A bounded dynamical network of curves and the stability of its steady states

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Abstract. In this article we study the dynamic behavior of a network that consists of curves that are in motion and bounded. We first focus on the construction of the model which is a system of non-linear partial differential equations (PDEs). This system is subject to four conditions: angle and intersection conditions between the curves at the point that they meet; angle and intersection conditions between the curves and the boundary from which the network is bounded. Then, we define a linear operator and study the stability of the steady states of the corresponding boundary value problem (BVP).

Key words dynamical network, curves, intersection conditions, angle conditions, geometry, boundary.

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1 Introduction

In the last few decades many authors have studied dynamical networks formed by curves in motion. The studies focus on the differential geometry of the problem, see [10], [12], [14], [18], [22], numerical methods for dynamical systems, see [2], [9], [16], [19], [24], [25], and the stability of the network, see [3], [5], [6], [26], [27].

Focus has also been given on the mathematical modelling and applications in material science and engineering, see [1], [11], [15], [23].

In this article we study a bounded network of curves that are in motion and intersect at a junction. We consider a parametric form of two variables for each curve and form a system of non-linear partial differential equations (PDEs) that describes the motion of the network. We also consider the equation of the curve that describes the boundary of the domain that bounds the network. The system of non-linear PDEs is subject to four conditions. Intersection, angle conditions at the junctions and at the boundary of the domain that are formed using properties from differential geometry.

In Section 2 we form the boundary value problem (BVP), in Section 3 we linearize the non-linear PDEs and reformulate efficiently the conditions of the problem, and in Section 4 we define the linear operator used for the linearization of the BVP, and by studying the eigenvalues of this operator we conclude that the stability of the steady states of the network depends on the sign of the curvature of the boundary of the domain. The results in this article aim to bring new ideas and insights that can be applied to other types of networks, such as hexagonal dynamical networks as they appear for example in soap bubbles, honeycomb, grain growth etc, see [4], [13], [17], [21], and also update geometrical properties that are used in applications in engineering such as the structure of electrical circuits and studies of elasticity, plasticity in material science, see [7], [8], [20].

2 Problem formulation

We will consider a dynamical network of three curves in motion in $\mathbb{R}^2$ that meet at a junction and are bounded from the boundary $\partial\Omega$ of a domain $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^2$.

Definition 2.1. With $T_i(x,t)$ we denote the unit tangent vector of curve i at x and time t, and with $N_i(x,t)$ the unit normal vector of curve i at x and time t.

Let $u_i(x,t) = (u_{i1}(x,t), u_{i2}(x,t))$, $i = 1,2,3$, be a parametric form of curve i with

$$u_i : [0,1] \times [0,+\infty) \to \Omega, \quad i = 1,2,3,$$

and

$$u_{ij} : [0,1] \times [0,+\infty) \to \Omega_j, \quad i = 1,2,3, \quad j = 1,2.$$

With $u_{ix} = \frac{\partial u_i}{\partial x}$, $u_{ixx} = \frac{\partial^2 u_i}{\partial x^2}$ we will denote the partial derivatives of first and second order, respectively, of u_i in respect to x, while with $u_{it} = \frac{\partial u_i}{\partial t}$ we will denote the partial derivative of first order of u_i in respect to t.

Let $k_i(x,t)$, $k_i : \mathbb{R}^2 \to \mathbb{R}$ be curvature of curve i. At the steady states, $k_i = 0$. For these curves, by taking into account that normal velocity equals to curvature we get:

$$\frac{\partial u_i}{\partial t} N_i = k_i, \quad i = 1,2,3,$$

where dependencies on x and t have been omitted for simplicity in notation. Furthermore

$$k_i N_i = \frac{|u_{ixx} \times u_{ix}|}{|u_{ix}|^3} \frac{1}{|u_{ixx}|} u_{ixx}.$$

Equivalently

$$k_i N_i = \frac{1}{|u_{ix}|} u_{ixx},$$

because $|u_{ixx} \times u_{ix}| = |u_{ixx}||u_{ix}|$. Hence

$$u_{it} = \frac{1}{|u_{ix}|} u_{ixx}, \quad i = 1,2,3, \tag{1}$$

which is a non-linear system of PDEs. We will now focus on the conditions.

The three curves meet at a junction. Thus for each parametric form $u_i(x,t)$ of curve i, $i=1,2,3$, and if we consider that each curve meets the other at one end at $x=0$, the following relation holds:

$$u_1(0,t)=u_2(0,t)=u_3(0,t). \tag{2}$$

At $x=0$, curve 1 forms an angle $\theta_{1,2}(t)=\theta_{1,2}$ with curve 2, and curve 2 forms an angle $\theta_{2,3}(t)=\theta_{2,3}$ with curve 3. Curve 3 forms an angle $2\pi-\theta_{1,2}-\theta_{2,3}$ rad with curve 1. Then

$$T_1(0,t)T_2(0,t)=\cos\theta_{1,2}, \quad T_2(0,t)T_3(0,t)=\cos\theta_{2,3},$$

or, equivalently,

$$\frac{u_{1x}(0,t)}{|u_{1x}(0,t)|}\frac{u_{2x}(0,t)}{|u_{2x}(0,t)|}=\cos\theta_{1,2}, \quad \frac{u_{2x}(0,t)}{|u_{2x}(0,t)|}\frac{u_{3x}(0,t)}{|u_{3x}(0,t)|}=\cos\theta_{2,3},$$

or, equivalently,

$$u_{1x}(0,t)u_{2x}(0,t)=s^t_{1x}(1)s^t_{2x}(1)\cos\theta_{1,2}, \quad u_{2x}(0,t)u_{3x}(0,t)=s^t_{2x}(1)s^t_{3x}(1)\cos\theta_{2,3}. \tag{3}$$

At the steady states we will have $\theta_{1,2}=\theta_{2,3}=\frac{2\pi}{3}$ rad. Where s^t_i denotes the arc length parameter, with

$$s^t_i := s^t_i(x)=\int_0^x |u_{ix}(\xi,t)|d\xi,$$

and consequently,

$$s^t_{ix}=|u_{ix}(x,t)|.$$

Definition 2.2. With $f=0$, $f:\mathbb{R}^2\to\mathbb{R}$, we denote the equation of the boundary $\partial\Omega$ of Ω.

The network that the curves form is bounded by Ω, i.e. each curve meets $\partial\Omega$ at $x=1$. Hence since $f=0$ is the equation of $\partial\Omega$, we get:

$$f(u_i(1,t))=0, \quad i=1,2,3. \tag{4}$$

Definition 2.3. With $N^i_{\partial\Omega}:=N_{\partial\Omega}(u_i(1,t))=\frac{\nabla f(u_i(1,t))}{|\nabla f(u_i(1,t))|}$ we denote the unit normal vector of the boundary $\partial\Omega$ of Ω at $u_i(1,t)$, and with $K^i_{\partial\Omega}$ is the curvature of the boundary $\partial\Omega$ of Ω at $u_i(1,t)$.

Each curve meets the boundary Ω with $\frac{\pi}{2}$ rad i.e at the point that they meet, the unit tangent of the curve and the unit tangent of the boundary are orthogonal. Hence

$$\frac{u_{ix}(1,t)}{|u_{ix}(1,t)|}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{\nabla f(u_i(1,t))}{|\nabla f(u_i(1,t))|}=0, \quad i=1,2,3,$$

or, equivalently,

$$u_{ix}(1,t)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}N^i_{\partial\Omega}=0, \quad i=1,2,3. \tag{5}$$

To sum up, we have the system of non-linear PDEs (1) and its conditions (2)–(5).

3 Linearization

The system of PDEs (1) is non-linear. In this subsection we will linearize effectively (1).

Theorem 3.1. We consider the system of non-linear PDEs (1) and its conditions (2)–(5). Then a linearization of this BVP consists of the linear system of PDEs:

$$v_{it} = v_{ixx}, \quad i = 1, 2, 3. \tag{6}$$

and the conditions:

1. Conditions for intersection at the junction at $x = 0$:

$$v_1(0, t) = v_2(0, t) = v_3(0, t). \tag{7}$$

2. Angle conditions at the junction at $x = 0$:

$$v_{11x}(0, t) - v_{21x}(0, t) = 0, \quad v_{21x}(0, t) - v_{31x}(0, t) = 0. \tag{8}$$

3. Conditions for intersection at the boundary $\partial\Omega$ at $x = 1$:

$$v_i(1, t)\nabla f = 0, \quad i = 1, 2, 3. \tag{9}$$

4. Angle conditions at the boundary $\partial\Omega$ at $x = 1$:

$$[K^i_{\partial\Omega} v_i(1, t) - v_{ix}(1, t)]N_i = 0. \tag{10}$$

Where $\tilde{u}_i(x, t) := \hat{u}_i(s^t_i, t)$, $s^t_i = s^t_i(x)$, solution of (1). Where $\hat{u}_i(s^t_i, t)$ is parametric form of curve i with arc length parameter s^t_i, i.e.

$$s^t_i := s^t_i(x) = \int_0^x |u_{ix}(\xi, t)| d\xi, \quad s^t_{ix} = |u_{ix}(x, t)|.$$

In addition:

$$v_i(x, t) = v_{i1}(x, t)N_i + v_{i2}(x, t)T_i, \tag{11}$$

with

$$v_i : [0, 1] \times [0, +\infty) \to \Omega, \quad v_{ij} : [0, 1] \times [0, +\infty) \to \Omega_j, \quad i = 1, 2, 3, \quad j = 1, 2.$$

Furthermore $T_i := T_i(s^t_i(x), t)$, $N_i := N_i(s^t_i(x), t)$, and $K^i_{\partial\Omega}$ the curvature of the boundary $\partial\Omega$ of Ω at $\tilde{u}_i(1, t)$.

Proof. Let

$$u_i(x, t) = \tilde{u}_i(x, t) + \epsilon_i v_i(x, t), \quad 0 < \epsilon_i \ll 1, \tag{12}$$

with $\tilde{u}_i(x, t) := \hat{u}_i(s^t_i, t)$, $s^t_i = s^t_i(x)$, solution of (1). Where $\hat{u}_i(s^t_i, t)$ is parametric form of curve i with s^t_i arc length parameter, i.e.

$$s^t_i := s^t_i(x) = \int_0^x |u_{ix}(\xi, t)| d\xi, \quad s^t_{ix} = |u_{ix}(x, t)|.$$

Furthermore let

$$v_i(x,t) = v_{i1}(x,t)N_i + v_{i2}(x,t)T_i,$$

with

$$v_i : [0,1] \times [0,+\infty) \to \Omega, \quad v_{ij} : [0,1] \times [0,+\infty) \to \Omega_j, \quad i = 1,2,3, \quad j = 1,2.$$

By substituting (12) into (1) we get

$$\tilde{u}_{it} + \epsilon_i v_{it} = \frac{1}{|\tilde{u}_{ix} + \epsilon_i v_{ix}|^2}(\tilde{u}_{ixx} + \epsilon_i v_{ixx}), \quad i = 1,2,3.$$

Let

$$F(\epsilon_i) = \frac{1}{|\tilde{u}_{ix} + \epsilon_i v_{ix}|^2}(\tilde{u}_{ixx} + \epsilon_i v_{ixx}), \quad i = 1,2,3,$$

whereby

$$F(\epsilon_i) = F(0) + \epsilon_i F'(0) + \mathcal{O}(\epsilon_i^2), \quad i = 1,2,3.$$

Where

$$F(0) = \frac{1}{|\tilde{u}_{ix}|}\tilde{u}_{ixx},$$

and

$$F'(0) = \frac{v_{ixx}|\tilde{u}_{ix}|^2 - 2v_{ix}\tilde{u}_{ix}\tilde{u}_{ixx}}{|\tilde{u}_{ix}|^4}.$$

Equivalently, since $\tilde{u}_i$ is defined as a parametric form of curve i, we have $|\tilde{u}_{ix}| = 1$, and $\tilde{u}_{ix}\tilde{u}_{ixx}$=0, we have:

$$F'(0) = v_{ixx}.$$

Hence

$$F(\epsilon_i) = \frac{1}{|\tilde{u}_{ix}|}\tilde{u}_{ixx} + \epsilon_i v_{ixx} + \mathcal{O}(\epsilon_i^2), \quad i = 1,2,3.$$

and equivalently a linearization of (1) is

$$\tilde{u}_{it} + \epsilon_i v_{it} = \frac{1}{|\tilde{u}_{ix}|}\tilde{u}_{ixx} + \epsilon_i v_{ixx}, \quad i = 1,2,3,$$

or, equivalently, since $\tilde{u}_i$ is assumed solution of (1):

$$v_{it} = v_{ixx}, \quad i = 1,2,3.$$

We will now define the conditions. From (2) and by using (12) we get

$$\tilde{u}_1(0,t) + \epsilon_1 v_1(0,t) = \tilde{u}_2(0,t) + \epsilon_2 v_2(0,t) = \tilde{u}_3(0,t) + \epsilon_3 v_3(0,t),$$

whereby, and since $\tilde{u}_i(0,t)$ is solution of (1), we arrive at

$$v_1(0,t) = v_2(0,t) = v_3(0,t).$$

From (3) and by using (12) we arrive at

$$[\tilde{u}_{1x}(0,t) + \epsilon_1 v_{1x}(0,t)][\tilde{u}_{2x}(0,t) + \epsilon_2 v_{2x}(0,t)] = s^t_{1x}(1)s^t_{2x}(1)\cos\theta_{1,2},$$

and

$$[\tilde{u}_{2x}(0,t)+\epsilon_2 v_{2x}(0,t)][\tilde{u}_{3x}(0,t)+\epsilon_3 v_{3x}(0,t)] = s^t_{2x}(1)s^t_{3x}(1)\cos\theta_{2,3}.$$

Equivalently, by ignoring the coefficients of $\epsilon_1\epsilon_2 \cong \epsilon^2$, $\epsilon_2\epsilon_3 \cong \epsilon^2$, with $0<\epsilon_i \ll 1$ for $i=1,2,3$, and by using $\tilde{u}_{1x}(0,t)=T_i$, for T_i at $x=0$, $\forall i=1,2,3$:

$$[1+\sum_{i=1}^{2}\epsilon_i v_{i2x}(0,t)]\tilde{u}_{1x}(0,t)\tilde{u}_{2x}(0,t)+\epsilon_1 v_{11x}(0,t)T_2N_1+\epsilon_2 v_{21x}(0,t)T_1N_2 = s^t_{1x}(1)s^t_{2x}(1)\cos\theta_{1,2},$$

and

$$[1+\sum_{i=2}^{3}\epsilon_i v_{i2x}(0,t)]\tilde{u}_{2x}(0,t)\tilde{u}_{3x}(0,t)+\epsilon_2 v_{21x}(0,t)T_3N_2+\epsilon_3 v_{31x}(0,t)T_2N_3 = s^t_{2x}(1)s^t_{3x}(1)\cos\theta_{2,3}.$$

whereby taking into account that $\tilde{u}_i(x,t)$ is solution of (1), and $\epsilon_i \cong \epsilon$ with $0<\epsilon\ll 1$, we have

$$v_{11x}(0,t)T_2N_1 + v_{21x}(0,t)T_1N_2 = 0,$$

and

$$v_{21x}(0,t)T_3N_2 + v_{31x}(0,t)T_2N_3 = 0.$$

Equivalently, since $T_1N_2 = T_2N_3 = cos(\theta_{1,2}+\frac{\pi}{2})$ and $T_2N_1 = T_3N_2 = cos(\theta_{1,2}-\frac{\pi}{2})$:

$$v_{11x}(0,t)cos(\theta_{1,2}-\frac{\pi}{2}) + v_{21x}(0,t)cos(\theta_{1,2}+\frac{\pi}{2}) = 0,$$

and

$$v_{21x}(0,t)cos(\theta_{2,3}-\frac{\pi}{2}) + v_{31x}(0,t)cos(\theta_{2,3}+\frac{\pi}{2}) = 0.$$

Consequently:

$$v_{11x}(0,t)-v_{21x}(0,t)=0, \quad v_{21x}(0,t)-v_{31x}(0,t)=0.$$

From (4) and by using (12) we arrive at

$$f(\tilde{u}_i(1,t)+\epsilon_i v_i(1,t)) = 0, \quad i=1,2,3.$$

or, equivalently,

$$f(\tilde{u}_{i1}(1,t)+\epsilon_i v_{i1}(1,t), \tilde{u}_{i2}(1,t)+\epsilon_i v_{i2}(1,t)) = 0, \quad i=1,2,3.$$

or, equivalently,

$$f(f^t_{i1}, f^t_{i2}) = 0, \quad i=1,2,3,$$

where

$$f^t_{i1} = \tilde{u}_{i1}(1,t)+\epsilon_i v_{i1}(1,t), \quad f^t_{i2} = \tilde{u}_{i2}(1,t)+\epsilon_i v_{i2}(1,t).$$

We now differentiate in respect to ϵ_i and we get:

$$\frac{df}{d\epsilon_i} = 0, \quad i=1,2,3,$$

or, equivalently,

$$\frac{\partial f}{\partial f^t_{i1}}\frac{df^t_{i1}}{d\epsilon_i}+\frac{df}{\partial f^t_{i2}}\frac{df^t_{i2}}{d\epsilon_i}=0,\quad i=1,2,3,$$

or, equivalently,

$$v_i(1,t)\nabla f=0,\quad i=1,2,3.$$

From (5) and by using (12) we arrive at

$$[\tilde{u}_{ix}(1,t)+\epsilon_i v_{ix}(1,t)]\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}N_{\partial\Omega}(\tilde{u}_i(1,t)+\epsilon_i v_i(1,t))=0,\quad i=1,2,3,$$

whereby using Taylor expansion at $\epsilon_i=0$:

$$[\tilde{u}_{ix}(1,t)+\epsilon_i v_{ix}(1,t)]\Big[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}N_{\partial\Omega}(\tilde{u}_i(1,t))+\epsilon_i\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\nabla N_{\partial\Omega}(\tilde{u}_i(1,t))v_i(1,t)\Big]=0.$$

Equivalently, if we ignore the coefficients of ϵ_i^2, and by using that $\tilde{u}_i(x,t)$ is a solution of (1), we arrive at:

$$\tilde{u}_{ix}(1,t)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\nabla N^t_{\partial\Omega}v_i(1,t)+v_{ix}(1,t)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}N^t_{\partial\Omega}=0,$$

or, equivalently, and by using Goldman's formulas, see [14], and by considering N_i at $x=1$, we arrive at:

$$K^i_{\partial\Omega}N_i v_i(1,t)-v_{ix}(1,t)N_i=0,$$

or, equivalently,

$$[K^i_{\partial\Omega}v_i(1,t)-v_{ix}(1,t)]N_i=0.$$

The proof is completed.

We now state the following corollary.

Corollary 3.1. We consider the linear BVP that consists of the system of linear PDEs (6) and the conditions (7)–(10). Then at the steady states ($k_i=0$) the system will take the form:

$$v_{i1t}=v_{i1xx},\quad v_{i2t}=v_{i2xx},\quad i=1,2,3. \tag{13}$$

Where $v_i:[0,1]\times[0,+\infty)\to\Omega$, $v_{ij}:[0,1]\times[0,+\infty)\to\Omega_j$, $i=1,2,3$, $j=1,2$, as defined in (11).

Proof. By using the Frenet formulas we get

$$v_{it}=[v_{i1t}+k_i v_{i2}]N_i+[v_{i2t}-k_i v_{i1}]T_i,\quad v_{ix}=[v_{i1x}+k_i v_{i2}]N_i+[v_{i2x}-k_i v_{i1}]T_i,$$

and

$$v_{ixx}=[v_{i1xx}+2k_i v_{i2x}+k_{ix}v_{i2}-k_i^2 v_{i1}]N_i+[v_{i2xx}-2k_i v_{i1x}-k_{ix}v_{i1}-k_i^2 v_{i2}]T_i.$$

Thus we can write (6) in the following form

$$v_{i1t} + k_i v_{i2} = v_{i1xx} + 2k_i v_{i2x} + k_{ix} v_{i2} - k_i^2 v_{i1}, \quad i = 1, 2, 3,$$

and

$$v_{i2t} - k_i v_{i1} = v_{i2xx} - 2k_i v_{i1x} - k_{ix} v_{i1} - k_i^2 v_{i2}, \quad i = 1, 2, 3.$$

At the steady states ($k_i = 0$) the system will take the form:

$$v_{i1t} = v_{i1xx}, \quad v_{i2t} = v_{i2xx}, \quad i = 1, 2, 3.$$

The proof is completed.

4 Stability

In this section we will study the stability of the steady states of (6) with conditions (7)–(10). System (6) at the steady states takes the form of (13) which is separable and the general solution can be expressed as

$$v_{ij}(x, t) = X_{ij}(x) Y_{ij}(t), \quad i = 1, 2, 3, \quad j = 1, 2.$$

where

$$X_{ij} : [0, 1] \to \Omega_j, \quad Y_{ij} : [0, +\infty) \to \Omega_j, \quad i = 1, 2, 3, \quad j = 1, 2.$$

For the eigenvalue problem that appears, we are interested in studying the stability and existence of eigenvalues with $\lambda \leq 0$. If $\lambda < 0$ we have:

$$v_{ij} = [c_{ij} \cosh(\sqrt{-\lambda} x) + d_{ij} \sinh(\sqrt{-\lambda} x)] e^{\lambda t}, \quad i = 1, 2, 3, \quad j = 1, 2,$$

while if $\lambda = 0$ we have:

$$v_{ij} = c_{ij} + d_{ij} x, \quad i = 1, 2, 3, \quad j = 1, 2.$$

If there do not exist eigenvalues with $\lambda \leq 0$, then $c_{ij} = d_{ij} = 0$, $\forall i = 1, 2, 3$, $\forall j = 1, 2$.

Theorem 4.1. We consider the linear BVP that consists of the system of linear PDEs (6) and the conditions (7)–(10). Then:

1. If $K^i_{\partial\Omega} > 0$, $\forall i = 1, 2, 3$, then there exist negative eigenvalues and the steady states are unstable. The eigenvalues are given from the solutions of the algebraic equations:

   $$K^i_{\partial\Omega} - \sqrt{-\lambda} \tanh(\sqrt{-\lambda}) = 0, \quad i = 1, 2, 3.$$

   The eigenfunctions are then given by:

   $$v_{i1} = c_{i1} \cosh(\sqrt{-\lambda} x) e^{\lambda t}, \quad i = 1, 2, 3,$$

   and

   $$v_{i2} = -d_{i2} [\tanh(\sqrt{-\lambda}) \cosh(\sqrt{-\lambda} x)] e^{\lambda t}, \quad i = 1, 2, 3.$$

   Where c_{i1}, d_{i2} are constant.

2. If $K^i_{\partial\Omega} = 0$, or, $K^i_{\partial\Omega} = 1$, $\forall i = 1, 2, 3$, then there exists the zero eigenvalue and the steady states are neutral stable. The eigenfunctions are then given by:

   - If $K^i_{\partial\Omega} = 0$:
   $$v_{i1} = c_{i1}, \quad i = 1, 2, 3,$$
   and
   $$v_{i2} = -d_{i2}(1 + x), \quad i = 1, 2, 3.$$
   Where c_{i1}, d_{i2} are constant.
   - If $K^i_{\partial\Omega} = 1$:
   $$v_{i1} = d, \quad i = 1, 2, 3.$$
   and
   $$v_{i2} = 0, \quad i = 1, 2, 3.$$
   Where d is constant.

Proof. We consider the linear system of PDEs (6) and the conditions (7)–(10). Then if $\lambda < 0$ we have:

$$v_{ij} = [c_{ij} \cosh(\sqrt{-\lambda}x) + d_{ij} \sinh(\sqrt{-\lambda}x)]e^{\lambda t}, \quad i = 1, 2, 3, \quad j = 1, 2.$$

From (7), (11) we have
$$v_1(0, t) = v_2(0, t) = v_3(0, t),$$
or, equivalently,
$$c_{11}N_1 + c_{12}T_1 = c_{21}N_2 + c_{22}T_2 = c_{31}N_1 + c_{32}T_3,$$
whereby we can get

$$c_{11} = c_{21}N_1N_2 + c_{22}N_1T_2, \quad c_{21} = c_{31}N_2N_3 + c_{32}N_2T_3, \quad c_{31} = c_{11}N_3N_1 + c_{12}N_3T_1,$$

and

$$c_{12} = c_{21}T_1N_2 + c_{22}T_1T_2, \quad c_{22} = c_{31}T_2N_3 + c_{32}T_2T_3 \quad c_{32} = c_{11}T_3N_1 + c_{12}T_3T_1.$$

Equivalently

$$c_{11} = -\frac{1}{2}c_{21} - \frac{\sqrt{3}}{2}c_{22}, \quad c_{21} = -\frac{1}{2}c_{31} - \frac{\sqrt{3}}{2}c_{32}, \quad c_{31} = -\frac{1}{2}c_{11} - \frac{\sqrt{3}}{2}c_{12},$$

and
$$c_{12} = -\frac{\sqrt{3}}{2}c_{21} - \frac{1}{2}c_{22}, \quad c_{22} = -\frac{\sqrt{3}}{2}c_{31} - \frac{1}{2}c_{32}, \quad c_{32} = -\frac{\sqrt{3}}{2}c_{11} - \frac{1}{2}c_{12},$$
or, equivalently,

$$3[\sum_{i=1}^{3} c_{i1}] + \sqrt{3}[\sum_{i=1}^{3} c_{i2}] = 0, \quad \sqrt{3}[\sum_{i=1}^{3} c_{i1}] + 3[\sum_{i=1}^{3} c_{i2}] = 0,$$

and hence

$$\sum_{i=1}^{3} c_{i1} = \sum_{i=1}^{3} c_{i2} = 0. \tag{14}$$

From (8):

$$d_{11} = d_{21} = d_{31}. \tag{15}$$

From (9) at $x = 1$:

$$v_i(1,t)T_i = 0, \quad i = 1,2,3,$$

or, equivalently using (11),

$$v_{i2}(1,t) = 0, \quad i = 1,2,3,$$

or, equivalently,

$$c_{i2}\cosh(\sqrt{-\lambda}) + d_{i2}\sinh(\sqrt{-\lambda}) = 0, \quad i = 1,2,3,$$

or, equivalently,

$$c_{i2} + d_{i2}\tanh(\sqrt{-\lambda}) = 0, \quad i = 1,2,3. \tag{16}$$

From (10) at $x = 1$ we have:

$$[K^i_{\partial\Omega} v_i(1,t) - v_{ix}(1,t)]N_i = 0,$$

or, equivalently using (11),

$$K^i_{\partial\Omega} v_{i1}(1,t) - v_{i1x}(1,t) = 0,$$

or, equivalently,

$$K^i_{\partial\Omega}[c_{i1}\cosh(\sqrt{-\lambda}) + d_{i1}\sinh(\sqrt{-\lambda})] - \sqrt{-\lambda}[c_{i1}\sinh(\sqrt{-\lambda}) + d_{i1}\cosh(\sqrt{-\lambda})] = 0,$$

or, equivalently,

$$K^i_{\partial\Omega}[c_{i1} + d_{i1}\tanh(\sqrt{-\lambda})] - \sqrt{-\lambda}[c_{i1}\tanh(\sqrt{-\lambda}) + d_{i1}] = 0,$$

or, equivalently,

$$[K^i_{\partial\Omega} - \sqrt{-\lambda}\tanh(\sqrt{-\lambda})]c_{i1} + [K^i_{\partial\Omega}\tanh(\sqrt{-\lambda}) - \sqrt{-\lambda}]d_{i1} = 0. \tag{17}$$

The eigenvalue λ exists in the equations (16) and (17) but its only (17) that affects it. Since $\lambda < 0$, in (17) we have that

$$K^i_{\partial\Omega}\tanh(\sqrt{-\lambda}) - \sqrt{-\lambda} \neq 0, \quad i = 1,2,3.$$

For $K^i_{\partial\Omega} \leq 0$, we have that

$$K^i_{\partial\Omega} - \sqrt{-\lambda}\tanh(\sqrt{-\lambda}) < 0, \quad i = 1,2,3.$$

Consequently if we use (15) and set $d_{i1} = d$, $\forall i = 1,2,3$, we conclude to $d = c_{i1} = 0$. If this would not hold, c_{i1} would have the opposite sign from d which is not possible from (14).

Hence from the algebraic equations (14)–(17) we have $d_i = c_{ij} = 0$, $i = 1, 2, 3$, $j = 1, 2$. This means that for $K^i_{\partial\Omega} \leq 0$ there do not exist negative eigenvalues.

For $K^i_{\partial\Omega} > 0$ however there exist eigenvalues λ such that

$$K^i_{\partial\Omega} - \sqrt{-\lambda}\tanh(\sqrt{-\lambda}) = 0.$$

In this case $d_{i1} = 0$, $\forall i = 1, 2, 3$ and the algebraic system of equations (14)–(16) is underdetermined. This means that for $K^i_{\partial\Omega} \leq 0$ there exist negative eigenvalues that are the solutions of the above equation, and the eigenfunctions are given by

$$v_{i1} = c_{i1}\cosh(\sqrt{-\lambda}x)e^{\lambda t}, \quad i = 1, 2, 3,$$

and

$$v_{i2} = -d_{i2}[\tanh(\sqrt{-\lambda})\cosh(\sqrt{-\lambda}x)]e^{\lambda t}, \quad i = 1, 2, 3.$$

We consider again the linear system of PDEs (6) and the conditions (7)–(10). Then if $\lambda = 0$ we have:

$$v_{ij} = c_{ij} + d_{ij}x, \quad i = 1, 2, 3, \quad j = 1, 2.$$

From (7) we have

$$v_1(0, t) = v_2(0, t) = v_3(0, t).$$

or, equivalently,

$$c_{11}N_1 + c_{12}T_1 = c_{21}N_2 + c_{22}T_2 = c_{31}N_1 + c_{32}T_3,$$

and consequently (14) holds. From (8) we have that (15) also holds. From (9) at $x = 1$ we have:

$$v_i(1, t)T_i = 0, \quad i = 1, 2, 3,$$

or, equivalently using (11),

$$v_{i2}(1, t) = 0, \quad i = 1, 2, 3,$$

or, equivalently,

$$c_{i2} + d_{i2} = 0, \quad i = 1, 2, 3. \tag{18}$$

From (10) at $x = 1$ we have:

$$[K^i_{\partial\Omega}v_i(1, t) - v_{ix}(1, t)]N_i = 0,$$

or, equivalently using (11),

$$K^i_{\partial\Omega}v_{i1}(1, t) - v_{i1x}(1, t) = 0,$$

or, equivalently,

$$K^i_{\partial\Omega}[c_{i1} + d_{i1}] - d_{i1} = 0,$$

or, equivalently,

$$K^i_{\partial\Omega}c_{i1} + [K^i_{\partial\Omega} - 1]d_{i1} = 0. \tag{19}$$

If $K^i_{\partial\Omega} \neq 0, 1$, then from the algebraic equations (14), (15), (18), (19) we have $d_{ij} = c_{ij} = 0$, $i = 1, 2, 3$, $j = 1, 2$.

For $K^i_{\partial\Omega} = 0$ however, from (19) we have that $d_{i1} = 0$, $\forall i = 1, 2, 3$. Then the algebraic system of equations (14)–(16) is underdetermined. This means that for $K^i_{\partial\Omega} = 0$ the eigenvalue $\lambda = 0$ exists. The eigenfunctions will then be given by

$$v_{i1} = c_{i1}, \quad i = 1, 2, 3,$$

and

$$v_{i2} = -d_{i2}(1 + x), \quad i = 1, 2, 3.$$

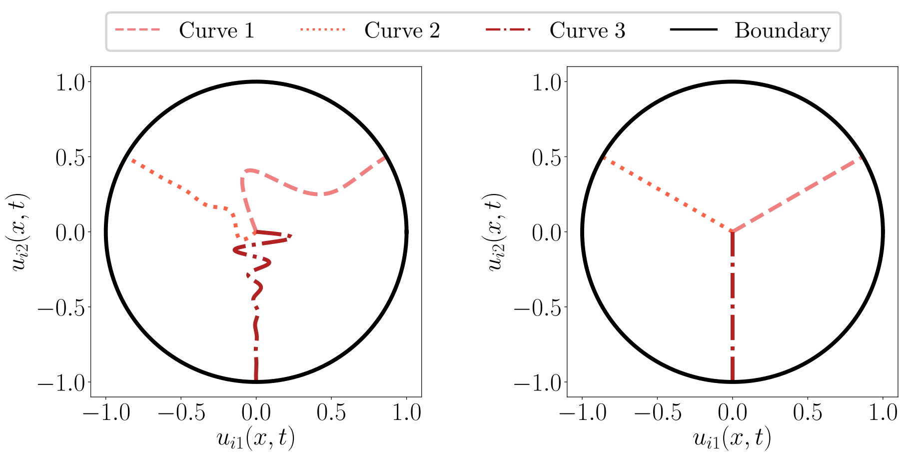
For $K^i_{\partial\Omega} = 1$ in (19) we have that $c_{i1} = 0$, $i = 1, 2, 3$. Then from the algebraic equations (14), (18), we get $d_{i2} = c_{ij} = 0$, $i = 1, 2, 3$, $j = 1, 2$. Consequently for $K^i_{\partial\Omega} = 1$ the eigenvalue $\lambda = 0$ exists. If we use (15) and set $d_{i1} = d$, $i = 1, 2, 3$, the eigenfunctions will be given by:

$$v_{i1} = d, \quad i = 1, 2, 3,$$

and

$$v_{i2} = 0, \quad i = 1, 2, 3.$$

The proof is completed.



(a) Transient condition.

(b) Steady state.

Figure 1: Illustration of simple network of three curves with circular boundary.

Numerical Example

For the sake of illustration, we consider the simple example of a dynamical network of three curves with boundary the unit circle, as shown in Fig. 1. In this case, we have that $K^i_{\partial\Omega} = 1 > 0$ and, thus, from Theorem 4.1, there exist negative eigenvalues of the corresponding BVP and the steady state of the network is unstable. Indeed, calculation of the eigenvalues from the solution of:

$$1 - \sqrt{-\lambda}\tanh(\sqrt{-\lambda}) = 0, \quad i = 1, 2, 3,$$

gives that $\lambda = -1.4392 < 0$, with multiplicity equal to three.

Conclusions

In this article we studied a network of curves that are in motion, meet at a junction, and are bounded. We first focused on the construction of the model which is a BVP that consists of the system of non-linear PDEs (1) and the conditions (2)–(5). We then linearized the PDEs, reformulated the conditions and for the new BVP that appeared, the system of linear PDEs (6) and the conditions (7)–(10), we studied the stability of the steady states.

We concluded that stability depends on the sign of the curvature of the boundary of the domain at the points that the boundary meets each curve. We also provided the negative eigenvalues and their eigenfunctions, as well as the eigenfunctions of the zero eigenvalue. As a further extension of this article, we aim to apply the results to hexagonal networks, and study the stability of this type of networks having also in mind applications in soap bubbles, honeycomb, grain growth. Additionally we aim to use the techniques and properties proved in the main results of this article to update the geometrical properties that are used in electrical circuit theory and elasticity, plasticity problems in material science. For all this there is already some ongoing research.

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Conflict of Interest

This work does not have any conflicts of interest.

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