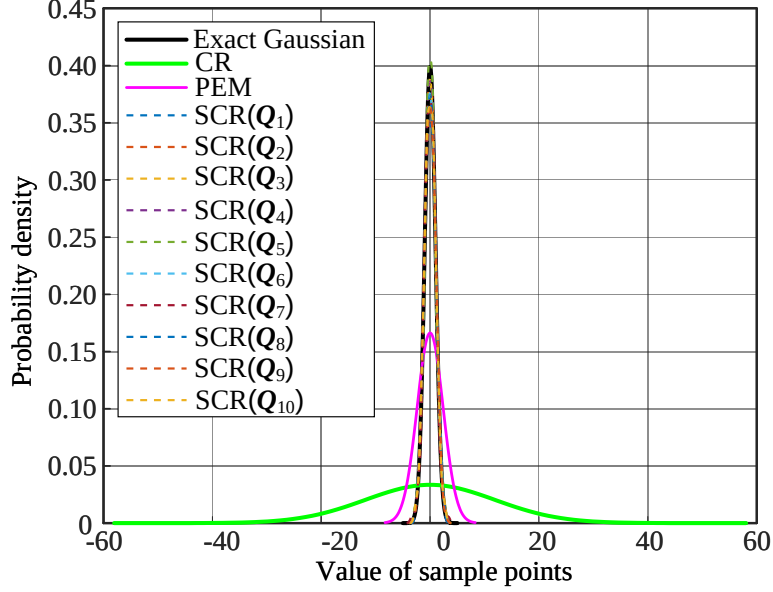


Figure 1: Sampling distributions in the 1000-dimensional standard Gaussian space.

Table 3: Moment-estimation performance in the 1000-dimensional Gaussian space.

Order	Gaussian true value	Estimated value			Average relative error		
		CR	PEM	SCR	CR	PEM	SCR
2	1	1	1	1	0	0	0
4	3	1000	3	2.99	332.33	0	0.04
6	15	1.0×10^6	9	14.81	66666	0.40	0.13
8	105	1.0×10^9	27	101.87	9.52×10^6	0.74	0.26
10	945	1.0×10^{12}	81	887.78	1.06×10^9	0.91	0.42
12	10395	1.0×10^{15}	243	9261.7	9.62×10^{10}	0.98	0.60

4.2.4 Handling correlated and non-Gaussian uncertainties

The CPs and corresponding weights of both CR and SCR are constructed for Gaussian-weighted integrals [20]. However, in power systems, uncertainties often exhibit significant non-Gaussian characteristics, and correlations may exist among multiple random variables [1, 31]. As noted in [17, 32], a symmetric point selection strategy applied directly to asymmetric distributions fails to capture the statistical characteristics accurately. Therefore, directly employing the SCR to handle correlated random variables with arbitrary distributions in PPF analysis may degrade computational accuracy. To extend SCR to realistic power-system uncertainties, Gaussian copula and Spearman correlation are incorporated into SCR framework. This establishes the mapping from the independent standard Gaussian space used for cubature-point construction to the correlated and non-Gaussian input space. Figure 2 illustrates the transformation of the Gaussian-distributed variables into actual variables in power system.

Spearman correlation (also called rank correlation) can effectively capture the monotonic dependence between random variables. Since both the cumulative distribution function (CDF) and its inverse are

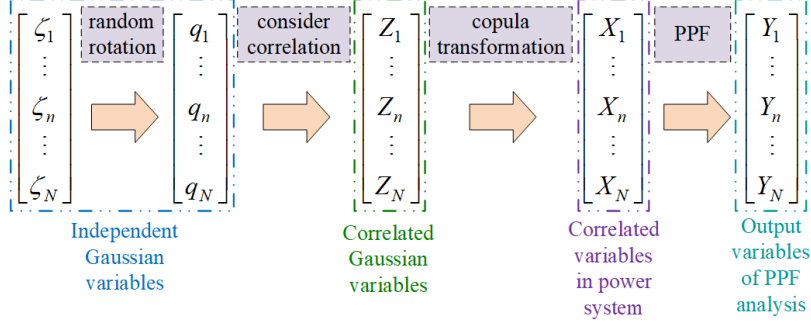


Figure 2: Schematic diagram of the transformation process among variables in the SCR method.

strictly monotonic, Spearman correlation remains invariant under changes in the marginal distributions of the variables. This property implies that the rank correlation is preserved when a set of random variables with arbitrary distributions is transformed to the standard Gaussian distribution [33].

Consider N_m pairs of observations (α_m, β_m) of random variables α and β . Let $r_{-\alpha_m}$ and $r_{-\beta_m}$ denote the ranks of α_m and β_m within their respective samples, and let $\bar{r}_{-\alpha}$ and $\bar{r}_{-\beta}$ represent the average ranks. The rank correlation coefficient $\rho_s(\alpha, \beta)$ between α and β is then defined as:

$$\rho_s(\alpha, \beta) = \frac{\sum_{m=1}^{N_m} [(r_{-\alpha_m} - \bar{r}_{-\alpha})(r_{-\beta_m} - \bar{r}_{-\beta})]}{\sqrt{\sum_{m=1}^{N_m} (r_{-\alpha_m} - \bar{r}_{-\alpha})^2} \sqrt{\sum_{m=1}^{N_m} (r_{-\beta_m} - \bar{r}_{-\beta})^2}}. \quad (25)$$

Copula theory states that an N -dimensional joint distribution function can be decomposed into N marginal distribution functions and a linking (copula) function, with the dependence among variables represented by this copula. For the N -dimensional random vector $\mathbf{X} = [X_1, X_2, \dots, X_N]^T$ with a joint CDF $H(X_1, X_2, \dots, X_N)$. If all marginal CDFs $F_i(X_i)$, ($i = 1, 2, \dots, N$) are continuous, then there exists a unique copula function, such that:

$$H(X_1, \dots, X_N) = C(F_1(X_1), \dots, F_N(X_N)). \quad (26)$$

Common copula families primarily include the elliptical and Archimedean classes. In this paper, the Gaussian copula (an elliptical copula) is employed to construct the mapping from stochastic CPs to the actual variables; other copulas can be implemented following the same procedure. The mathematical form of the Gaussian copula under the bivariate case is given as follows:

$$C(\alpha, \beta) = \iint_{-\infty}^{\varphi_{\alpha}^{-1} \varphi_{\beta}^{-1}} \frac{1}{2\pi\kappa} \exp\left(\frac{2\rho st - s^2 - t^2}{2\kappa^2}\right) ds dt, \quad (27)$$

where s and t are the variables of integration, ρ is the Pearson correlation coefficient between α and β , and $\kappa = \sqrt{1 - \rho^2}$.

If α and β follow the Gaussian distribution, a well-defined relationship exists between the Pearson correlation coefficient ρ and the Spearman correlation coefficient ρ_s , which is given by [34]:

$$\rho(\alpha, \beta) = 2 \sin\left(\frac{\pi}{6} \rho_s(\alpha, \beta)\right) . \quad (28)$$

The Gaussian copula introduces the dependence structure through the correlated Gaussian vector $\mathbf{Z}$. Specifically, the correlation matrix $\mathbf{\Sigma}_Z$ is constructed according to the Pearson correlation coefficients obtained from (28). Subsequently, the Cholesky decomposition is applied to $\mathbf{\Sigma}_Z$ to transform independent standard Gaussian variables into the correlated Gaussian variables. Then, the correlated Gaussian variables $\mathbf{Z}$ are transformed into standard uniform variables as

$$U_i = \Phi(Z_i), \quad i = 1, \dots, N, \quad (29)$$

where $\Phi(\cdot)$ denotes the CDF of the standard Gaussian distribution. For a given set of marginal distributions, the original random variables can then be obtained through the inverse marginal transformation:

$$X_i = F_i^{-1}(U_i), \quad i = 1, \dots, N, \quad (30)$$

where $F_i^{-1}(\cdot)$ represents the inverse CDF of the i -th marginal distribution.

4.2.5 Procedure of the proposed SCR-based PPF

The detailed procedural steps for implementing SCR-based PPF are outlined below.

Step 1: Estimate the marginal PDFs of the uncertain inputs and their pairwise Spearman correlation coefficients from historical data.

Step 2: Convert the Spearman correlation coefficients to the corresponding Pearson correlations using (28).

Step 3: Construct the correlation matrix $\mathbf{\Sigma}_Z$ in the Gaussian space and obtain its Cholesky factor $\mathbf{L}$, such that $\mathbf{\Sigma}_Z = \mathbf{L}\mathbf{L}^T$.

Step 4: Generate the $2N$ conventional cubature directions $\{\zeta_n\}_{n=1}^{2N}$.

Step 5: Generate the Haar-distributed orthogonal matrix $\mathbf{Q}$ and construct the SCR points $\{\tilde{\xi}_n\}_{n=1}^{2N}$ according to (17).

Step 6: Introduce the Gaussian dependence structure through $\mathbf{Z}_n = \mathbf{L}\tilde{\xi}_n$, $n = 1, 2, \dots, 2N$.

Step 7: Transform each component of the correlated Gaussian vector into the uniform space using $U_{n,i} = \Phi(Z_{n,i})$, where $U_{n,i}$ and $Z_{n,i}$ denote the i -th components of the n -th sample point in the uniform and correlated Gaussian spaces, respectively.

Step 8: Transform the uniform variables into the original distributions using $X_{n,i} = F_i^{-1}(U_{n,i})$,

where $F_i^{-1}(\cdot)$ denotes the inverse CDF of the i -th original distribution.

Step 9: Evaluate the DPF model for the resulting $2N$ input points to obtain $\{\mathbf{Y}_n\}_{n=1}^{2N}$.

Step 10: Estimate the required output statistics using $\mathbb{E}[\mathbf{h}(\mathbf{Y})] \approx \sum_{n=1}^{2N} W_n \mathbf{h}(\mathbf{Y}_n)$.

5 Case study

To validate the performance of the proposed method, simulations are conducted on the IEEE 300-bus and 3120-bus systems. The PPF results obtained by MC are adopted as the benchmark solution. Comparative studies are performed against representative PPF approaches, including the CR, PEM, three surrogate-model methods (PCE, LRA, and GP), and MC-based method (FT). MC employs 20,000 samples as the reference solution. PEM uses its prescribed $2N+1$ estimating points. For CR, PCE, LRA, GP, and FT, the number of DPF evaluations is set equal to that of SCR to ensure a fair comparison, with 10,000 independent samples are propagated through the corresponding surrogate models. The PCE model adopts polynomial degrees from 1 to 3, the LRA model is constructed using rank-one polynomial expansion, and the GP model employs a squared-exponential kernel. For FT, the output probability density is reconstructed first, and the statistical moments are calculated from the estimated density. All simulations are implemented in MATLAB R2022a with MATPOWER 7.1 on a computer equipped with an AMD Ryzen 9 7940HX (2.40 GHz) processor and 16.0 GB memory.

5.1 IEEE 300-bus system

The IEEE 300-bus system includes three test cases to comprehensively evaluate the performance of the proposed method. Section 5.1.1 investigates the scalability and accuracy of the proposed method with increasing uncertainty dimensions. Section 5.1.2 evaluates the adaptability of the proposed method under different renewable energy penetration levels. Section 5.1.3 further examines the robustness of the proposed method under different correlation levels among uncertain variables.

5.1.1 Case 1 – Tests with different uncertainty dimensions

Assume that the active and reactive power fluctuations of loads follow Gaussian distributions, where the mean equals the original load value and the STD is set to 5% of the mean. To evaluate the performance of different methods under increasing uncertainty dimensions, 10, 40, 80, and 150 loads are treated as uncertain sources in Scenarios 1–4, respectively.

Figure 3 shows the two-dimensional projections of the sample points generated by MC, CR, and SCR onto the dimensions associated with Loads 1 and 2. Due to the axis-aligned construction of CR points, multiple samples may overlap after projection onto a low-dimensional subspace, resulting in limited directional coverage. As the uncertainty dimension increases, CR points tend to concentrate along several

predefined directions and deviate further from the high-probability region of the Gaussian space, which weakens their capability to represent high-dimensional uncertainty characteristics. In contrast, SCR introduces Haar-distributed orthogonal rotations to redistribute directional components across dimensions while preserving the original cubature structure. Consequently, SCR points provide broader directional coverage and more closely follow the distribution of the MC samples.

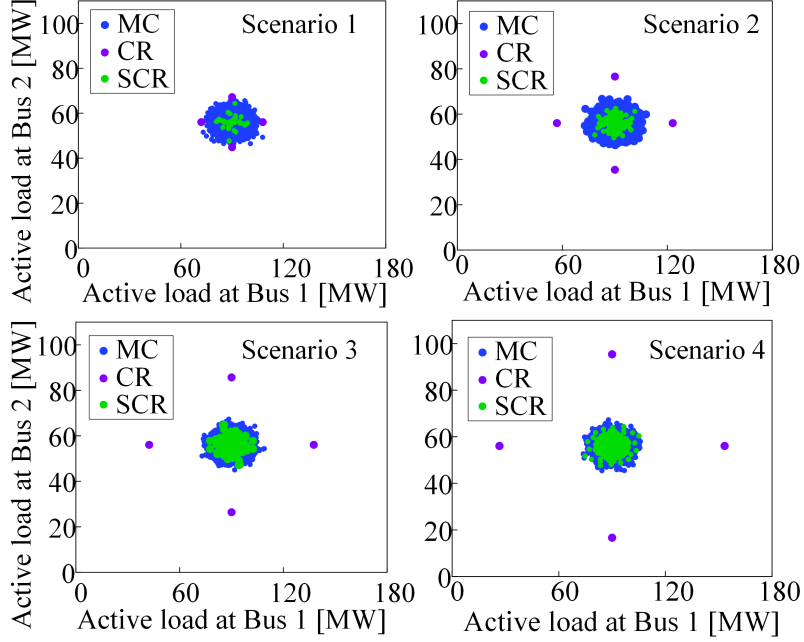


Figure 3: IEEE 300-bus system, Case 1 – Sample-point projections generated by MC, CR, and SCR under different uncertainty dimensions.

To quantitatively evaluate the overall accuracy of moment estimation across all evaluated buses, the total relative error of the k -th order moment is defined as

$$\varepsilon_{mk} = \sum_{i=1}^{N_b} \frac{|\widetilde{M}_{k,i} - M_{k,i}^*|}{|M_{k,i}^*|}, \quad (31)$$

where ε_{mk} denotes the total relative error of the k -th order moment over all evaluated buses; N_b is the number of evaluated buses (i.e., all PQ buses in the test system); $\widetilde{M}_{k,i}$ is the estimated k -th order moment at bus i obtained by the considered method; and $M_{k,i}^*$ is the corresponding reference value obtained by MC simulation.

Tables 4 and 5 compare the relative errors of the third- and fourth-order, respectively, voltage-magnitude moment estimations under different uncertainty dimensions. As the uncertainty dimension increases, the errors of all methods generally increase, indicating the increasing difficulty of capturing probabilistic characteristics in high-dimensional spaces. SCR consistently achieves the lowest errors, with only 0.36% and 0.50% errors for the third- and fourth-order moments under 150 uncertain inputs, respectively. CR performs well in low-dimensional cases due to its third-degree algebraic accuracy, but its

fixed-direction cubature structure limits the representation of high-order statistical features as the dimension increases. PEM achieves comparable accuracy to SCR in low-dimensional scenarios through moment matching, while its accuracy degrades for higher-order moments in high-dimensional cases. With the same DPF evaluation budget, PCE, LRA, and GP exhibit progressively larger errors as the uncertainty dimension increases, reflecting the increasing difficulty of constructing accurate high-dimensional surrogates from a limited training set. FT also exhibits increased moment errors at higher dimensions, as its moment estimates are obtained from the reconstructed output density. The superior performance of SCR is attributed to the Haar-distributed orthogonal rotation, which improves the directional representation of cubature points in high-dimensional probability spaces while preserving third-degree accuracy. Consequently, SCR provides more accurate higher-order moment estimates across the investigated uncertainty dimensions.

Table 4: IEEE 300-bus system, Case 1 – Errors of third-order voltage-magnitude moments [%].

Method	Scen. 1	Scen. 2	Scen. 3	Scen. 4
CR	0.02	0.13	0.32	1.21
SCR	0.02	0.09	0.29	0.36
PEM	0.02	0.09	0.29	0.43
PCE	0.09	0.30	0.45	1.01
LRA	0.03	0.34	0.38	1.51
GP	0.13	0.22	0.49	1.20
FT	0.22	0.24	1.52	1.95

Table 5: IEEE 300-bus system, Case 1 – Errors of fourth-order voltage-magnitude moments [%].

Method	Scen. 1	Scen. 2	Scen. 3	Scen. 4
CR	0.02	0.17	0.39	1.47
SCR	0.02	0.12	0.38	0.50
PEM	0.02	0.12	0.39	0.58
PCE	0.12	0.40	0.60	1.37
LRA	0.04	0.45	0.51	2.01
GP	0.18	0.30	0.66	1.60
FT	0.30	1.22	2.42	2.59

5.1.2 Case 2 – Tests with different renewable energy penetration levels

To further evaluate the proposed method under more complex uncertainty configurations, multiple wind farms (WFs), PV plants, residential, and charging-station EV loads are incorporated into Scenario 4 of Section 5.1.1. Four renewable location penetration levels, i.e., 0.1, 0.3, 0.6, and 0.9, are considered, which are the ratio of buses equipped with renewable generation to the total number of candidate buses. Detailed parameters of renewable generation and EV loads are provided in Appendix B.

The averaged relative error of the 95% confidence-interval (CI) width ε_{CI} is used to quantify the

accuracy of interval estimation. It is defined as

$$\varepsilon_{\text{CI}} = \frac{1}{N_b} \sum_{i=1}^{N_b} \frac{|w_i - w_i^*|}{w_i^*}, \quad (32)$$

where w_i denotes the 95% CI width of the voltage magnitude at bus i estimated by the evaluated method, and w_i^* is the corresponding width obtained by MC. Fig. 4 compares ε_{CI} under different renewable penetration levels. As the penetration level increases, most methods exhibit increased errors, indicating greater difficulty in accurately capturing the output uncertainty range. In contrast, SCR consistently achieves the lowest error, remaining below 4% over all penetration levels. This result confirms that the proposed method maintains accurate and stable interval estimation under varying renewable integration conditions.

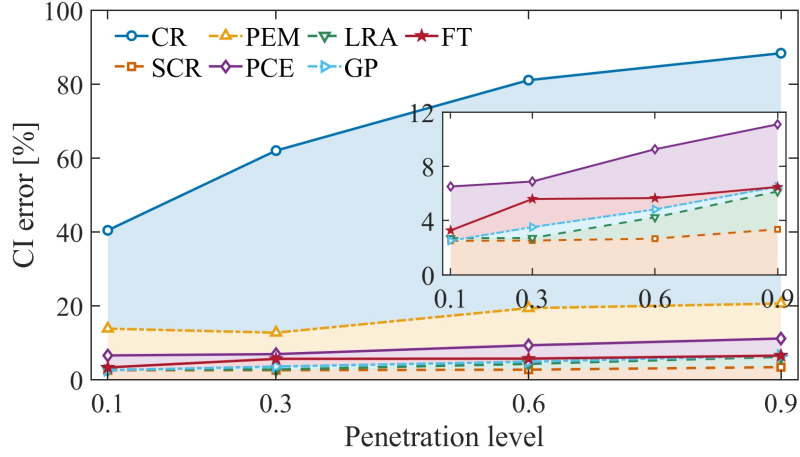


Figure 4: IEEE 300-bus system, Case 2 – Comparison of 95% CI errors under different penetration levels.

Fig. 5 compares PDFs of the voltage magnitude at bus 168. As the penetration level increases, the distribution shifts toward lower voltages and becomes broader, indicating increased voltage uncertainty and higher risk of limit violations. To quantitatively assess the PDF estimation accuracy, the Kullback–Leibler divergence (KLD) is adopted [35]:

$$\varepsilon_{\text{KLD}} = \int \vartheta(x) \log \frac{\vartheta(x)}{\vartheta^*(x)} dx, \quad (33)$$

where $\vartheta(x)$ is the PDF estimated by the evaluated method, and $\vartheta^*(x)$ denotes the reference PDF obtained by MC. The KLD is non-negative. Smaller KLD values indicate closer agreement with the MC reference. As shown in Fig. 6, SCR consistently yields the lowest KLD across all penetration levels, while the errors of the surrogate-based methods generally increase with penetration. These results demonstrate that SCR accurately captures the voltage distribution and maintains stable PDF estimation accuracy as renewable penetration increases.

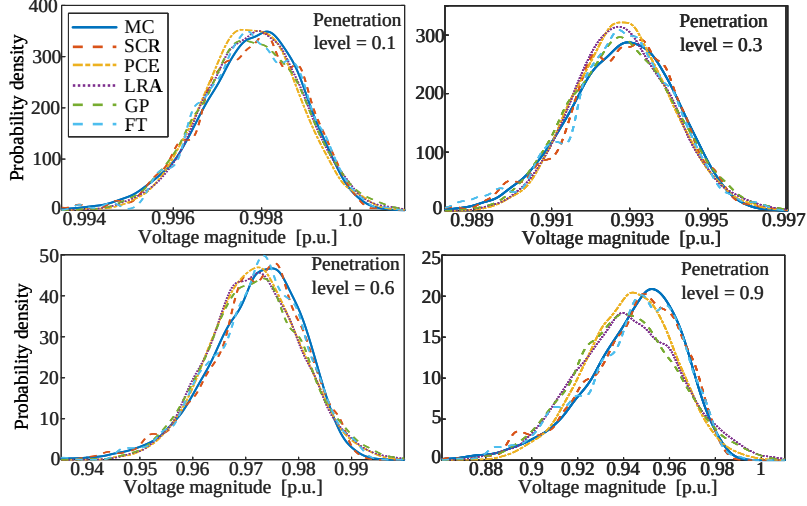


Figure 5: IEEE 300-bus system, Case 2 – Comparison of PDFs estimated by different methods.

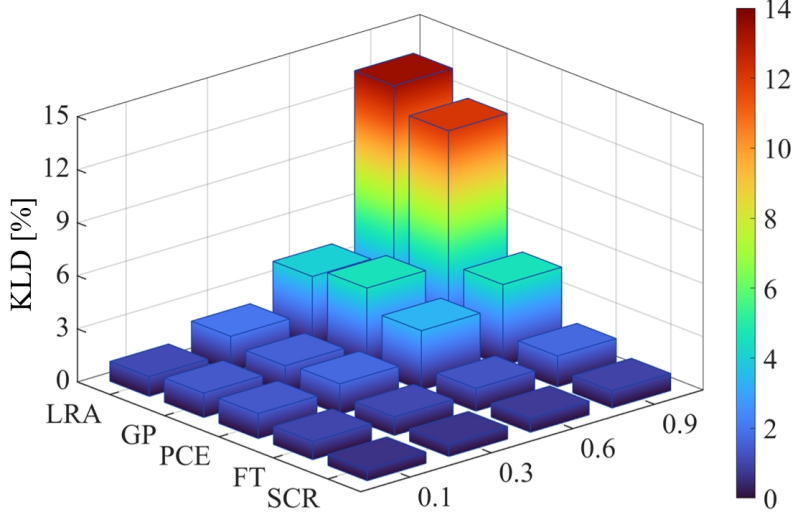


Figure 6: IEEE 300-bus system, Case 2 – KLD values under different penetration levels.

5.1.3 Case 3 – Tests with different correlation levels

To further evaluate the performance of the proposed method under correlated uncertainties, the scenario with 0.3 renewable penetration level in Section 5.1.2 is adopted as the base case. Spearman correlation coefficients among the first 20 stochastic loads are set to 0.24, 0.48, 0.72, and 0.96, representing four correlation levels.

Figure 7 illustrates the sample-point projections under different correlation levels. As the correlation increases, the MC samples exhibit progressively stronger dependence, while the SCR samples remain well aligned with the main probability region characterized by MC. Fig. 8 further compares the errors of the first four moments. The errors generally increase with both correlation level and moment order, reflecting the increasing difficulty of accurately propagating strongly correlated uncertainties. Among the compared methods, SCR consistently yields the lowest errors and exhibits the least sensitivity to increasing correlation, particularly for the third- and fourth-order moments. This performance can be

attributed to the Haar-distributed orthogonal transformation, which preserves the low-order accuracy of the cubature rule while improving the representation of higher-order probabilistic characteristics. In contrast, the errors of PCE and FT increase more noticeably under strong correlation, whereas LRA and GP show relatively moderate degradation. Overall, the results confirm that SCR maintains stable moment-estimation accuracy under different correlation levels.

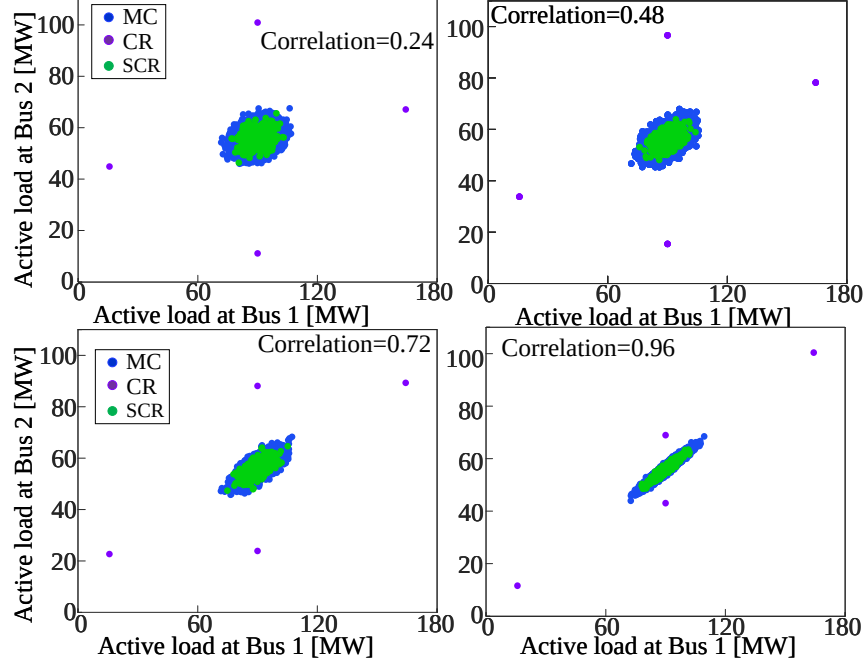


Figure 7: IEEE 300-bus system, Case 3 – Sample-point projections under different correlation levels.

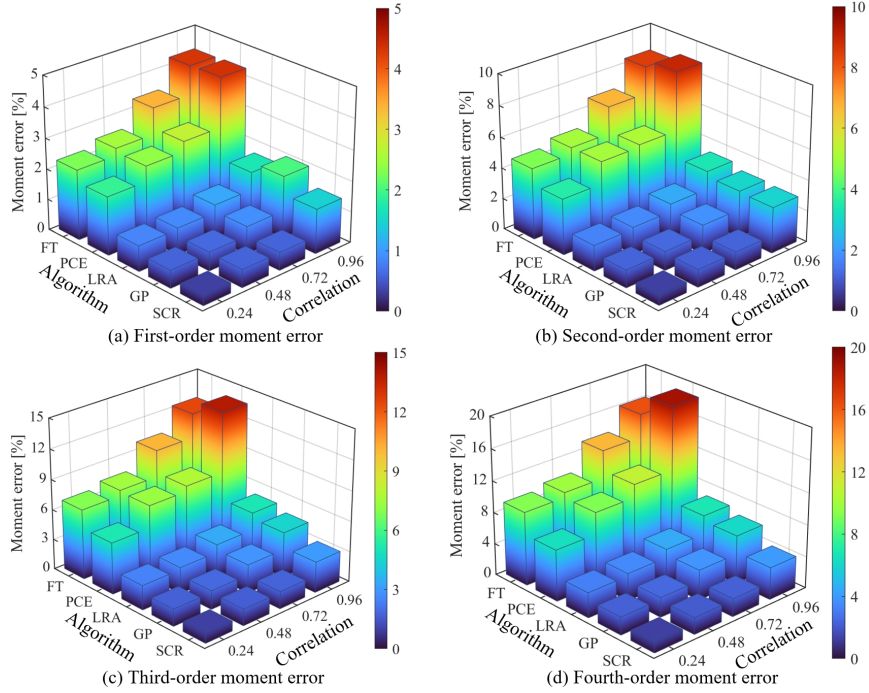


Figure 8: IEEE 300-bus system, Case 3 – Errors of the first four voltage-magnitude moments under different correlation levels.

Table 6 compares the total computation times under different correlation levels. The runtime of each method exhibits relatively limited fluctuations across the four correlation levels, indicating that the correlation strength has no substantial impact on the computational cost. SCR requires only 3.10~3.83 s, which is substantially faster than MC and the surrogate-based methods. Averaged over the four correlation levels, SCR achieves speedups of approximately 36.8, 327.6, and 428.3 times over MC, PCE, and LRA, respectively.

Fig. 9 further decomposes the computational cost at 0.96 correlation level. For MC, the runtime is dominated by DPF evaluations due to its large sample size. CR, SCR, and PEM require only a small number of DPF evaluations and introduce negligible sampling and probabilistic-transformation overhead, resulting in similar computational costs. FT also avoids expensive model construction, although its density reconstruction introduces additional post-processing cost. In contrast, PCE, LRA, and GP incur substantial computational overhead because surrogate models must be constructed before moment evaluation. For SCR, the computational cost is almost entirely attributed to DPF evaluations, whereas point generation, copula transformation, and moment calculation contribute negligibly. These results demonstrate that the improved accuracy of SCR is achieved without introducing appreciable computational overhead.

Table 6: IEEE 300-bus system, Case 3 – Total computation times under different correlation levels [s].

Correlation	MC	CR	SCR	PEM	PCE	LRA	GP	FT
0.24	120.52	3.15	3.10	3.05	1054.5	1360.7	195.79	5.36
0.48	121.32	3.24	3.17	3.15	1010.3	1421.3	193.83	5.14
0.72	121.81	3.19	3.11	3.11	1110.3	1411.6	230.33	6.27
0.96	121.82	3.80	3.83	3.79	1152.7	1464.8	219.65	5.93

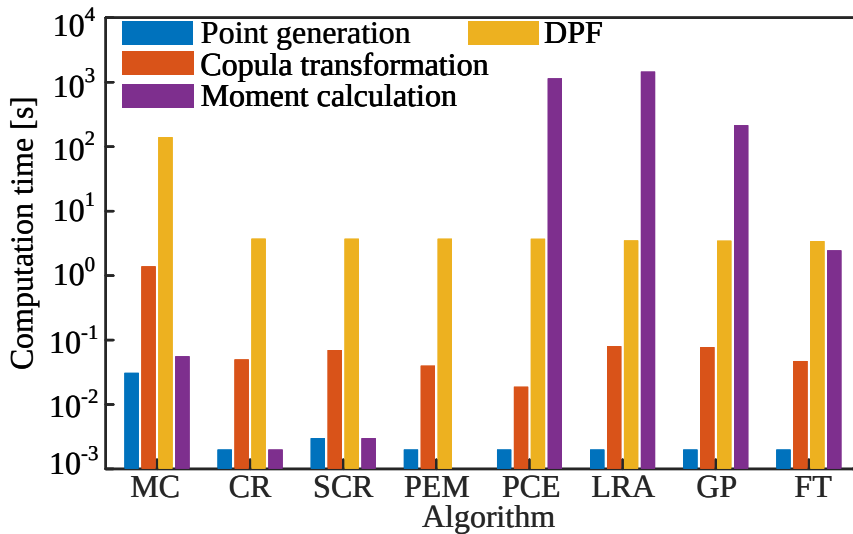


Figure 9: IEEE 300-bus system, Case 3 – Computational time comparison among different methods.

5.2 IEEE 3120-bus system

To further evaluate the applicability of the proposed method to large-scale systems, simulations are conducted on the IEEE 3120-bus system. Active and reactive power demands at 250 randomly selected buses are modeled as Gaussian random variables, with means equal to their nominal values and STDs equal to 8% times the corresponding means. And their correlation coefficients are set to 0.8. EV charging stations are additionally connected at buses 133 and 271, with the parameters given in Table B4 of Appendix B. To examine the adaptability of SCR to general probability distributions, wind farms, PV plants, and park loads are further incorporated into the system. Their stochastic characteristics are derived from measurement data collected from a region in China, and Gaussian mixture models (GMMs) [36] are employed to fit the corresponding marginal distributions. The fitted PDFs and rank-correlation structure are shown in Figs. 10 and 11, respectively, while locations of the stochastic sources are illustrated in Fig. 12. The rated capacities of the wind farm and PV plant are set to 120 MW and 37.8 MW, respectively. The resulting input uncertainty dimension is 520.

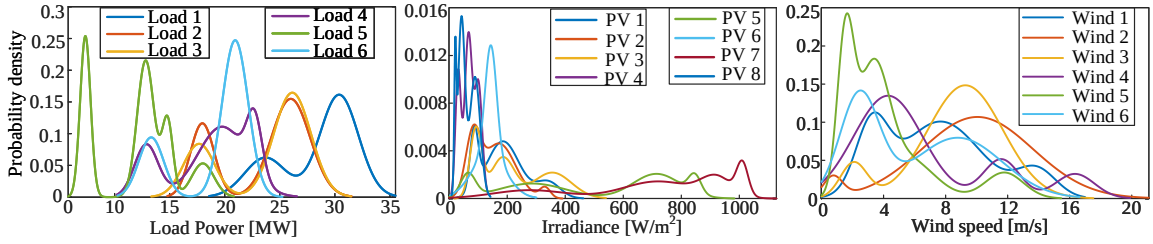


Figure 10: IEEE 3120-bus system – GMM-fitted PDFs of measurement-based stochastic sources.

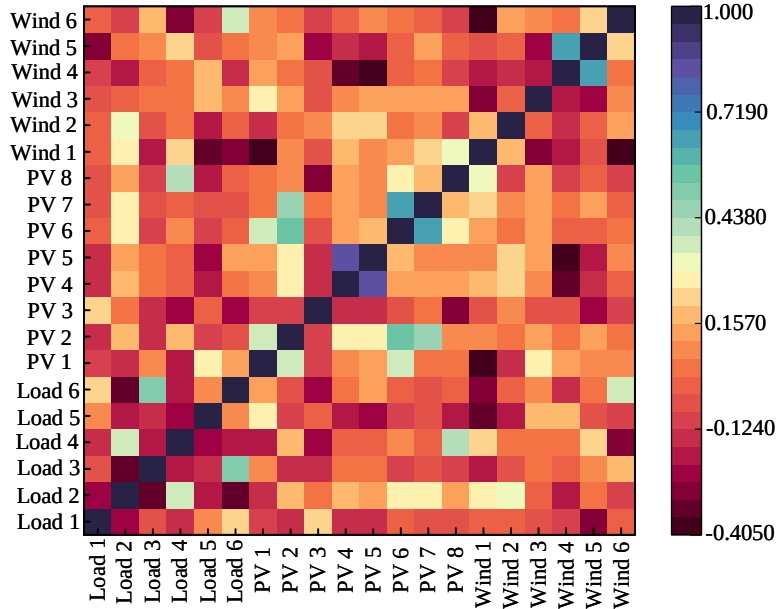


Figure 11: IEEE 3120-bus system – Spearman correlation matrix of measurement-based stochastic sources.

Table 7 compares the total relative errors of the first six voltage-magnitude moments obtained by different methods. The errors generally increase with the moment order, reflecting the increasing difficulty

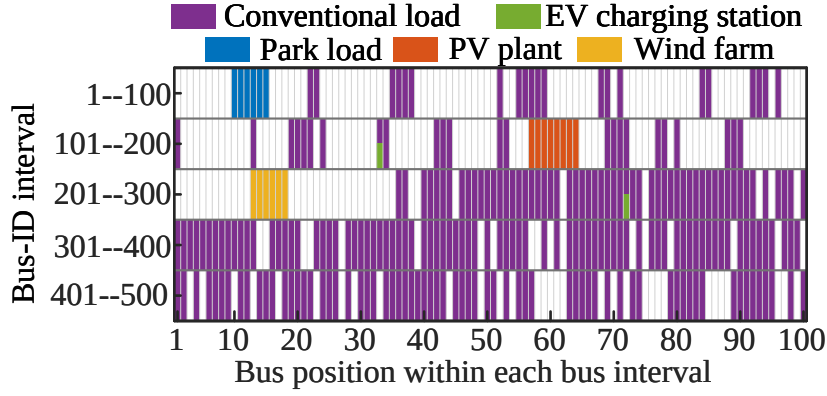


Figure 12: IEEE 3120-bus system – Locations of stochastic sources.

of accurately capturing higher-order statistics. CR exhibits the most pronounced error growth, increasing from 95.73% for the first moment to 678.72% for the sixth moment, while PEM also shows substantial degradation at higher orders. In contrast, SCR consistently yields the lowest errors, ranging from 5.05% to 37.85%, and retains its accuracy advantage as the moment order increases. PCE, LRA, GP, and FT provide substantially better estimates than CR and PEM, but their approximation errors become more evident for higher-order moments. These results are consistent with the theoretical analysis in Section 4.2.2 and confirm the improved higher-order moment-estimation capability of SCR in the large-scale PPF problem.

Table 7: IEEE 3120-bus system – Errors comparison of the first six voltage-magnitude moments [%].

Order	CR	SCR	PEM	PCE	LRA	GP	FT
1	95.73	5.05	25.21	6.34	6.07	6.76	6.31
2	197.80	10.55	53.34	13.48	12.03	14.09	12.95
3	306.65	16.54	84.76	21.49	19.50	22.08	19.95
4	422.74	23.05	119.85	30.41	29.96	30.82	27.34
5	546.59	30.14	159.01	40.32	40.83	40.41	35.14
6	678.72	37.85	202.72	51.29	64.33	50.93	43.38

Fig. 13 compares the voltage-magnitude PDFs estimated by different methods at representative buses of the IEEE 3120-bus system. Compared with the relatively smooth and near-symmetric PDFs observed in Fig. 5, these PDFs exhibit more pronounced non-Gaussian features, including asymmetric shapes and sharper local variations, providing a more challenging test of distribution estimation. Despite such complexity, SCR closely reproduces the MC reference in terms of peak location, peak magnitude, and overall distribution shape, whereas noticeable deviations are observed for the surrogate-based methods at several buses. Quantitatively, the KLD values in Table 6 further confirm this observation. SCR achieves the lowest KLD at all selected buses, ranging from 0.57% to 1.51%, while the other methods exhibit larger and more bus-dependent errors. These results demonstrate the stable PDF estimation accuracy of SCR for complex stochastic inputs in large-scale PPF problems.

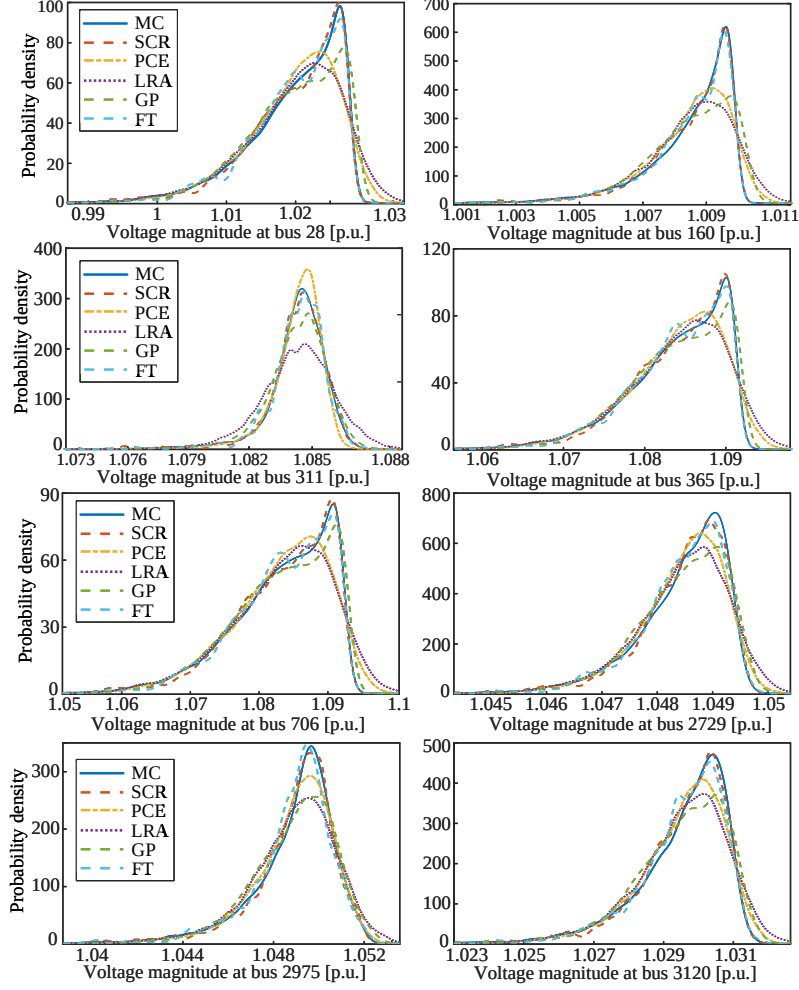


Figure 13: IEEE 3120-bus system – Comparison of PDFs estimated by different methods.

Table 8: IEEE 3120-bus system – KLD values of estimated PDFs at selected buses [%].

ID	SCR	PCE	LRA	GP	FT
28	0.79	18.36	44.36	5.73	1.05
160	1.02	24.44	62.48	21.54	2.41
311	1.51	2.37	27.58	2.87	3.95
365	0.59	16.14	36.27	2.77	1.10
706	0.57	13.02	31.14	1.73	1.07
2729	0.76	3.79	14.58	3.53	1.67
2975	0.99	1.09	10.01	3.91	1.92
3120	0.77	3.84	15.61	4.54	1.90

Table 9 further compares the accuracy of different methods in estimating the 95% CI widths of bus voltage magnitudes. SCR achieves the lowest average relative error of 1.81%, substantially outperforming the comparison methods. Its maximum error is limited to 9.69%, whereas those of the other methods range from 17.89% to 99.70%. These results indicate that SCR provides both accurate and consistent estimates of voltage uncertainty intervals across buses in the large-scale system.

Figure 14 compares the stage-wise and total computational costs of different methods. The compu-

Table 9: IEEE 3120-bus system – Errors of the 95% CI widths estimated by different methods [%].

Statistic	SCR	CR	PEM	PCE	LRA	GP	FT
Average	1.81	73.19	97.54	4.61	12.96	3.29	6.00
Maximum	9.69	96.94	99.70	31.23	51.14	34.24	17.89

tational burden of MC is dominated by repeated DPF evaluations, whereas CR, SCR, and PEM require substantially fewer DPF calculations and therefore exhibit comparable computational efficiency. In particular, the Haar-based orthogonal point generation in SCR requires only 0.019 s, introducing negligible additional overhead. In contrast, PCE, LRA, and GP incur substantial costs in surrogate construction and subsequent statistical evaluation, while FT requires additional field-theory-based processing. Overall, SCR retains a low computational cost while consistently achieving high estimation accuracy in the large-scale system.

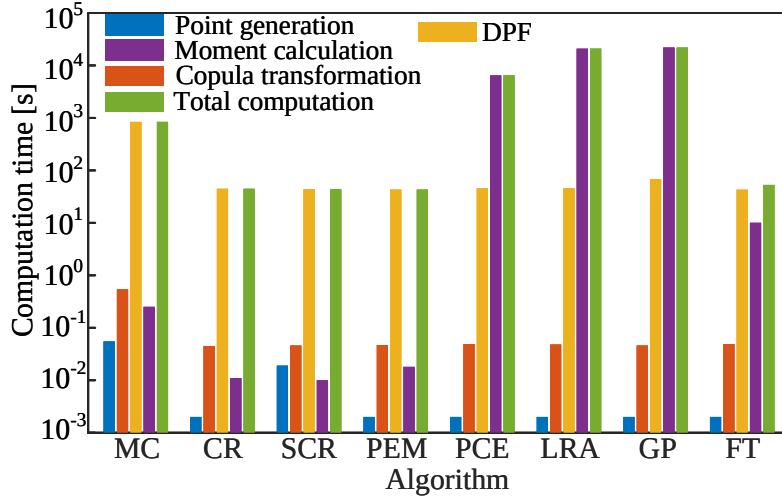


Figure 14: IEEE 3120-bus system – Computational time comparison among different methods.

To further evaluate the robustness of SCR estimates with respect to different matrices $\mathbf{Q}$, 200 independent repeated runs are performed. For each comparison method and each moment order, SCR is considered to achieve a win when its error is lower than that of the comparison method. The win rate measures how frequently SCR provides a more accurate estimate, with a higher value indicating a more consistent accuracy advantage. The minimum win rate is defined as the lowest order-specific win rate and represents the worst-case repeat-run performance of SCR. Minimum win rate above 50% indicates that SCR outperforms the competing method in the majority of runs even for its least favorable moment order. The 95% Wilson CI quantifies the uncertainty associated with the estimated win rate [37]. The narrower interval indicates a more precise and stable estimate, while the higher lower bound provides stronger evidence that the accuracy advantage of SCR persists after accounting for sampling variability. As shown in Table 10, SCR achieves minimum win rates of 100% against CR and PEM and 92.5%, 77.0%, 98.0%, and 78.0% against PCE, LRA, GP, and FT, respectively. The lower bounds of all 95%

Wilson CIs remain above 50%, and widths are below 12%. These results demonstrate that the accuracy advantage of SCR is maintained across different moment orders and $\mathbf{Q}$ realizations, and is unlikely to result from random variation.

Table 10: IEEE 3120-bus system – Repeated-run stability and statistical significance analysis of the SCR method [%].

Methods	Minimum win rate	95% Wilson CI
CR	100.0	[98.12, 100.00]
PEM	100.0	[98.12, 100.00]
PCE	92.5	[88.00, 95.40]
LRA	77.0	[70.69, 82.29]
GP	98.0	[94.97, 99.22]
FT	78.0	[71.76, 83.18]

6 Conclusions

This paper proposes the stochastic cubature rule to solve the probabilistic power flow problem for power systems with high-dimensional uncertainties. Performance of the proposed method is systematically evaluated on the IEEE 300-bus and 3120-bus systems under different cases. Main conclusions are as follows.

- SCR preserves third-degree algebraic accuracy while improving higher-order moment estimation accuracy through Haar-distributed orthogonal transformations. The IEEE 300-bus tests with increasing uncertainty dimensions confirm that this improvement translates into consistently lower higher-order moment errors in high-dimensional PPF problems.
- The copula-based extension enables SCR to accommodate correlated and complex uncertainties. Simulation results under varying renewable penetration and correlation levels demonstrate stable accuracy in moment, PDF, and confidence-interval estimation, indicating good robustness under complex uncertainty conditions.
- Tests on the modified IEEE 3120-bus system demonstrate that SCR retains high estimation accuracy and low computational cost under large-scale, high-dimensional, strongly correlated, and measurement-based uncertainties. Repeated tests with independent Haar realizations further confirm the statistical robustness of its estimation performance.

Overall, the proposed method achieves a balance between accuracy and computational efficiency. As a nonintrusive method, SCR can be readily integrated with existing deterministic power-flow solvers. Future work will investigate the performance of SCR in estimating tail-related quantities, such as violation

probabilities and quantiles.

7 Funding

This work is supported by the China Scholarship Council (CSC) by funding H. Hou for 8-month study at the University College Dublin.

A Derivation of the variance of the fourth-order SCR moment estimate

According to (22), the key to deriving $\text{Var}[\widetilde{M}_{4,j}]$ is to obtain the expression for $\mathbb{E}[(\widetilde{M}_{4,j})^2]$. Squaring yields

$$\begin{aligned}\mathbb{E}\left[(\widetilde{M}_{4,j})^2\right] &= \frac{1}{4N^2}\mathbb{E}\left[\sum_{n=1}^{2N}\sum_{m=1}^{2N}\widetilde{\xi}_{n,j}^4\widetilde{\xi}_{m,j}^4\right] \\ &= \frac{1}{4N^2}\mathbb{E}\left[\sum_{n=1}^{2N}\widetilde{\xi}_{n,j}^8 + \sum_{n=1}^{2N}\widetilde{\xi}_{n,j}^4\widetilde{\xi}_{\varpi(n),j}^4 + \sum_{n=1}^{2N}\sum_{\substack{m=1 \\ m \neq n, m \neq \varpi(n)}}^{2N}\widetilde{\xi}_{n,j}^4\widetilde{\xi}_{m,j}^4\right] \\ &= \frac{1}{4N^2}\left[4N\mathbb{E}\left(\widetilde{\xi}_{n,j}^8\right) + 4N(N-1)\mathbb{E}\left(\widetilde{\xi}_{n,j}^4\widetilde{\xi}_{m,j}^4\right)\right] \\ &= \frac{1}{N}\mathbb{E}\left(\widetilde{\xi}_{n,j}^8\right) + \frac{N-1}{N}\mathbb{E}\left(\widetilde{\xi}_{n,j}^4\widetilde{\xi}_{m,j}^4\right),\end{aligned}\tag{34}$$

where

$$\varpi(n) = \begin{cases} n + N, & 1 \leq n \leq N, \\ n - N, & N + 1 \leq n \leq 2N. \end{cases}\tag{35}$$

Due to the symmetric positive-negative pairing of the CPs, i.e., $\widetilde{\xi}_{\varpi(n),j} = -\widetilde{\xi}_{n,j}$, the double summation in (34) contains $2N$ diagonal terms with $m = n$ and $2N$ paired terms with $m = \varpi(n)$, both yielding $\widetilde{\xi}_{n,j}^8$. The remaining $4N(N-1)$ terms correspond to distinct unpaired CPs.

For the eighth-order moment, setting $N_\ell = 1$, $a_1 = 4$, and $A = 4$ in (19) gives

$$\mathbb{E}(\widetilde{\xi}_{n,j}^8) = \frac{105N^4}{N(N+2)(N+4)(N+6)}.\tag{36}$$

Since the Haar-distributed orthogonal matrix has identically distributed rows and columns, for any fixed dimension j , the row vector $[q_{j1}, \dots, q_{jN}]$ is also uniformly distributed on S^{N-1} . Therefore, for two unpaired CP indices satisfying $m \neq n$ and $m \neq \varpi(n)$, setting $N_\ell = 2$, $a_1 = a_2 = 2$, and $A = 4$ in (19)

gives

$$\mathbb{E} \left(\widetilde{\xi}_{n,j}^4 \widetilde{\xi}_{m,j}^4 \right) = \frac{9N^4}{N(N+2)(N+4)(N+6)}. \quad (37)$$

Substituting (36) and (37) into (34) gives

$$\begin{aligned} \mathbb{E} \left[\left(\widetilde{M}_{4,j} \right)^2 \right] &= \frac{105N^4}{N^2(N+2)(N+4)(N+6)} + \frac{9N^4(N-1)}{N^2(N+2)(N+4)(N+6)} \\ &= \frac{3N^2(3N+32)}{(N+2)(N+4)(N+6)}. \end{aligned} \quad (38)$$

$\mathbb{E} \left[\widetilde{M}_{4,j} \right]$ has been given in (20). Therefore, $\text{Var} \left[\widetilde{M}_{4,j} \right]$ is

$$\begin{aligned} \text{Var} \left[\widetilde{M}_{4,j} \right] &= \mathbb{E} \left[\left(\widetilde{M}_{4,j} \right)^2 \right] - \left[\mathbb{E} \left(\widetilde{M}_{4,j} \right) \right]^2 \\ &= \frac{3N^2(3N+32)}{(N+2)(N+4)(N+6)} - \frac{9N^2}{(N+2)^2} \\ &= \frac{24N^2(N-1)}{(N+2)^2(N+4)(N+6)}. \end{aligned} \quad (39)$$

B Parameters of stochastic sources

Parameters of stochastic sources considered in Section 5.1.2 and 5.2 are summarized in Tables B1 - B4.

Table B1: IEEE 300-bus system, Case 2 – Parameters of wind farm and wind speed.

Weibull parameter		Rated power [MW]	Power factor	v_{ci} [m/s]	v_r [m/s]	v_{co} [m/s]
α_w	β_w					
7	2	120	1	3	10	22

Table B2: IEEE 300-bus system, Case 2 – Parameters of PV power plant and solar irradiance.

Beta parameter		Area of array [m ²]	Efficiency [%]	Power factor	Mean power [MW]
α_b	β_b				
2.83	7.94	252000	15	1	7.80

References

Table B3: IEEE 300-bus system, Case 2 – Parameters of EV load in residential areas.

Bus ID	Gaussian parameter		Power factor
	Mean [MW]	STD [MW]	
7	0.20	0.03	0.97
19	0.10	0.10	0.95
35	0.11	0.11	0.95

Table B4: IEEE 300-bus system, Case 2, and IEEE 3120-bus system – Parameters of EV load in charging stations.

Bus ID	Weibull parameter		Mean power [MW]	Power factor
	α	β		
133	0.19	1.87	17.08	0.95
271	0.11	2.33	9.79	0.95

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