On hybrid dynamical systems of differential-difference equations

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Abstract: In this paper, we define and study a class of linear hybrid dynamical systems characterized by differential-difference equations. We introduce two operators that facilitate the analysis of these systems and derive explicit formulas for their solutions. We examine the transfer function matrix and characteristic polynomial to assess stability. Our theoretical findings are supported by numerical examples, demonstrating their application in power systems stability analysis. Specifically, we substantiate our theory within the context of power systems stability analysis, incorporating elements of discrete behavior.

Keywords: Hybrid dynamical systems, Differential-difference equations, Stability analysis, Transfer function matrix, Power systems stability.

1 Introduction

A hybrid dynamical system is a system that involves functions of two variables. One variable is continuous, while the other is discrete. Hybrid models appear in many physical processes and systems. For example, it is relevant to mention electrical power systems include continuous devices, e.g., electrical machines; discrete events, e.g., faults; and devices with discrete variables, e.g., under-load tap changers, automatic generation control and power electronic converters [3, 29]. In this vein, in [19], the author develops the concept of *hybrid automaton* as a systematic approach model hybrid power system models. The control and stability analysis of hybrid systems is particularly challenging because the methodologies and techniques available for fully continuous or fully discrete systems do not apply to hybrid systems [18]. A usual approach is relaxing the discrete variables approximating them as continuous ones. This is, for example a common solution for the small-signal stability analysis of power systems [27]. However, in this way, some unstable phenomena that may arise from the interactions between continuous and discrete variables are lost. Relevant example of such instabilities are discussed, for example, in [15] and [22]. Hybrid systems also arise as a byproduct of control hard limits, e.g., grazing phenomena [14] and Filippov approach for discontinous right-hand side [32], or of nonlinear control approaches, e.g., model predictive control [7], reset control [5], and sliding-mode control [36]. Finally, it is also relevant to note that hybrid continuous-discrete arise in stochastic systems [8]. An example of these systems are, again, electric power systems where the discrete random events, such as load variations and line disconnections can trigger system dynamics [26], or continuous disturbances, such as wind generation fluctuations, trigger the variations of discrete variables, such as tap-changer positions [33].

The dynamic behavior of a hybrid dynamical system can be described using both differential and difference equations, allowing for greater flexibility in modeling dynamic phenomena. The concept of a hybrid dynamical system and the study of its solutions were first introduced in [17], followed by several interesting works, as seen in [16, 20, 21, 23, 35, 37]. Additionally, there are other notable works in the literature that investigate solutions, stability, controllability, and more, as documented in [6, 25, 34].

Despite the extensive research on hybrid systems, there remains a significant gap in the literature concerning the analysis and stability of systems that integrate both continuous and discrete dynamics in a seamless manner. Traditional methods often approximate discrete variables as continuous, which can overlook critical instability phenomena arising from the interaction of these variables. This paper aims to bridge this gap by defining and studying a class of linear hybrid dynamical systems of differential-difference equations, providing a more accurate representation of hybrid systems.

Our scientific contributions are threefold:

1. **Definition and Analysis:** We define a novel class of linear hybrid dynamical systems characterized by differential-difference equations, providing a robust framework for their analysis.

2. **Operator Development:** We introduce two new operators that facilitate the study of solutions to these hybrid systems, enhancing analytical capabilities.

3. **Stability Investigation:** We explore the transfer function matrix and characteristic polynomial of these systems, offering new insights into their stability properties.

We substantiate our theoretical developments within the context of power systems stability analysis, highlighting how our approach can capture discrete behaviors often overlooked by conventional methods.

By addressing these challenges and contributing new methodologies, our work advances the understanding and control of hybrid dynamical systems, particularly in applications such as power systems where discrete and continuous dynamics are inherently intertwined.

In this article we are initially interested in studying the solutions of the system:

$$\begin{aligned} &\tfrac{d}{dt}x_k(t) = Ax_k(t) + By_k(t) + u_k(t), \\ &y_{k+1}(t) = Cx_k(t) + Dy_k(t) + v_k(t). \end{aligned} \tag{1}$$

subject to given initial conditions $x_k(0)$, $y_0(t)$. Where $x_k(t) \in \mathbb{R}^{r\times 1}$ and $y_k(t) \in \mathbb{R}^{m\times 1}$ are unknown vectors of functions of two variables supported on $[0, +\infty] \times \mathbb{N}^*$, i.e., $t \in [0, +\infty]$ and $k \in \mathbb{N}$, meaning t is a continuous variable and k is a discrete variable. In addition, $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times m}$, $C \in \mathbb{R}^{m\times r}$, and $D \in \mathbb{R}^{m\times m}$ are constant known matrices, and $u_k(t) \in \mathbb{R}^{r\times 1}$ and $v_k(t) \in \mathbb{R}^{m\times 1}$ are known vectors of functions of two variables supported on $[0, +\infty] \times \mathbb{N}^*$.

In mathematical analysis, a function $f(x)$ is indeed said to be continuous at x_0 if the limit of f as x tends to x_0 is equal to $f(x_0)$. This notion of continuity refers to the property that small changes in the input x result in small changes in the output $f(x)$.

In the context of hybrid dynamical systems, when we refer to a "continuous variable," we mean a variable that can take any value within a given range, typically representing physical quantities that change smoothly over time. For example, in our system, the variable t represents time and is considered continuous because it can take any non-negative real value ($t \in [0, +\infty]$).

On the other hand, a "discrete variable" is one that can only take distinct, separate values, usually integers. In our system, the variable k is discrete because it represents distinct events or states indexed by integers ($k \in \mathbb{N}$).

Thus, the term "continuous variable" aligns with the broader mathematical understanding of continuity but is used here to differentiate between variables that vary smoothly over time (continuous) and those that change in discrete steps (discrete).

The equations (1) are differential–difference equations. If we discard the variable $y_{k+1}(t)$ then (1) would be an hybrid system in a form of differential–

algebraic equation:

$$\begin{aligned} \tfrac{d}{dt}x_k(t) &= Ax_k(t) + By_k(t) + u_k(t), \\ 0_{n,1} &= Cx_k(t) + Dy_k(t) + v_k(t). \end{aligned} \tag{2}$$

Additionally if we discard the variable $\frac{d}{dt}x_k(t)$ then (1) would be an hybrid system in a form of difference–algebraic equation:

$$\begin{aligned} 0_{n,1} &= Ax_k(t) + By_k(t) + u_k(t), \\ y_{k+1}(t) &= Cx_k(t) + Dy_k(t) + v_k(t). \end{aligned} \tag{3}$$

The differential–equation (2) is subject to given initial conditions $x_k(0)$, $y_k(0)$ while the difference–equation (3) is subject to given initial conditions $x_0(t)$, $y_0(t)$. Where $x_k(t), y_k(t)$, and A, B, C, D, $u_k(t), v_k(t)$ are defined as previously for (1). In the rest of this paper, we will denote the $n \times n$ identity matrix as I_n, the $n \times n$ zero matrix as 0_n, and the $n \times 1$ zero column vector as $0_{n,1}$. However, in some cases, the identity matrix will be denoted as I with appropriate dimensions for better readability.

In Section 2, we present explicit solution formulas for equations (1) through (3), along with a discussion on the stability and characteristic equations of these systems. Section 3 includes numerical examples and an application of our theoretical findings to the practical problem of hierarchical frequency control utilized in high-voltage electrical power systems. We illustrate how the operation of discrete elements can lead to instability and demonstrate the relevance of our approach in addressing these issues.

2 Main Results

In this Section we present our main results. We give first a definition.

Definition 2.1. We define with $\mathcal{J}_{A,B}$ the operator:

$$\mathcal{J}_{A,B}x_k(t) = \int_0^t e^{A(t-s)}Bx_k(s)ds, \tag{4}$$

and with $\mathcal{S}_{A,B}$ the operator:

$$\mathcal{S}_{A,B}x_k(t) = \sum_{j=0}^{k-1} A^{k-1-j}Bx_j(t). \tag{5}$$

Where $x_k(t) \in \mathbb{R}^{m\times 1}$ is unknown vector of functions of two variables supported on $[0, +\infty] \times \mathbb{N}^*$, and $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times m}$.

We now state the following Theorem:

Theorem 2.1. We consider the operators $\mathcal{J}_{A,B}$ and $\mathcal{S}_{A,B}$ as defined in (4) and (5), respectively. Let

$$w_k(t) = C\left[e^{At}x_k(0) + \mathcal{J}_{A,I}u_k(t)\right] + v_k(t),$$

$$\omega_k(t) = B\left[D^k y_0(t) + \mathcal{S}_{D,I}v_k(t)\right] + u_k(t),$$

and

$$\tilde{A} = C\mathcal{J}_{A,B} + D, \quad \hat{A} = A + B\mathcal{S}_{D,C}.$$

Then, two equivalent solutions of the differential-difference equation (1) with given initial conditions, the sequence $x_k(0)$, and the function $y_0(t)$ are given by:

$$\begin{aligned} x_k(t) &= e^{At}x_k(0) + \mathcal{J}_{A,B}[\tilde{A}^k y_0(t) + \mathcal{S}_{\tilde{A},I}w_k(t)] + \mathcal{J}_{A,I}u_k(t), \\ y_k(t) &= \tilde{A}^k y_0(t) + \mathcal{S}_{\tilde{A},I}w_k(t), \end{aligned} \tag{6}$$

and

$$\begin{aligned} x_k(t) &= e^{\hat{A}t}x_k(0) + \mathcal{J}_{\hat{A},I}\omega_k(t), \\ y_k(t) &= D^k y_0(t) + \mathcal{S}_{D,C}[e^{\hat{A}t}x_k(0) + \mathcal{J}_{\hat{A},I}\omega_k(t)] + \mathcal{S}_{D,I}v_k(t). \end{aligned} \tag{7}$$

Proof. By solving the differential equation in (1) we get

$$x_k(t) = e^{At}x_k(0) + \int_0^t e^{A(t-s)}[By_k(s) + u_k(s)]ds,$$

or, equivalently, by using (4),

$$x_k(t) = e^{At}x_k(0) + \mathcal{J}_{A,B}y_k(t) + \mathcal{J}_{A,I}u_k(t).$$

By substituting the above expression into the difference equation of (1) we get

$$y_{k+1}(t) = C[e^{At}x_k(0) + \mathcal{J}_{A,B}y_k(t) + \mathcal{J}_{A,I}u_k(t)] + Dy_k(t) + v_k(t),$$

or, equivalently,

$$y_{k+1}(t) = [C\mathcal{J}_{A,B} + D]y_k(t) + w_k(t).$$

The solution of the above difference equation is:

$$y_k(t) = [C\mathcal{J}_{A,B} + D]^k y_0(t) + \sum_{j=0}^{k-1} [C\mathcal{J}_{A,B} + D]^{k-1-j} w_j(t),$$

or, equivalently, by using (5),

$$y_k(t) = [C\mathcal{J}_{A,B} + D]^k y_0(t) + \mathcal{S}_{C\mathcal{J}_{A,B}+D,I} w_k(t).$$

Additionally,

$$x_k(t) = e^{At}x_k(0) + \mathcal{J}_{A,B}[(C\mathcal{J}_{A,B} + D)^k y_0(t) + \mathcal{S}_{C\mathcal{J}_{A,B}+D,I} w_k(t)] + \mathcal{J}_{A,I} u_k(t).$$

If we set $\tilde{A} = C\mathcal{J}_{A,B} + D$ we arrive at (6). We now solve the difference equation in (1):

$$y_k(t) = D^k y_0(t) + \sum_{j=0}^{k-1} D^{k-1-j}[Cx_j(t) + v_j(t)]$$

or, equivalently, by using (5),

$$y_k(t) = D^k y_0(t) + \mathcal{S}_{D,C} x_k(t) + \mathcal{S}_{D,I} v_k(t).$$

By substituting the above expression into the differential equation of (1) we get

$$\frac{d}{dt}x_k(t) = Ax_k(t) + B[D^k y_0(t) + \mathcal{S}_{D,C} x_k(t) + \mathcal{S}_{D,I} v_k(t)] + u_k(t)$$

or, equivalently,

$$\frac{d}{dt}x_k(t) = [A + B\mathcal{S}_{D,C}]x_k(t) + \omega_k(t).$$

The solution of the above differential equation is:

$$x_k(t) = e^{[A+B\mathcal{S}_{D,C}]t}x_k(0) + \int_0^t e^{[A+B\mathcal{S}_{D,C}](t-s)}\omega_k(s),$$

or, equivalently, by using (4),

$$x_k(t) = e^{[A+B\mathcal{S}_{D,C}]t}x_k(0) + \mathcal{J}_{A+B\mathcal{S}_{D,C},I}\,\omega_k(t).$$

Additionally,

$$y_k(t) = D^k y_0(t) + \mathcal{S}_{D,C}[e^{[A+B\mathcal{S}_{D,C}]t}x_k(0) + +\mathcal{J}_{A+B\mathcal{S}_{D,C},I}\omega_k(t)] + \mathcal{S}_{D,I}v_k(t).$$

If we set $\hat{A} = A + B\mathcal{S}_{D,C}$ we arrive at (7). The proof is completed.

Note that (1) is a differential–difference equation. As mentioned in the introduction if we discard the variable $y_{k+1}(t)$ then (1) can be written in the form of (2) which is an hybrid system in the form of differential–algebraic equation. Additionally if we discard the variable $\frac{d}{dt}x_k(t)$ then (1) can be written in the form of (3) which is an hybrid system in the form of difference–algebraic equation.

Lemma 2.1. The hybrid system (2) can be written as

$$E\frac{d}{dt}Y_k(t) = A_0 Y_k(t) + V_k(t),$$

where

$$E = \begin{bmatrix} I & 0_n \\ 0_n & 0_n \end{bmatrix}, \quad A_0 = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

and

$$Y_k(t) = \begin{bmatrix} x_k(t) \\ y_k(t) \end{bmatrix}, \quad V_k(t) = \begin{bmatrix} u_k(t) \\ v_k(t) \end{bmatrix}.$$

For $E = \begin{bmatrix} 0_n & 0_n \\ 0_n & I \end{bmatrix}$ an equivalent form of the hybrid system (3) is given by:

$$EY_{k+1}(t) = A_0 Y_k(t) + V_k(t).$$

Proof. The hybrid system (2) is in the form of differential–algebraic equation and can be written as:

$$\begin{bmatrix} I & 0_n \\ 0_n & 0_n \end{bmatrix}\begin{bmatrix} \frac{d}{dt}x_k(t) \\ \frac{d}{dt}y_k(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_k(t) \\ y_k(t) \end{bmatrix} + \begin{bmatrix} u_k(t) \\ v_k(t) \end{bmatrix},$$

or, equivalently,

$$E\frac{d}{dt}Y_k(t) = A_0 Y_k(t) + V_k(t).$$

The hybrid system (3) can be written as:

$$\begin{bmatrix} 0_n & 0_n \\ 0_n & I \end{bmatrix}\begin{bmatrix} x_{k+1}(t) \\ y_{k+1}(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_k(t) \\ y_k(t) \end{bmatrix} + \begin{bmatrix} u_k(t) \\ v_k(t) \end{bmatrix},$$

or, equivalently,

$$EY_{k+1}(t) = A_0Y_k(t) + V_k(t).$$

The proof is completed.

Since in both cases E is a singular matrix we will use in our next result matrix pencil theory. The pencil $sE - A_0$ is regular for which there exist regular square matrices P, Q such that:

$$\begin{aligned} PEQ &= I_p \oplus H_q, \\ PA_0Q &= J_p \oplus I_q. \end{aligned} \tag{8}$$

Where $p+q = n$, with p being the sum of the algebraic multiplicities of the finite eigenvalues of the pencil, H_q a $q \times q$ nilpotent matrix with index q_*, with q being the algebraic multiplicity of the infinite eigenvalue, and J_p the $p \times p$ Jordan matrix constructed from the finite eigenvalues of the pencil and their algebraic multiplicities. Furthermore

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}, \tag{9}$$

with P_1, P_2 being $p \times n$, $q \times n$ matrices respectively, and Q_p, Q_q being $n \times p$, $n \times q$ matrices respectively. The matrices P_1, Q_p are constructed by left, right respectively linear independent eigenvectors of the finite eigenvalues while matrices P_2, Q_q are constructed by left, right respectively linear independent eigenvectors of the infinite eigenvalues. We state the following Theorem:

Theorem 2.2. Consider the hybrid systems (2), (3) as written in Lemma 2.1. Then the solution of (2) is given by:

$$Y_k(t) = Q_p[e^{J_pt}X_k^p(0) + \mathcal{J}_{J_p,P_1}V_k(t)] - Q_qP_2V_k(t), \tag{10}$$

and the solution of (3) is given by:

$$Y_k(t) = Q_p[J_p^kX_0^p(t) + \mathcal{S}_{J_p,P_1}V_k(t)] - Q_qP_2V_k(t). \tag{11}$$

where J_p, H_q Jordan matrices of the finite, infinite eigenvalues of the pencil $sE - A_0$ respectively and Q, P matrices related to the eigevectors of the finite, infinite eigenvalues of the pencil $sE - A_0$ respectively as defined in (8), (9). The operators $\mathcal{J}_{A,B}$, $\mathcal{S}_{A,B}$ are defined in (4), (5), and $X_k(t) = Q^{-1}Y_k(t)$, with $X_k(t) = \begin{bmatrix} X_k^p(t) \\ X_k^q(t) \end{bmatrix}$.

Proof. Using Lemma 2.1, (2) can be written as:

$$E\frac{d}{dt}Y_k(t) = A_0 Y_k(t) + V_k(t).$$

Let $Y_k(t) = QX_k(t)$ with $X_k(t) = \begin{bmatrix} X_k^p(t) \\ X_k^q(t) \end{bmatrix}$. Then by using this expression and multiplying by P we get

$$PEQ\frac{d}{dt}X_k(t) = PA_0QX_k(t) + PV_k(t),$$

whereby using the properties of the pencil we arrive at two subsystems

$$\frac{d}{dt}X_k^p(t) = J_pX_k^p(t) + P_1V_k(t),$$

and

$$H_q\frac{d}{dt}X_k^q(t) = X_k^q(t) + P_2V_k(t).$$

These two subsystems have the following solutions, see [9, 10]:

$$X_k^p(t) = e^{J_pt}X_k^p(0) + \mathcal{J}_{J_p,P_1}V_k(t),$$

and

$$X_k^q(t) = -\sum_{j=0}^{q*-1} H_q^j P_2 V_k^{(j)}(t).$$

In our case and because of the structure of E we have $q^* = 1$. Hence the solution of (2) is:

$$Y_k(t) = QX_k(t) = Q_pX_k^p(t) + Q_qX_k^q(t),$$

or, equivalently,

$$Y_k(t) = Q_p[e^{J_pt}X_k^p(0) + \mathcal{J}_{J_p,P_1}V_k(t)] - Q_qP_2V_k(t),$$

Additionally, (2) can be written as

$$EY_{k+1}(t) = A_0Y_k(t) + V_k(t).$$

or, equivalently,

$$PEQX_{k+1}(t) = PA_0QX_k(t) + PV_k(t),$$

whereby using the properties of the pencil we arrive at two subsystems

$$X_{k+1}^p(t) = J_p X_k^p(t) + P_1 V_k(t),$$

and

$$H_q X_{k+1}^q(t) = X_k^q(t) + P_2 V_k(t).$$

These two subsystems have the following solutions, see [9, 10]:

$$X_k^p(t) = J_p^k X_0^p(t) + \mathcal{S}_{J_p,P_1} V_k(t),$$

and

$$X_k^q(t) = -\sum_{j=0}^{q*-1} H_q^j P_2 V_{k+j}(t).$$

In our case and because of the structure of E we have $q^* = 1$. Hence the solution of (3) is:

$$Y_k(t) = Q_p[J_p^k X_0^p(t) + \mathcal{S}_{J_p,P_1} V_k(t)] - Q_q P_2 V_k(t).$$

The proof is completed.

Next, we will provide the characteristic polynomial of two variables for the hybrid dynamical system (1). We state the following Proposition.

Proposition 2.1. The characteristic polynomial of two variables for (1) is given by:

$$p(s,\hat{s}) = \begin{vmatrix} (\hat{s}-1)A - s(\hat{s}-1)I_r & (\hat{s}-1)B \\ (\hat{s}-1)C & (\hat{s}-1)D + (\hat{s}+1)I_m \end{vmatrix}, \tag{12}$$

where $(s,\hat{s}) \in \mathbb{C}^2$, and $\hat{s} \neq 1$.

Proof. Let $\mathcal{L}$ be the Laplace transform, and $\mathcal{Z}$ the Z-transform. If we apply the Laplace and Z-transform to (1) while discarding the initial conditions $x_k(0)$, $y_0(t)$, etc we get:

$$sX(s,z) = AX(s,z) + BY(s,z) + U(s,z),$$

$$zY(s,z) = CX(s,z) + DY(s,z) + V(s,z),$$

or, equivalently,

$$(sI_r - A)X(s,z) - BY(s,z) = U(s,z),$$

$$-CX(s,z) + (zI_m - D)Y(s,z) = V(s,z),$$

or, equivalently,

$$\begin{bmatrix} sI_r - A & -B \\ -C & zI_m - D \end{bmatrix} \begin{bmatrix} X(s,z) \\ Y(s,z) \end{bmatrix} = \begin{bmatrix} U(s,z) \\ V(s,z) \end{bmatrix}.$$

Hence, a characteristic equation of (1) is given by:

$$\begin{vmatrix} sI_r - A & -B \\ -C & zI_m - D \end{vmatrix} = 0. \tag{13}$$

By using $z = \frac{\hat{s}+1}{1-\hat{s}}$, with $\hat{s} \neq 1$, see [11], the characteristic equation of (1) takes the form:

$$\begin{vmatrix} (\hat{s}-1)A - s(\hat{s}-1)I_r & (\hat{s}-1)B \\ (\hat{s}-1)C & (\hat{s}-1)D + (\hat{s}+1)I_m \end{vmatrix} = 0,$$

and the pencil of (1) is given by:

$$\begin{bmatrix} (\hat{s}-1)A - s(\hat{s}-1)I_r & (\hat{s}-1)B \\ (\hat{s}-1)C & (\hat{s}-1)D + (\hat{s}+1)I_m \end{bmatrix}.$$

The characteristic polynomial of two variables of (1) is given by (12). The proof is completed.

We consider the solution (6) of system (1). To have asymptotic stability for system (1), the solutions $x_k(t)$ and $y_k(t)$ have to exist and not be infinite as k and t tend to infinity. Let's first focus on the solution $x_k(t)$ in (6), which can be written in the form:

$$x_k(t) = e^{At} x_k(0) + \mathcal{J}_{A,B} y_k(t) + \mathcal{J}_{A,I} u_k(t).$$

Obviously, we need the terms $e^{At}x_k(0)$, $\mathcal{J}_{A,B}y_k(t)$, and $\mathcal{J}_{A,I}u_k(t)$ to exist and not be infinite as k and t tend to infinity. If λ is an eigenvalue of A then $e^{At}x_k(0)$ converges to $0_{r,1}$ as $k, t \to \infty$ if $x_k(0)$ is bounded, i.e. there exists $\mu_1 > 0$ such that

$$\|x_k(0)\| < \mu_1,$$

and if

$$\mathrm{Re}(\lambda) < 0.$$

Under this assumption $\mathcal{J}_{A,B}y_k(t)$,$\mathcal{J}_{A,I}u_k(t)$ converge to $0_{r,1}$ as $k, t \to \infty$ if $u_k(t)$ is bounded, i.e. there exists $\mu_2 > 0$ such that

$$\|u_k(t)\| < \mu_2,$$

and if $y_k(t)$ is bounded. We have

$$y_k(t) = \tilde{A}^k y_0(t) + \mathcal{S}_{\tilde{A},I} w_k(t).$$

If $\tilde{\lambda}$ is an eigenvalue of the operator $\tilde{A}$ then $\tilde{A}^k y_0(t)$ converges to $0_{m,1}$ as $k, t \to \infty$ if $y_0(t)$ is bounded, i.e. there exists $\mu_3 > 0$ such that

$$\|y_0(t)\| < \mu_3,$$

and if

$$|\tilde{\lambda}| < 1.$$

Under this assumption $\mathcal{S}_{\tilde{A},I} w_k(t)$, converges to $0_{m,1}$ as $k, t \to \infty$ if $w_k(t)$ is bounded, or, equivalently, if $v_k(t)$ is bounded, i.e. there exists $\mu_5 > 0$ such that

$$\|v_k(t)\| < \mu_4.$$

If $\tilde{\lambda}$ is an eigenvalue of the operator $\tilde{A}$ and $\tilde{u}$ eigenfunction then:

$$\tilde{\lambda}\tilde{A} = \tilde{\lambda}\tilde{u},$$

or, equivalently from (6),

$$(\tilde{\lambda} I_m - D)\tilde{u} = C\mathcal{J}_{A,B}\tilde{u},$$

whereby under the assumption that $\mathrm{Re}(\lambda) < 0$ and $\tilde{u}$ is bounded we have that $C\mathcal{J}_{A,B}\tilde{u}$ tends to $0_{m,1}$ as $k, t \to \infty$. In this case we have that

$$(\tilde{\lambda} I_m - D)\tilde{u} = 0_{m,1},$$

or, equivalently,

$$\tilde{\lambda}\tilde{u} = D\tilde{u},$$

which means that in this case $\tilde{\lambda}$ is eigenvalue of D. We proved the following Theorem:

Theorem 2.3. Consider system (1) with bounded initial conditions $x_k(0)$, $y_0(t)$ and bounded input vectors $u_k(t)$, $v_k(t)$. Let λ be an eigenvalue of A and $\tilde{\lambda}$ an eigenvalue of D. Then (1) is asymptotically stable if:

$$\mathrm{Re}(\lambda) < 0, \quad |\tilde{\lambda}| < 1. \tag{14}$$

Having investigated a hybrid system of differential-difference equations represented by (1) and its solutions, as well as exploring the stability and characteristic equation of (1), it would be interesting as a future work to extend other types of differential or difference equations into hybrid equations. These could include fractional differential operators, both discrete and continuous (see [12, 13]), and stochastic differential equations (see [1, 2, 30, 31]).

3 Numerical Examples

Electrical power systems are among the most complex modern systems. Due to their intricate nature and unique operational characteristics, they have been the focus of extensive research. Power system stability is defined as "the ability of an electric power system, for a given initial operating condition, to regain a state of operating equilibrium after being subjected to a physical disturbance, with most system variables bounded so that practically the entire system remains intact" [24]. In this section, we will showcase compelling examples to substantiate and support our theoretical framework.

Example 1 We consider the hybrid system of differential–difference equations (1) with $u_k(t) = 0$, $v_k(t) = 0_{2,1}$, and

$$A = -1, \quad B = \begin{bmatrix} 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad D = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The initial conditions of (1) are:

$$x_k(0) = \frac{1}{k^2+1}, \quad y_0(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}.$$

Based on (6), we can solve (1) by following these steps. We first compute $w_k(t)$:

$$w_k(t) = \frac{e^{-t}}{k^2+1}\begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Then $\tilde{A}$:

$$\tilde{A} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \mathcal{J}_{-1,\begin{bmatrix} 2 & 0 \end{bmatrix}} + \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hence we have

$$y_k(t) = \tilde{A}^k \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + \sum_{j_0}^{k-1} \frac{e^{-t}}{j^2+1}\tilde{A}^{k-1-j}\begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

and

$$x_k(t) = \frac{e^{-t}}{k^2+1} + \int_0^t e^{-(t-s)} \begin{bmatrix} 2 & 0 \end{bmatrix} y_k(s)ds.$$

If, for instance, we wish to obtain $x_k(t)$ and $y_k(t)$ for $k = 1, 2$, we can proceed as follows. For $k = 1$:

$$y_1(t) = \tilde{A}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + e^{-t}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

or, equivalently,

$$y_1(t) = (\begin{bmatrix} 0 \\ 3 \end{bmatrix} \mathcal{J}_{-1,\begin{bmatrix} 2 & 0 \end{bmatrix}} + \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}) \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + e^{-t}\begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

or, equivalently,

$$y_1(t) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \mathcal{J}_{-1,\begin{bmatrix} 2 & 0 \end{bmatrix}} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \sin(t) \\ 2\cos(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 3e^{-t} \end{bmatrix},$$

or, equivalently,

$$y_1(t) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \int_0^t e^{-(t-s)} \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} \sin(s) \\ \cos(s) \end{bmatrix} ds + \begin{bmatrix} \frac{1}{4}\sin(t) \\ 2cos(t) + 3e^{-t} \end{bmatrix},$$

or, equivalently,

$$y_1(t) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \int_0^t 2e^{-(t-s)} \sin(s) ds + \begin{bmatrix} \frac{1}{4}\sin(t) \\ 2\cos(t) + 3e^{-t} \end{bmatrix},$$

or, equivalently,

$$y_1(t) = \begin{bmatrix} 0 \\ 6e^{-t} \end{bmatrix} \int_0^t e^{s} \sin(s) ds + \begin{bmatrix} \frac{1}{4}\sin(t) \\ 2\cos(t) + 3e^{-t} \end{bmatrix},$$

or, equivalently,

$$y_1(t) = \begin{bmatrix} 0 \\ 6e^{-t} \end{bmatrix} \left(\frac{e^t}{2}(\sin(t) - \cos(t)) + \frac{1}{2} \right) + \begin{bmatrix} \frac{1}{4}\sin(t) \\ 2\cos(t) + 3e^{-t} \end{bmatrix},$$

or, equivalently,

$$y_1(t) = \begin{bmatrix} y_1^{(1)}(t) \\ y_1^{(2)}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\sin(t) \\ 3(\sin(t) + \cos(t)) - 3e^{-t} + 2\cos(t) + 3e^{-t} \end{bmatrix}.$$

For $x_1(t)$ we have:

$$x_1(t) = \frac{e^{-t}}{2} + \int_0^t e^{-(t-s)} \begin{bmatrix} 2 & 0 \end{bmatrix} y_1(s) ds,$$

or, equivalently,

$$x_1(t) = \frac{e^{-t}}{2} + \frac{e^{-t}}{2} \int_0^t e^{s} \sin(s) ds,$$

or, equivalently,

$$x_1(t) = \frac{e^{-t}}{2} + \frac{e^{-t}}{2}\left(\frac{e^t}{2}(\sin(t) - \cos(t)) + \frac{1}{2}\right),$$

or, equivalently,

$$x_1(t) = \frac{e^{-t}}{4} + \frac{1}{4}(\sin(t) - \cos(t)).$$

For $k = 2$ and (1) we have:

$$y_2(t) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} (\frac{e^{-t}}{4} + \frac{1}{4}(\sin(t) - \cos(t)) + \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} y_1(t),$$

or, equivalently,

$$y_2(t) = \begin{bmatrix} y_2^{(1)}(t) \\ y_2^{(2)}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{16}\sin(t) \\ \frac{e^{-t}}{2} + \sin(t) + \cos(t) + \frac{3}{4}\cos(t)e^{-t} \end{bmatrix}.$$

For $x_2(t)$ we have:

$$x_2(t) = \frac{e^{-t}}{5} + \int_0^t e^{-(t-s)} \begin{bmatrix} 2 & 0 \end{bmatrix} y_2(s)ds,$$

or, equivalently,

$$x_2(t) = \frac{e^{-t}}{5} + \frac{e^{-t}}{8}\int_0^t e^s \sin(s)ds,$$

or, equivalently,

$$x_2(t) = \frac{e^{-t}}{5} + \frac{e^{-t}}{8}\left(\frac{e^t}{2}(\sin(t) + \cos(t)) - \frac{1}{2}\right),$$

or, equivalently,

$$x_2(t) = \frac{11e^{-t}}{80} + \frac{1}{16}(\sin(t) + \cos(t)).$$

The response of the states in time, for $k = 1$, is shown in Figure 1.

For the eigenvalue analysis, we consider the characteristic equation (13) of this system:

$$\begin{vmatrix} sI_r - A & -B \\ -C & zI_m - D \end{vmatrix} = 0,$$

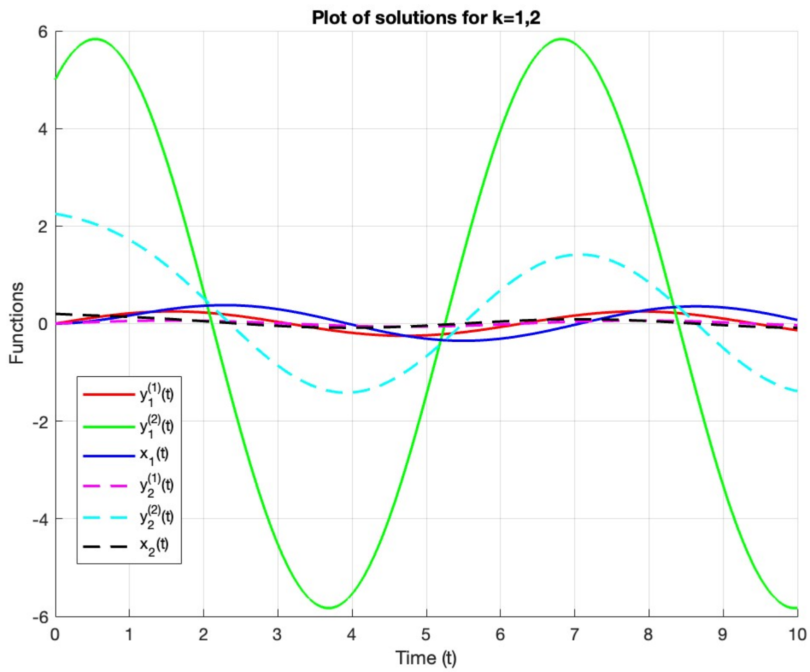

Figure 1: Plot of solutions for $k = 1, 2$ in Example 1

or, equivalently,

$$\begin{vmatrix} s+1 & -2 & 0 \\ 0 & z-\frac{1}{4} & 0 \\ -3 & 0 & z-\frac{1}{2} \end{vmatrix} = 0,$$

or, equivalently,

$$(2z-1)(4z-1)(s+1) = 0.$$

The roots of this equation are:

$$z_1 = \frac{1}{4}, \quad z_2 = \frac{1}{2}, \quad s_1 = -1.$$

Thus, $s_1 < 0$, and $|z_1|, |z_2| < 1$, and the system is asymptotically stable.

Example 2 We consider now the following hybrid system:

$$2\tfrac{d}{dt}x_k(t) = 5x_k(t) - y_k(t) + 2kt,$$

$$0_{n,1} = -x_k(t) + y_k(t) + 2k + 2t,$$

with initial conditions $x_k(0) = y_k(0) = k$. Equivalently, in matrix form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} Y_k(t) = \frac{1}{2} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} Y_k(t) + \begin{bmatrix} kt \\ k+t \end{bmatrix},$$

which is the hybrid system (2) given in the form in Lemma 2.1. The pencil $sE - A_0$ is given by:

$$sE - A_0 = \begin{bmatrix} s - \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

This pencil has a finite eigenvalue $s = 2$ and an infinite. In addition

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The solution of (2) is given by (10) with

$$X_k(0) = Q^{-1} Y_k(0) = \begin{bmatrix} k \\ 0 \end{bmatrix}, \quad X_k^p(0) = k.$$

In addition since $J_p = 2$:

$$\mathcal{J}_{J_p,P_1} V_k(t) = \int_0^s e^{2(t-s)} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} ks \\ k+s \end{bmatrix} ds = \int_0^s e^{2(t-s)} (ks + k + s) ds,$$

or, equivalently,

$$\mathcal{J}_{J_p,P_1} V_k(t) = \frac{3k+1}{4} e^{2t} - \frac{(2t+3)k + (2t+1)}{4}.$$

Hence from (10) we have:

$$Y_k(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [e^{2t} k + \frac{3k+1}{4} e^{2t} - \frac{(2t+3)k + (2t+1)}{4}] - \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} kt \\ k+t \end{bmatrix},$$

or, equivalently,

$$Y_k(t) = \begin{bmatrix} \frac{7k+1}{4} e^{2t} - \frac{(2t+3)k}{4} - \frac{2t+1}{4} \\ \frac{7k+1}{4} e^{2t} + \frac{(5-2t)k}{4} + \frac{6t-1}{4} \end{bmatrix},$$

or, equivalently,

$$x_k(t) = \tfrac{7k+1}{4} e^{2t} - \tfrac{(2t+3)k}{4} - \tfrac{2t+1}{4},$$

$$y_k(t) = \tfrac{7k+1}{4} e^{2t} + \tfrac{(5-2t)k}{4} + \tfrac{6t-1}{4}.$$

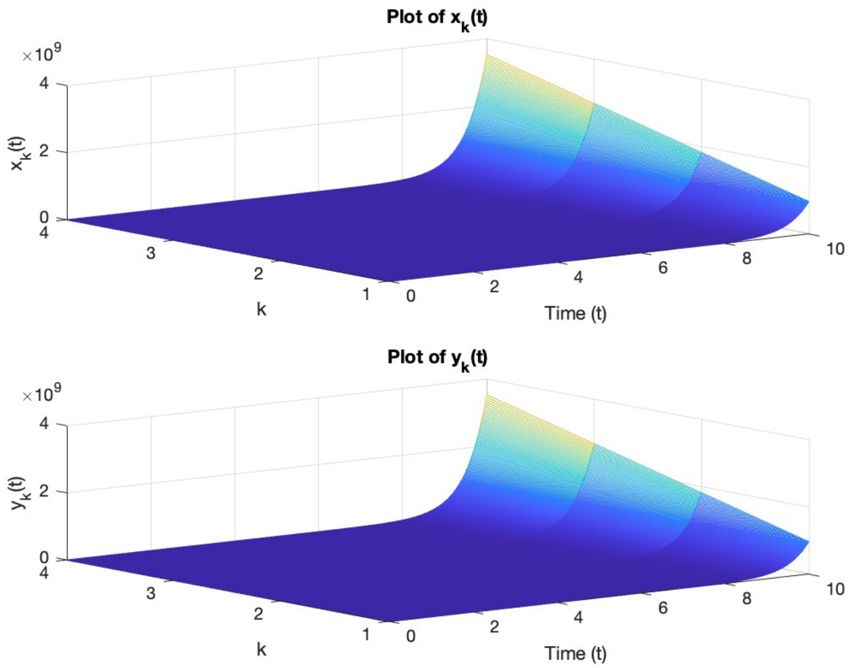

Figure 2: Plots of solutions for $x_k(t)$, $y_k(t)$ in Example 2

For a better perception of the states' behavior, Figure 2 graphically represents their response in time, where the system's instability can be observed.

For the eigenvalue analysis, we consider the characteristic equation (13) of this system given by:

$$\begin{vmatrix} sI_r - A & -B \\ -C & zI_m - D \end{vmatrix} = 0.$$

or, equivalently,

$$sz - \frac{5z}{2} - s + 2 = 0.$$

The response of the eigenvalues s and z follows Figure 3, which illustrates the system's instability, evident through the presence of positive s values and $|z|$ greater than one.

Example 3 In this example, we consider the power system dynamic model exposed in Figure 4. This represents the so called "single node model" and it

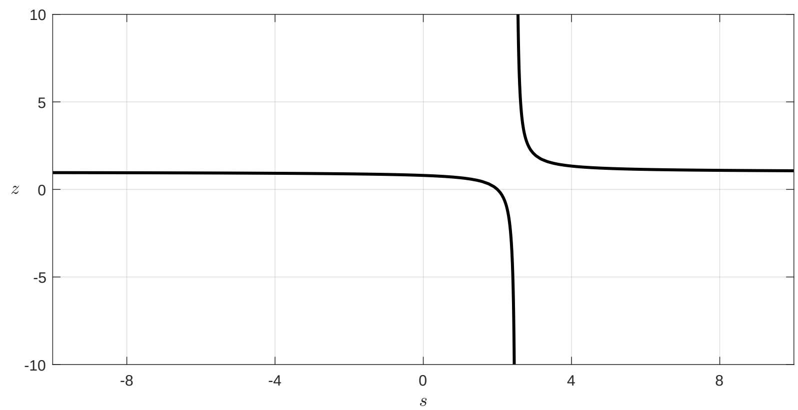

Figure 3: Eigenvalue Response of Example 2

has been proposed to study dynamic performance of primary and secondary frequency regulation in conventional power systems the dynamics of which are dominated by the synchronous generators [4]. In Figure 4, the generator is represented through its rotating inertia (M), the primary regulation is a simple low pass with gain k_1 and time constant T and the secondary frequency control is an integrator with gain k_0.

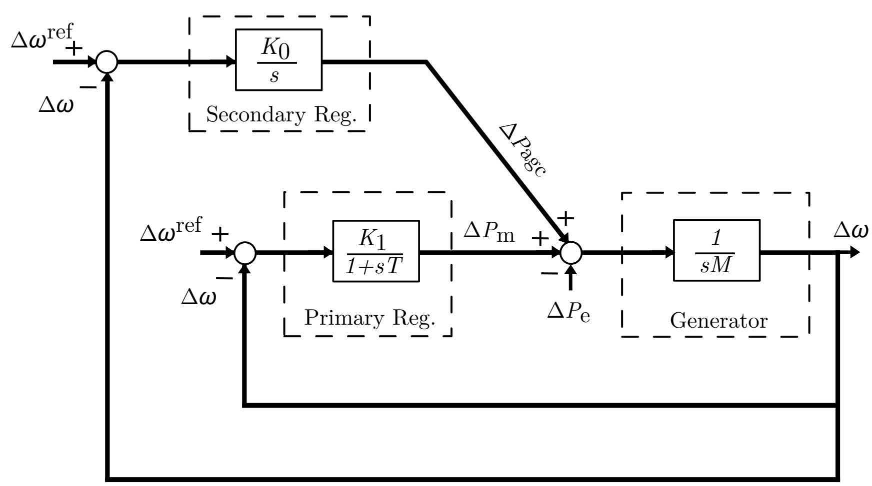

Figure 4: AGC Model

The focus of this example is on the modelling of the secondary frequency control, usually called Automatic Generation Control (AGC). The objective of the AGC is to restore the system frequency to its nominal value following a contingency. This is obtained by changing the power set point of the synchronous machine through the signal (ΔP_{agc}). Without the AGC, the primary control is unable to restore the initial frequency as it is not, in general, a perfect tracking control. Figure 5 illustrates the effect of the primary control alone and the combined action of primary frequency control and AGC. We note that, while the AGC is able to restore the nominal frequency of the system, the gain K_0 has to be properly tuned to avoid the overall system becoming unstable. If K_0 is too high, in fact, the AGC dynamic can couple with that of the primary frequency control and lead to unstable oscillations [27].

Modelling the AGC as a continuous perfect-tracking controller is a simplification. In practice, however, as it is relatively slow and centralized and the signal ΔP_{agc} has to be transmitted over long distances, the AGC is implemented as a discrete controller that updates its output at regular time intervals. Figure 5 shows the different trajectories of the system frequency as obtained with the continuous and the discrete AGC models, both with a gain $K_0 = 5$, and for the discrete AGC an updating time of 5 s. The contingency is a step-negative variation of the electrical load $\Delta P_e(t)$. We note that, for discrete AGC, the instability can arise as a combination of the gain K_0 and the time interval with which the signal ΔP_{agc} is updated [22], as observed Figure in 6.

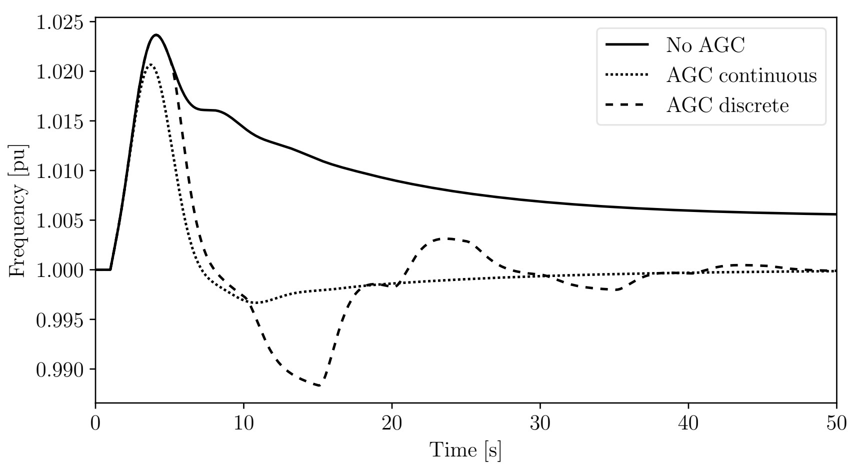

Figure 5: AGC within a Power System

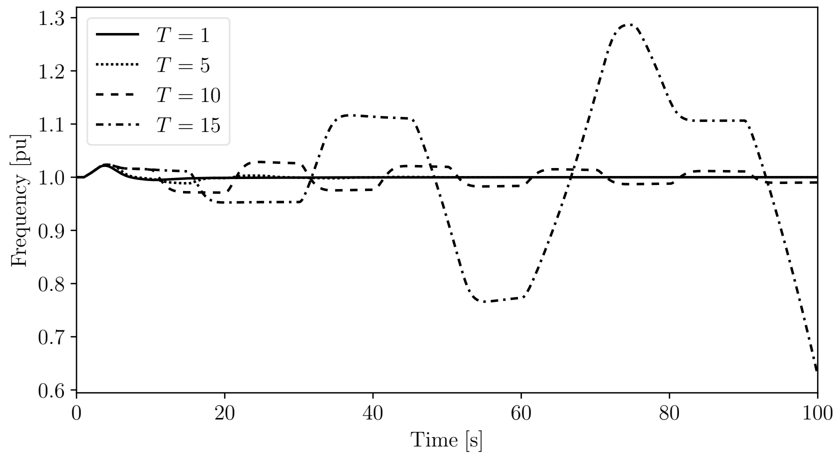

Figure 6: System response to different time updates of Discrete AGC

The continuous AGC model is practical for small-signal stability analysis as it allows calculating the eigenvalues of the system using the conventional state matrix. However, the dynamic response of the discrete AGC model has a more complex behavior than the continuous linear one as the instability depends on the combined effect of K_0 and T. We show in this example that, with the proposed technique, it is possible to analyze the small-signal stability of the power system model while retaining the discrete model of the AGC.

The continuous model of the system presented in Figure 4 is described by the following set of equations:

$$\begin{aligned} M\Delta\dot{\omega}(t) &= \Delta P_m(t) - \Delta P_e(t) + \Delta P_{\mathrm{agc}}(t), \\ T\Delta\dot{P}_m(t) &= K_1(\Delta\omega^{\mathrm{ref}} - \Delta\omega(t)) - \Delta P_m(t), \\ \Delta P_{\mathrm{agc}} &= K_0(\Delta\omega^{\mathrm{ref}} - \Delta\omega(t)), \end{aligned} \tag{15}$$

which describe, respectively, the generator, the primary frequency control provided by the turbine governor of the machine, and the AGC. If we model the AGC as a discrete controller, (15) becomes an hybrid system, as follows:

$$
\begin{aligned}
M\Delta\dot{\omega}(t) = \Delta P_m(t) - \Delta P_e(t) + \mathcal{H}(t-kT)\Delta P_{\text{agc},k} \\
+(1-\mathcal{H}(t-kT))\Delta P_{\text{agc},k-1}, \\
T\Delta\dot{P}_m(t) = K_1(\Delta\omega^{\text{ref}} - \Delta\omega(t)) - \Delta P_m(t), \\
\Delta P_{\text{agc},k} = K_0(\Delta\omega^{\text{ref}} - \Delta\omega_k),
\end{aligned} \tag{16}
$$

where T is the sampling period of the AGC signal; $\omega_k = \omega(kT)$ is the sampled value of the system frequency; $k \in \mathbb{N}_+$; and $H(t)$ is the Heaviside step function:

$$
\mathcal{H}(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0, \end{cases}
$$

The dashed line in Figure 5 shows the dynamic performance of model (16).

Rewriting the system equations using the notation introduced in this work leads to:

$$
\mathcal{H}(t-kT)\Delta P_{\text{agc},k} + (1-\mathcal{H}(t-kT))\Delta P_{\text{agc},k-1} = \\
K_0\mathcal{H}(t-kT)\nabla\omega_k + K_0(\Delta\omega^{\text{ref}} - \Delta\omega_{k-1}).
$$

Then, let us define:

$$
u_k(t) = \left[\begin{array}{c} \frac{1}{M}(K_0\mathcal{H}(t-kT)\nabla\omega_k + K_0(\Delta\omega^{\text{ref}} - \delta\omega_{k-1})) \\ \frac{K_1}{T}\Delta\omega^{\text{ref}} \end{array} \right].
$$

By using the notation of (1), we have:

$$
\dot{x}(t) = Ax(t) + u_k(t),
$$

$$
y_{k+1} = v_k(t),
$$

where:

$$
x_k(t) := x(t) = \left[\begin{array}{c} \omega(t) \\ \Delta P_m(t) \end{array} \right], \quad y_k(t) := y(t) = \Delta P_{\text{agc},k},
$$

and

$$
A = \left[\begin{array}{cc} 0 & \frac{1}{M} \\ -\frac{K_1}{T} & -\frac{1}{T} \end{array} \right], \quad v_k(t) := v_k = K_0(\Delta\omega^{\text{ref}} - \Delta\omega_{k+1}).
$$

Compared to (1) we have $B = 0_{2,1}$, $C = 0_{1,2}$, $D = 0_{2,2}$. From (12) the characteristic polynomial of (15) is given by:

$$
p(s,\hat{s}) = \begin{vmatrix} (\hat{s}-1)A - s(\hat{s}-1)I_2 & 0_{2,1} \\ 0_{1,2} & \hat{s}+1 \end{vmatrix}.
$$

where

$$(\hat{s}-1)A - s(\hat{s}-1)I_r = (\hat{s}-1)[A - sI_r],$$

or, equivalently,

$$(\hat{s}-1)A - s(\hat{s}-1)I_r = (\hat{s}-1)\begin{bmatrix} -s & \frac{1}{M} \\ -\frac{K_1}{T} & -\frac{1}{T}-s \end{bmatrix}.$$

Hence

$$p(s,\hat{s}) = \begin{vmatrix} -s & \frac{1}{M} & 0 \\ -\frac{K_1}{T} & -\frac{1}{T}-s & 0 \\ 0 & 0 & \hat{s}+1 \end{vmatrix},$$

where $\hat{s} \neq 1$. For the eigenvalue analysis, we consider the characteristic equation $p(s,\hat{s}) = 0$ of this system, or, equivalently:

$$(s^2 + \frac{1}{T}s + \frac{K_1}{MT})(\hat{s}+1) = 0.$$

The roots of this equation are:

$$s_{1,2} = -\frac{1}{2T} \pm \frac{1}{2}\sqrt{\frac{1}{T^2} - \frac{2K_1}{MT}}, \quad \hat{s}_1 = -1.$$

If $\frac{1}{T} - \frac{2K_1}{M} < 0$ then, $\mathrm{Re}(s_{1,2}) < 0$, and $\hat{s}_1 < 0$, and the system is asymptotically stable. If $\frac{1}{T} - \frac{2K_1}{M} > 0$, since $-\frac{2K_1}{MT} < 0$ we have $\sqrt{\frac{1}{T^2} - \frac{2K_1}{MT}} < \frac{1}{T}$, hence, $\mathrm{Re}(s_{1,2}) < 0$, and $\hat{s}_1 < 0$, and the system is asymptotically stable.

Conclusions

In this paper, we investigate a hybrid system of differential-difference equations represented by (1) and derive formulas for its solutions. We explore the stability and characteristic equation of (1), and conclude by providing numerical examples to illustrate our theory. Our analysis includes applications demonstrating the relevance of the proposed methodology in modeling devices and controllers within electric power systems.

In future work, our objectives include the incorporation of fractional operators, both discrete and continuous, into the system described by (1), leading to the construction of a class of fractional hybrid dynamical systems. Additionally, we plan to delve deeper into the stability analysis of electrical networks modeled as hybrid systems. Another promising direction is to employ operator theory, such as semigroups and resolvents, instead of matrices for a more general case. This approach could potentially extend the applicability and robustness of our results in more complex and diverse hybrid dynamical systems.

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