Delay-based Numerical Stability of the Partitioned-Solution Approach

Federico Milano, *Fellow, IEEE*

***Abstract*— This paper proposes a technique to evaluate the numerical stability and accuracy of the Partitioned-Solution Method (PSA) for the time domain integration of the Differential Algebraic Equations (DAEs) that are used for power system angle and voltage stability analyses. The partitioned approach consists in solving differential equations and algebraic constraints separately. While efficient, this technique introduces a "mismatch" between state and algebraic variables. The paper states the formal analogy of such a mismatch with a time delay of algebraic variables. Then a Small-signal Stability Analysis (SSSA) for Delay Differential Algebraic Equation (DDAE) is applied to define the numerical stability of the PSA as a function of the integration time step. Results are tested using the IEEE 14-bus system as well as a 1,479-bus model of the all-island Irish system.**

I. Introduction

Time domain integration is the most important tool for the stability analysis of power systems. This paper discusses a novel application of the stability analysis of DAEs. Based on the SSSA discussed in [1], the paper proposes a tool to evaluate the numerical stability of the well-known PSA for time-domain integration of power systems [2]. The paper provides a systematic quantitative tool to evaluate how precise such an approach is.

In recent years, there has been a growing interest in the modelling of time delays for power system dynamic analysis and control applications. Such an interest in mainly due to the increasing relevance of wide area control systems as well as of phasor measurement units for which communication delays of control signals cannot be ignored.

The focus of most research papers on time delays is, as natural, devoted to the design of robust controllers that are able to reduce the impact of communication delays. The following papers are recent contributions to the robust control of wide area control schemes [3]–[7]. The main goal of such papers is to improve the effect of power system stabilizers to damp inter-area oscillations. Another emerging area where delays are relevant is the load frequency control [8], [9].

Another, less common, category of papers focus on the evaluation of the small-signal stability of large DDAEs. Delays transform the classical problem of finding the roots of the state matrix of the system at the equilibrium point into the solution of a transcendental characteristic equation, with *infinitely many roots*. Since an explicit solution cannot be found in general, the spectrum can be only approximated based on some discretization scheme [10]–[12]; bounded using based on the definition of a Lyapunov function [13]; or reformulated as the solution of a linear matrix inequality (LMI) problem [14], whose demanding computational burden can be acceptable for some applications [15].

Based on the experience matured in previous works [1], [16], this paper will use the Chebyshev discretization that has proven to be very accurate, to scale well with size of the system and to have a reasonable computational burden. The latter, however, increases more than linearly with the size of the problem. Hence proper numerical schemes and implementations have to be used. This paper exploits a GPU-based numerical library, namely, MAGMA, that provides an efficient parallel implementation of LAPACK functions and QR factorization for solving the linear eigenvalue problem [17].

The contributions of the paper are twofold, as follows.

- Identify a formal analogy between the PSA for the time domain integration of power systems modelled as DAEs and a set of DDAEs where the delay is the time step of the integration method.
- Provide a quantitative tool to define both the numerical stability of the PSA as well as its accuracy in terms of capturing the modes of the system dynamics. Such a tool is based on the SSSA analysis of the equivalent DDAE above.

The proposed technique allows taking advantage of the efficiency of the PSA while preventing numerical instabilities. The tool also allows understanding whether the PSA is not adequate – e.g., because the time step has to be too small to maintain the accuracy at the desired level – and hence helps decide whether other approaches, e.g., an iterative predictor-corrector method or the Simultaneous-Solution Method (SSA), has to be preferred.

II. Partitioned-Solution Approach

The conventional power system model for voltage and transient stability analyses consists of a set of DAEs as follows:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \end{aligned} \tag{1}$$

where $\boldsymbol{f}$ ($\boldsymbol{f} : \mathbb{R}^{n+m} \mapsto \mathbb{R}^n$) are the differential equations, $\boldsymbol{g}$ ($\boldsymbol{g} : \mathbb{R}^{n+m} \mapsto \mathbb{R}^m$) are the algebraic equations, $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathbb{R}^n$) are the state variables, and $\boldsymbol{y}$ ($\boldsymbol{y} \in \mathbb{R}^m$) are the algebraic variables. Discrete events in (1) are not included explicitly but modelled as if-then rules that changes the structure of the DAE while preserving its continuity and differentiability [18].

There is a huge variety of numerical integration methods. However, only a few are adequate to solve the DAEs that model power systems. Such DAEs, in fact, proves to be highly nonlinear and stiff, i.e., the time constant span several orders of magnitude. A comprehensive discussion on this topic is provided in [19].

In order to numerically integrate (1), the first issue that has to be solved is how to handle algebraic constraints $\boldsymbol{g}$. There are mainly two approaches [2]:

1) *Partitioned-solution approach* (PSA). Variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are updated sequentially.
2) *Simultaneous-solution approach* (SSA). Variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are solved together in a single step using a solver such as the Newton method.

As expected, both approaches have advantages and drawbacks. The features of the PSA are briefly outlined below.

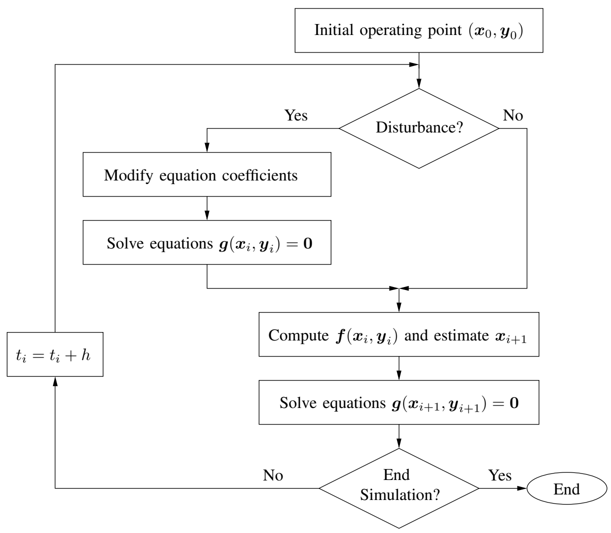

Fig. 1: Partitioned-solution approach for time domain integration.

In the partitioned approach, $\boldsymbol{x}$ and $\boldsymbol{y}$ are updated independently (see Fig. 1). Hence, any numerical integration method can be used. Due to its lower computational burden, the PSA was the first to be adopted for power system analysis and was typically used combined with explicit numerical methods, e.g., Runge-Kutta's formulæ, which do not require computing and factorizing the Jacobian matrix $\boldsymbol{f}_{\boldsymbol{x}}$. However, the partitioned approach introduces a mismatch between $\boldsymbol{x}$ and $\boldsymbol{y}$. In fact, for a generic step i, while computing $\boldsymbol{x}_{i+1}$, algebraic variables are frozen to the old value $\boldsymbol{y}_i$. Moreover, the state variables $\boldsymbol{x}_{i+1}$ are not modified when computing $\boldsymbol{y}_{i+1}$. To avoid the mismatch between $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$, a possible solution is to iterate over $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ for each time step. A typical solution is to use a predictor-corrector method as discussed in [20]. However, iterative processes, apart from being time consuming, can lead to numerical instabilities, e.g., cycling. Another, common solution consists in reducing the integration step length. Clearly, in this way, the computational burden increases but the advantage is that the PSA is numerically stable if the time step is sufficiently small.

To update algebraic variables requires the solution of the algebraic constraints, namely $\boldsymbol{g} = \boldsymbol{0}$. This is a set of linear equations if the current-injection formulation and standard transmission line and transformer models are used. However, in general, $\boldsymbol{g} = \boldsymbol{0}$ are nonlinear, e.g., if under-load tap changers are considered, and thus require computing and factorizing iteratively the Jacobian matrix $\boldsymbol{g}_{\boldsymbol{y}}$. With this aim, to reduce the computational effort, one can use a "dishonest Newton method", which consists in updating the Jacobian matrix only whenever the number of iterations required to solve $\boldsymbol{g} = \boldsymbol{0}$ increases beyond a given threshold or whenever a structural change of the system occurs, e.g., a fault. Such a threshold is typically defined based on heuristics.

III. Small-Signal Stability of Delayed Power Systems Models

The DDAE formulation is obtained by introducing time delays in (1). Let

$$x_d = x(t-\tau), \qquad y_d = y(t-\tau) \tag{2}$$

be the *retarded* or *delayed* state and algebraic variables, respectively, where t is the current simulation time, and τ ($\tau > 0$) is the time delay. In the remainder of this paper, since the main focus is on SSSA, time delays are assumed to be constant.

If some state and or algebraic variables in (1) are affected by a time delay as in (2), one obtains:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d, \boldsymbol{y}_d) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_d) \end{aligned} \tag{3}$$

which is the index-1 Hessenberg form of DDAE given in [21]. Note that $\boldsymbol{g}$ do not depend on $\boldsymbol{y}_d$. This allows obtaining a closed form for the small-signal stability analysis and, as discussed in [1], (3) is adequate to model, without lack of generality, power system models.

Differentiating (3) at the stationary solution yields:

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{f}_{\boldsymbol{x}}\Delta\boldsymbol{x} + \boldsymbol{f}_{\boldsymbol{x}_d}\Delta\boldsymbol{x}_d + \boldsymbol{f}_{\boldsymbol{y}}\Delta\boldsymbol{y} + \boldsymbol{f}_{\boldsymbol{y}_d}\Delta\boldsymbol{y}_d \tag{4}$$

$$\boldsymbol{0} = \boldsymbol{g}_{\boldsymbol{x}}\Delta\boldsymbol{x} + \boldsymbol{g}_{\boldsymbol{x}_d}\Delta\boldsymbol{x}_d + \boldsymbol{g}_{\boldsymbol{y}}\Delta\boldsymbol{y} \tag{5}$$

where, neglecting without loss of generality singularity-induced bifurcation points, it can be assumed that $\boldsymbol{g}_{\boldsymbol{y}}$ is non-singular. Substituting (5) into (4), one obtains:[1]

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}_0\Delta\boldsymbol{x} + \boldsymbol{A}_1\Delta\boldsymbol{x}(t-\tau) + \boldsymbol{A}_2\Delta\boldsymbol{x}(t-2\tau) \tag{6}$$

where:

$$\boldsymbol{A}_0 = \boldsymbol{f}_{\boldsymbol{x}} - \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}} \tag{7}$$

$$\boldsymbol{A}_1 = \boldsymbol{f}_{\boldsymbol{x}_d} - \boldsymbol{f}_{\boldsymbol{y}}\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}_d} - \boldsymbol{f}_{\boldsymbol{y}_d}\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}} \tag{8}$$

$$\boldsymbol{A}_2 = -\boldsymbol{f}_{\boldsymbol{y}_d}\boldsymbol{g}_{\boldsymbol{y}}^{-1}\boldsymbol{g}_{\boldsymbol{x}_d} \tag{9}$$

The first matrix $\boldsymbol{A}_0$ is the well-known state matrix that is computed for standard DAEs of the form (1). The other two matrices are not null only if the system is of retarded type. The matrix $\boldsymbol{A}_1$ is found in any delay differential equations, while $\boldsymbol{A}_2$ appears specifically in DDAEs, although it can be

[1] The interested reader can find in [1] the details on how to determine (7)-(9) from (4) and (5).

null if either $\boldsymbol{f}$ does not depend on $\boldsymbol{y}_d$ or $\boldsymbol{g}$ does not depend on $\boldsymbol{x}_d$. The substitution in (6) of a sample solution of the form $e^{\lambda t}\boldsymbol{v}$, with $\boldsymbol{v}$ a non-trivial possibly complex vector of order n, leads to the *characteristic equation*:

$$\det \Delta(\lambda) = 0 \tag{10}$$

where

$$\Delta(\lambda) = \lambda \boldsymbol{I}_n - \boldsymbol{A}_0 - \boldsymbol{A}_1 e^{-\lambda\tau} - \boldsymbol{A}_2 e^{-2\lambda\tau} \tag{11}$$

is called the *characteristic matrix*. In (11), $\boldsymbol{I}_n$ is the identity matrix of order n. The solutions of (11) are called the *characteristic roots* or *spectrum*, similar to the finite-dimensional case. The stability of (6) can be defined based on the sign of the roots of (11), i.e., the stationary point is stable if all roots have negative real part, and unstable if there exists at least one eigenvalue with positive real part.

Equation (11) is transcendental and, hence, shows infinitely many roots. In general, the explicit solution of (11) is not known and only approximated numerical solutions of a subset of the roots of (11) can be found. The case study is based on the Chebyshev discretization approach proposed in [10]. Such a discretization leads to solve an eigenvalue problem of an augmented matrix of order $(n \cdot N) \times (n \cdot N)$ where n is the dynamic order of the DDAE and, hence, the size of matrices $\boldsymbol{A}_0$, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$; and N is the number of points of the Chebyshev grid. The interested reader can find further details on this technique in [1].

IV. Proposed Numerical Stability Analysis

The purpose of this section is twofold. The formal analogy between the PSA and a DDAE system with the structure given in (3) is stated first. Then the implications on the numerical stability of the PSA in terms of the small-signal stability analysis outlined in Section III are outlined. It is important to note that the proposed technique discusses exclusively the properties of the steady-state operating point since it is based on a small-signal stability analysis. Numerical instabilities due to transient phenomena during the time domain integration cannot be captured using the technique described below and are beyond the scope of this paper.

Let's consider the flowchart shown in Fig. 1. The estimation of $\boldsymbol{x}_{i+1}$ based on $\boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{y}_i)$ can be viewed as the standard numerical integration of a set of ordinary differential equations. Algebraic variables, in fact, are constant and equal to $\boldsymbol{y}_i$, i.e., the value obtained at the previous step. One may argue that also state variables $\boldsymbol{x}_i$ used to evaluate $\boldsymbol{f}$ are obtained from the previous step of the algorithm. However, since any integration scheme can be used to estimate $\boldsymbol{x}_i$, the accuracy with which $\boldsymbol{x}_{i+1}$ can be estimated depends only on the properties of such a scheme. Algebraic variables $\boldsymbol{y}_i$, on the other hand, are implicit functions of the state ones but this functional dependence is neglected. The following step of the partitioned approach is the solution of $\boldsymbol{0} = \boldsymbol{g}(\boldsymbol{x}_{i+1}, \boldsymbol{y}_{i+1})$, which allows determining the new value of the algebraic variables. In this step, the state variables $\boldsymbol{x}_{i+1}$ are constant. Their value is not exact as they are estimated for an old value of the algebraic variables, but since $\boldsymbol{x}_{i+1}$ will be the value used to evaluate the next step, i.e., $\boldsymbol{x}_{i+2}$, one can assume that the solution of $\boldsymbol{0} = \boldsymbol{g}(\boldsymbol{x}_{i+1}, \boldsymbol{y}_{i+1})$ provides the *right* value of the algebraic variables at the $(i+1)$-th step. Based on the discussion above, the PSA is formally equivalent to the following DDAE:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}_d) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \end{aligned} \tag{12}$$

where $\boldsymbol{y}_d = \boldsymbol{y}(t-h)$, with h the integration time step of the PSA. Linearizing (12), one has:

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= \boldsymbol{f}_{\boldsymbol{x}} \Delta\boldsymbol{x} + \boldsymbol{f}_{\boldsymbol{y}_d} \Delta\boldsymbol{y}_d \\ \boldsymbol{0} &= \boldsymbol{g}_{\boldsymbol{x}} \Delta\boldsymbol{x} + \boldsymbol{g}_{\boldsymbol{y}} \Delta\boldsymbol{y} \end{aligned} \tag{13}$$

where $\boldsymbol{f}_{\boldsymbol{y}_d} = \boldsymbol{f}_{\boldsymbol{y}}$ of (1), as all algebraic variables are assumed to be delayed by h. From (13) one can deduce the following matrices of the characteristic equation (11):

$$\boldsymbol{A}_0 = \boldsymbol{f}_{\boldsymbol{x}}, \quad \boldsymbol{A}_1 = -\boldsymbol{f}_{\boldsymbol{y}} \boldsymbol{g}_{\boldsymbol{y}}^{-1} \boldsymbol{g}_{\boldsymbol{x}}, \quad \boldsymbol{A}_2 = \boldsymbol{0} \tag{14}$$

The numerical stability of the PSA can be thus evaluated based on the spectrum of (6). Moreover, the accuracy of the PSA can be also defined based on the comparison of the eigenvalues of (12) versus the eigenvalues of the original DAE (1).

It is important to note that the stability of the PSA depends on both the stability of the numerical integration method and the time step h. The eigenvalue analysis of (12) defines the stability related to the sole h. If A-stable numerical methods are utilized, e.g., the implicit trapezoidal method, the integration method does not affect results. But, if other integration methods are used, e.g., an explicit Runge-Kutta formula, these can be unstable for values of h lower than that indicated by the proposed analysis.

V. Case Study

This case study discusses the numerical stability and accuracy of the PSA through two power systems, namely the well-known IEEE 14-bus system and a dynamic model of the all-island Irish transmission system. Simulations are obtained using Dome, a Python-based power system analysis toolbox [22]. The Dome version used for in this case study is based on Python 3.4.2, Nvidia Cuda 7.5, Numpy 1.9.2, CVXOPT 1.1.8, MAGMA 1.6.1, and has been executed on a 64-bit Linux Fedora 21 operating system running on a two Intel Xeon 10 Core 2.2 GHz CPUs, 64 GB of RAM, and a 64-bit Nvidia Tesla K20X GPU.

A. IEEE 14-bus System

The model of the IEEE 14-bus system considered in this section is that described in [23]. If no Power System Stabilizer (PSS) is included, such a system is poorly damped due to the interaction between the subtransient dynamics of the synchronous machine connected at bus 1 and its Automatic Voltage Regulator (AVR). Table I shows how the step size h affects the eigenvalues. Values in bold face indicate the poorly damped mode whose real part gets closer to the imaginary axis as h increases. For $h = 0.005$ s, the equivalent DDAE that represent the PSA is unstable. Note, however, that for $h = 0.001$ s the dynamics of the system are already significantly modified with respect to the original DAE. Hence an iterative PSA is required.

Table II shows results for the IEEE 14-bus system with inclusion of a PSS connected to the AVR of generator 1. In this case, the original DAE is well damped. The impact of the PSA is also much lower and the time step can be increased up to 0.01 s without significantly affecting the time response of the system. For higher values of the time step the PSA becomes unstable. It is interesting to note that the stiffness of the system with and without PSS is the same. However, the stabilizing effect of the PSS allows a larger time step. Hence, the accuracy of the PSA does not depend directly on the stiffness of the system *per se* but, rather, on its stability margin.

B. All-island Irish System

In this subsection the robustness of the proposed technique is tested on a relatively large real-world grid, namely the all-island Irish system. The model includes $1,479$ buses, $1,851$ transmission lines and transformers, 245 loads, 22 conventional synchronous power plants with AVRs and turbine governors, 6 PSSs and 176 wind power plants. The topology and the data of the transmission system are based on the actual real-world system provided to the UCD Energy Institute by the Irish TSO, EirGrid, but dynamic data are guessed and based on the knowledge of the technology of power plants.

The total number of state variables is $1,360$. The considered system is very stiff. The eigenvalues range from about -0.1 to about -2000, which means that the time constants spans 4 orders of magnitude. For this case study, a 5 point Chebyshev grid is used. Thus the size of the matrix used for the eigenvalue analysis is $6,800 \times 6,800$.

Table III shows a selection of the rightmost eigenvalues for the considered dynamic model of the all-island Irish grid. Also in this case, due to the inclusion of PSS devices, the PSA is stable for relatively high values of the time step h. For $h = 0.01$ s one mode is poorly damped (see bold value in the fourth column of Table III). For $h = 0.05$ s a number of unstable eigenvalues shows up.

The numerical method to determine the eigenvalues of the DDAE shows some numerical issues for this large system. In particular, extraneous positive eigenvalues appear for low values of the time step. Such spurious eigenvalues can be identified using a sensitivity analysis. In fact, based on the properties of continuous functions, if an eigenvalue is *physical*, small variations of h lead to small variations of the eigenvalue. On the other hand, if an eigenvalue is originated by *numerical* issues, its magnitude is volatile and tends to change erratically for small variations of h. This property has been exploited to identify positive eigenvalues arising from numerical issues.

VI. Conclusions

This paper proposes an application to the PSA time integration method of the small-signal stability analysis of DDAE. The main idea of the paper relies on the fact that the PSA is formally analogous to a DDAE where *all* algebraic variables that appears within the differential equations are delayed by a time delay equal to the integration step length. Numerical results indicate that the stability of the system, not only its stiffness, plays a crucial role in the numerical stability of the PSA. This means, in turn, that the higher the stability margin the larger the allowed integration time step.

Future work will investigate the stability of iterative PSA and the functional relation between iterations and the time delay of its equivalent DDAE. Another aspect that appears worth further study is the impact of transient phenomena on the stability of the PSA, as these cannot be captured using the steady-state analysis proposed in this paper.

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TABLE I: Selection of relevant rightmost eigenvalues of the IEEE 14-bus system without PSS for different values of the integration time step h

No delay	$h = 0.0001$ s	$h = 0.001$ s	$h = 0.002$ s	$h = 0.005$ s
-0.01999	-0.01999	-0.01999	-0.01999	$\mathbf{0.15555 \pm 9.15342}$i
-0.06639	-0.06639	-0.06640	$\mathbf{-0.06521 \pm 9.25923}$i	-0.01999
$\mathbf{-0.22187 \pm 9.32160}$i	$\mathbf{-0.21386 \pm 9.31866}$i	$\mathbf{-0.14259 \pm 9.29132}$i	-0.06640	-0.06640
-0.58171 ± 0.34146i	-0.58171 ± 0.34148i	-0.58172 ± 0.34167i	-0.58173 ± 0.34189i	-0.58175 ± 0.34253i
-0.60521 ± 0.73574i	-0.60518 ± 0.73579i	-0.60488 ± 0.73629i	-0.60455 ± 0.73684i	-0.60355 ± 0.73849i
-0.71363	-0.71364	-0.71370	-0.71377	-0.71398
-0.74760	-0.74761	-0.74768	-0.74775	-0.74798
-0.80957	-0.80969	-0.81081	-0.81205	-0.81586
-1.00247 ± 1.39928i	-1.00228 ± 1.39943i	-1.00059 ± 1.40074i	-0.99871 ± 1.40220i	-0.99302 ± 1.40655i
-1.00250	-1.00250	-1.00248	-1.00246	-1.00240
-1.00398	-1.00398	-1.00396	-1.00394	-1.00390
-1.00437	-1.00437	-1.00436	-1.00435	-1.00433
-1.00869	-1.00868	-1.00867	-1.00865	-1.00860
-1.01665	-1.01665	-1.01662	-1.01660	-1.01653
-1.15127 ± 0.95355i	-1.15121 ± 0.95367i	-1.15060 ± 0.95481i	-1.14992 ± 0.95606i	-1.14785 ± 0.95983i

TABLE II: Selection of relevant rightmost eigenvalues of the IEEE 14-bus system with PSS for different values of the integration time step h

No delay	$h = 0.0001$ s	$h = 0.001$ s	$h = 0.01$ s	$h = 0.05$ s
-0.01999	-0.01999	-0.01999	-0.01999	$\mathbf{3.75831 \pm 2.16516}$i
-0.06371	-0.02616	-0.02616	-0.02616	-0.01999
-0.10852	-0.10046	-0.10046	-0.10046	-0.02615
-0.58420 ± 0.33877i	-0.58147 ± 0.39222i	-0.58144 ± 0.39245i	$\mathbf{-0.38509 \pm 19.8256}$i	-0.10046
-0.60543 ± 0.73556i	-0.60383 ± 0.74578i	-0.60351 ± 0.74629i	-0.58115 ± 0.39476i	-0.57955 ± 0.40512i
-0.71397	-0.71293	-0.71300	-0.60030 ± 0.75136i	-0.58464 ± 0.77361i
-0.74310 ± 7.51790i	-0.74644	-0.74652	-0.71367	-0.71671
-0.74886	-0.99973	-0.99974	-0.74735	-0.75113
-0.77844	-1.00243	-1.00179 ± 1.40936i	-0.98405 ± 1.42227i	-0.89958 ± 1.47221i
-1.00250 ± 1.39916i	-1.00353 ± 1.40803i	-1.00242	-0.99977	-0.99991
-1.00250	-1.00373	-1.00372	-1.00224	-1.00153
-1.00403	-1.00471	-1.00470	-1.00359	-1.00306
-1.00430	-1.01157	-1.01156	-1.00460	-1.00415
-1.00823	-1.15666 ± 0.95231i	-1.15605 ± 0.95345i	-1.01142	-1.01081
-1.01545	-1.80135 ± 0.65369i	-1.80170 ± 0.65496i	-1.14985 ± 0.96479i	-1.11891 ± 1.01417i

TABLE III: Selection of rightmost eigenvalues of the all-island Irish system for different values of the integration time step h

No delay	$h = 0.0001$ s	$h = 0.001$ s	$h = 0.01$ s	$h = 0.05$ s
-0.09292	-0.09292	-0.09293	-0.09300	$\mathbf{0.80487 \pm 8.06482}$i
-0.10060	-0.10060	-0.10061	-0.10070	$\mathbf{0.69993 \pm 8.10911}$i
-0.12660 ± 0.34157i	-0.12666 ± 0.34176i	-0.12658 ± 0.34183i	-0.12573 ± 0.34252i	$\mathbf{0.68919 \pm 8.99941}$i
-0.13038 ± 0.17137i	-0.13125 ± 0.17577i	-0.13121 ± 0.17580i	-0.13087 ± 0.17603i	$\mathbf{0.51084 \pm 9.33566}$i
-0.15804 ± 5.24372i	-0.15805 ± 5.24381i	-0.15809 ± 5.24458i	-0.15834 ± 5.25229i	$\mathbf{0.48806 \pm 9.19641}$i
-0.16693 ± 0.27365i	-0.21587	-0.21593	-0.21654	$\mathbf{0.37544 \pm 7.11867}$i
-0.20039 ± 0.36001i	-0.21767 ± 0.42686i	-0.21757 ± 0.42701i	-0.21655 ± 0.42858i	$\mathbf{0.10606 \pm 6.21403}$i
-0.21585	-0.22191 ± 0.43812i	-0.22180 ± 0.43828i	-0.22074 ± 0.43994i	$\mathbf{0.09777 \pm 5.73908}$i
-0.21769 ± 0.42685i	-0.22473	-0.22479	-0.22538	$\mathbf{0.02961 \pm 6.50779}$i
-0.22194 ± 0.43817i	-0.23055 ± 0.18482i	-0.23054 ± 0.18487i	-0.23042 ± 0.18531i	-0.09335
-0.22472	-0.23823	-0.23829	-0.23897	-0.10112
-0.23069 ± 0.18482i	-0.24063 ± 0.22612i	-0.24062 ± 0.22618i	-0.24043 ± 0.22680i	-0.11405 ± 9.74783i
-0.23370 ± 0.30331i	-0.24689 ± 0.26828i	-0.24686 ± 0.26836i	-0.24658 ± 0.26910i	-0.12183 ± 0.34552i
-0.23822	-0.25472 ± 0.46528i	-0.25465 ± 0.46545i	-0.25402 ± 0.46715i	-0.12933 ± 0.17708i
-0.23992 ± 0.22504i	-0.25702	-0.25710	-0.25788	-0.15463 ± 5.28682i
-0.24651 ± 0.26647i	-0.26853 ± 0.30406i	-0.26850 ± 0.30416i	-0.26813 ± 0.30510i	-0.21173 ± 0.43553i
-0.25701	-0.27048 ± 39.4356i	-0.28013 ± 0.49970i	-0.27938 ± 0.50200i	-0.21569 ± 0.44730i
-0.26817 ± 0.30342i	-0.28020 ± 0.49947i	-0.29060	-0.29157	-0.21935
-0.27275 ± 0.49776i	-0.29050	-0.29207 ± 0.17598i	-0.29218 ± 0.17645i	-0.22806
-0.27374 ± 0.50316i	-0.29205 ± 0.17594i	-0.29886 ± 0.46883i	-0.29769 ± 0.47150i	-0.22987 ± 0.18731i

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