Instantaneous Power Theory Revisited with Classical Mechanics

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***Abstract*—The paper revisits the concepts of instantaneous active and reactive powers and provides a novel definition for basic circuit elements based on quantities utilized in classical mechanics, such as absolute and relative velocity, momentum density, angular momentum and apparent forces. The discussion leverages from recent publications by the authors that interpret the voltage and current as *velocities* in generalized Lagrangian coordinates. The main result of the paper is a general and compact expression for the instantaneous active and reactive power of inductances, capacitances and resistances as a multivector proportional to the generalized kinetic energy and the geometric frequency multivector. Several numerical examples considering stationary and transient sinusoidal and non-sinusoidal conditions are discussed in the case study.**

***Index Terms*—Reactive power, differential geometry, Frenet frame, non-inertial frame of reference, angular momentum, apparent forces.**

Notation

Unless otherwise indicated, scalars are represented in Italic, e.g., x, X; vectors in lower case bold, e.g., $\boldsymbol{x} = (x_1, x_2, x_3)$; pseudovectors in upper case bold, e.g., $\boldsymbol{X}$; and multivectors in upper case Italic with a hat, e.g., $\hat{X}$. Vectors and pseudovectors have order 3.

Scalars:

C	capacitance
ℓ	momentum density
I	momentum of inertia
$\mathcal{L}$	Lagrangian
L	inductance
m	mass
p	instantaneous active power
s	arc length of a curve
R	resistance
t	time
T	kinetic energy
U	potential energy
θ	voltage phase angle
α_r, α_u	radial components of 2^{nd} time derivatives
ϱ_r, ϱ_u	radial components of 1^{st} time derivatives
ω_{d}^2	Lancret curvature ($= \omega_\kappa^2 + \omega_\tau^2$)
ω_κ	azimuthal frequency (related to curvature)
ω_τ	torsional frequency (related to torsion)

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Vectors:

$\mathbf{0}$	null vector
$\boldsymbol{f}$	force
$\boldsymbol{\imath}$	current
$\boldsymbol{n}$	normal vector before normalization
$\mathbf{p}$	momentum
$\boldsymbol{q}$	electric charge
$\boldsymbol{r}$	position
$\boldsymbol{u}$	velocity
$\boldsymbol{v}$	voltage
$\boldsymbol{\alpha}$	relative acceleration
$\boldsymbol{\nu}$	relative velocity
$\boldsymbol{\nu}_{\mathrm{d}}$	projection of $\boldsymbol{\xi}$ onto the Darboux vector
$\boldsymbol{\xi}$	relative position
$\boldsymbol{\pi}$	relative momentum
$\boldsymbol{\varphi}$	magnetic flux

Pseudovectors:

$\boldsymbol{L}$	angular momentum
$\boldsymbol{N}$	torque
$\boldsymbol{R}$	rotatum residual
$\boldsymbol{Q}$	instantaneous reactive power
$\boldsymbol{\Lambda}$	relative angular momentum
$\boldsymbol{\omega}_{\mathrm{d}}$	Darboux vector
$\boldsymbol{\omega}_r$	orbital angular velocity vector
$\boldsymbol{\omega}_u$	binormal vector before normalization
$\boldsymbol{\omega}_\xi$	relative rotation vector

Multivectors:

$\hat{E}$	energy multivector
$\hat{L}$	momentum multivector
$\hat{R}$	power residual multivector
$\hat{S}$	instantaneous power multivector
$\hat{W}$	power multivector

Operators:

$\nabla_{\boldsymbol{a}}$	directional derivative
$\mathbf{F}$	Frenet-frame transformation matrix
$\boldsymbol{I}_{\boldsymbol{a}}$	inertia operator
$\boldsymbol{\Omega}_{\mathrm{d}}$	rotation matrix ($= \boldsymbol{\omega}_{\mathrm{d}} \times$)
$\hat{\Omega}_{\boldsymbol{a}}$	geometric frequency multivector

Coordinates:

$\mathbf{e}_i$	i-th vector of an orthonormal basis
$\mathbf{B}$	binormal vector of the Frenet frame
$\mathbf{N}$	normal vector of the Frenet frame
$\mathbf{T}$	tangent vector of the Frenet frame

I. Introduction

A. Motivation

The definition of instantaneous power, in particular, of the reactive one, has been object of study since the early years of electrification. It still periodically raises intense discussions on public fora of the power system community. The aim of most definitions and theories proposed so far has been pragmatic, that is, to properly calculate losses and the power factor. This is, for example, the main goal of international standards, e.g., [1]. Despite all these discussions, there remain unresolved issues [2]. For example, the definitions in [1] require stationary conditions and the knowledge of the period T of voltages and currents. But the period is not defined during transients and, even in stationary conditions, it is rarely exactly the nominal one. Another underlying issue is that, in most works, instantaneous power is formulated through equations without an attempt to define its physical meaning. This is particularly true for reactive power which is often defined as what is not, e.g., *non-active* or *powerless* [3], rather than what it is. This work addresses these issues and proposes a physical meaning of the instantaneous power based on classical mechanics, variational principles and differential geometry.

B. Literature Review

The literature and discussions on active and reactive power are as old as AC power systems themselves. The topic has always been controversial. Already in [4], Steinmetz expresses his concern on the definition of P and Q in terms of the product of the voltage and current phasor magnitudes. The issue is that, as the instantaneous power has the double of the frequency of the voltage and current, the product of the phasors, which are defined at the fundamental frequency, cannot be a consistent quantity. This impasse was later solved by defining the active and reactive powers as averages over a period, but as mentioned above, this requires assuming stationary conditions and to know *a priori* the period.

To solve these inconsistencies, developments on the theory of active and reactive power recognized that the key of the problem was the identification of two components of the current, one parallel and one in quadrature with the voltage. This led to the concepts of *active* and *non-active* current [3], *power* and *powerless* current [5]; α and β current components [6]; and *instantaneous active* and *instantaneous reactive* currents [7]. All these approaches, starting from Fryze in 1932 [3], are substantially attempts to define a set of coordinates. However, it is more recent the proposal of considering voltage and currents as generalized vectors [8]. This has put the instantaneous power theory developed in [9] in the context of vector algebra and paved the way to the application of geometric algebra [10]–[20]. In this work, we use as starting point the geometric approach of these works and combine it with two related but, so far, not considered together, concepts.

The first concept is the recent interpretation by the first author of "frequency" as a geometric quantity [21]. This definition requires the utilization of differential geometry, which, differently from geometric algebra, studies the kinematics of a curve (or a surface) in three or higher dimensions. The ultimate goal of differential geometry is to determine the *invariants* of a curve. The authors have shown that certain invariants such as arc length, curvature, and torsion are intrinsically related to the frequency and transient behavior of voltages and currents in three- and multi-phase electric circuits [22]–[24]. Another feature of differential geometry is that it considers instantaneous quantities, and, thus it is fully consistent with the instantaneous power theory. More importantly for the developments of this work, differential geometry allows defining a set of intrinsic coordinates, called Frenet frame, that identifies the parallel and quadrature components of voltage and current as they evolve in time. In this work, we exploit this property to define active and reactive components of the instantaneous power.

The second concept is the Lagrangian approach to describe the dynamics of a physical system in terms of generalized coordinates, namely, positions and momenta. That is, not only we assume that voltages and currents are vectors with respect to some set of orthogonal coordinates, but that the components of the voltage and current vectors are the generalized coordinates in the sense of Lagrange. Using the Lagrangian equation to describe electric circuits can be found in several textbooks, e.g., [25]. However, for our aims, we need that voltages and currents have to be both velocities and forces, depending on the "domain", electrical or magnetic. This approach, which is not usual, was proposed for the first time in [26] and assumes that the energy stored in capacitances and inductances are both kinetic, and then defines potential energy using fluxes and electric charges as generalized positions. This formulation has never really become popular, mostly because — we believe — the Lagrangian function provides the same information that can be obtained from Kirchhoff laws, only in more involved way. However, the formulation in [26] appears as the missing link between the kinematic approach given by differential geometry and the dynamic (i.e., generalized laws of motion) of an electric circuit.

C. Contributions

This work proposes a "physical" interpretation and definition of instantaneous active and reactive powers for basic circuit elements, i.e., capacitances, inductances and resistances, based on differential geometry and classical mechanics. This definition is compatible with that provided by the well-known instantaneous power theory [7] and, thus also with the FBD-method [3]. Considering voltages and currents as generalized velocities and forces, the proposed approach allows decomposing the instantaneous active power in a variety of terms linked to the time-derivative of generalized kinetic energy and the Lagrangian function. We also show that the instantaneous active and reactive powers can be viewed as components of the second time derivative of generalized momentum density and angular momentum, respectively. The main contribution of the paper is a formula that links the instantaneous power, both active and reactive, with the generalized kinetic energy and the geometric frequency defined by the second author in [21]. The use of differential geometry and, in concrete, of the Frenet frame, allows describing the various terms in which the instantaneous power is decomposed in terms of geometrical invariants, that is, quantities that are independent from the coordinate system utilized to measure them. With this aim, we show the link between the orbital angular velocity that appears in the expression of the angular momentum and of the kinetic energy and invariants obtained from the Frenet frame apparatus. Referring voltage and currents onto the Frenet frame also allows expressing the instantaneous power components in terms of generalized apparent accelerations.

D. Organization

The remainder of the paper is organized as follows. Section II states the correspondence between mechanical, electrical and magnetic domains. Electrical and magnetic quantities are treated as generalized positions, velocities, momenta and forces. Based on this framework, Section III focuses on the mechanical domain, recalls the definitions of momentum density and angular momentum, and defines the instantaneous active and reactive powers in terms of the kinetic energy and the second time derivative of the angular momentum and momentum density. Section IV discusses the link between the orbital angular velocity that appears in the proposed expression of the instantaneous power with the curvature and torsion given by the Frenet apparatus. The extension to electrical and magnetic domains is also provided in this section. Section V illustrates the formulas derived in Section III through a series of examples. The examples are aimed at showing relevant special cases, including stationary balanced and unbalanced, as well as sinusoidal and non-sinusoidal systems. Section VI draws conclusions and outlines future work.

II. Generalized Quantities

The definition and interpretation of instantaneous active and reactive power given in this work are based on an analogy with quantities that appear in classical mechanics, such as angular momentum and torque. Before presenting the proposed interpretation, we first provide the concept of generalized quantities, which is typical of the Lagrangian formulation, but we extend it to electrical and magnetic quantities.

In the Lagrangian approach, generalized quantities are positions and momenta. With this aim, we use as starting point the correspondences assumed in [26] and, more recently, in [27], where the generalized quantities are defined as shown in Table I. In the table, conventional mechanical quantities are as position $\boldsymbol{r}$, velocity $\boldsymbol{u}$, momentum $\mathbf{p}$, and force $\boldsymbol{f}$. The electrical and magnetic quantities are the electric charge $\boldsymbol{q}$ and current $\boldsymbol{\imath}$, the flux $\boldsymbol{\varphi}$, and the voltage $\boldsymbol{v}$. All these quantities are considered as *vectors* in some opportunely defined coordinates. The definition of the coordinates is a critical aspect of the generalized approach and of this work in particular and is discussed in detail in following sections. Finally, the physical properties of mechanical, magnetic and electrical media are represented by mass m, inductance L and capacitance C, which, for simplicity, are assumed constant.

The duality of electric and magnetic equations is apparent in this formulation. The constitutive equations of condensers and inductors can be all viewed as generalized Newton's second law of motion, where C and L take the meaning of generalized masses. Moreover, considering that the conventional mechanical kinetic and potential energies are defined as:

TABLE I: Generalized positions, velocities, momenta and forces

Domain	Position	Velocity	Momentum	Force
Mechanical	$\boldsymbol{r}$	$\boldsymbol{u} = \boldsymbol{r}'$	$\mathbf{p} = m\boldsymbol{u}$	$\boldsymbol{f} = \mathbf{p}'$
Electrical	$\boldsymbol{\varphi}$	$\boldsymbol{v} = \boldsymbol{\varphi}'$	$\boldsymbol{q} = C\boldsymbol{v}$	$\boldsymbol{\imath} = \boldsymbol{q}'$
Magnetic	$\boldsymbol{q}$	$\boldsymbol{\imath} = \boldsymbol{q}'$	$\boldsymbol{\varphi} = L\boldsymbol{\imath}$	$\boldsymbol{v} = \boldsymbol{\varphi}'$

$$T = \frac{1}{2} m |\boldsymbol{u}|^2 \,, \qquad U = -\boldsymbol{f} \cdot \boldsymbol{r} \,, \tag{1}$$

the electrical kinetic and potential energies are:

$$T_e = \frac{1}{2} C |\boldsymbol{v}|^2 \,, \qquad U_e = -\boldsymbol{\imath} \cdot \boldsymbol{\varphi} \,, \tag{2}$$

and the magnetic kinetic and potential energies are:

$$T_m = \frac{1}{2} L |\boldsymbol{\imath}|^2 \,, \qquad U_m = -\boldsymbol{v} \cdot \boldsymbol{q} \,. \tag{3}$$

It is worth noting that this is not the usual notation utilized in most works and textbooks. More often, the correspondence between mechanical and electrical systems is done by assuming only the charge as position and the current as velocity, and defining T_e as a potential energy (see, for example, Chapter 13 of [25]). The advantages of the notation adopted in [26] will be evident in the remainder of this paper.

III. Instantaneous Power

This section defines the instantaneous active and reactive powers based on classical mechanics and differential geometry. We have chosen to present the mathematical developments in Sections III-A to III-D using mechanical quantities, since these are more intuitive and can be visualized better than electrical and magnetic ones. The extension to electrical and magnetic quantities is then carried out in Section III-E based on the correspondences given in Table I. Moreover, before introducing the proposed definition of instantaneous active and reactive powers, we need to introduce another set of mechanical quantities, namely momentum, angular momentum and momentum density of a point particle with mass m. We use the latter two quantities to define a momentum multivector. We then define the first and second time derivatives of the angular momentum and momentum density, as well as the corresponding energy and power multivectors. Finally, we show that the power multivector includes the instantaneous active and reactive powers and that these are functions of the kinetic energy and the curvature of the particle's trajectory.

A. Momentum Multivector

Figure 1 illustrates the relationships between position vector $\boldsymbol{r}$, momentum $\mathbf{p} = m\boldsymbol{u} = m\boldsymbol{r}'$ and angular momentum $\boldsymbol{L}$ for a point particle of mass m moving along a space curve $\boldsymbol{\gamma}(t)$. The orbital angular velocity vector $\boldsymbol{\omega}_r$ with respect to the origin O is given by:

$$\boldsymbol{\omega}_r = \frac{\boldsymbol{r} \times \boldsymbol{u}}{|\boldsymbol{r}|^2} \,. \tag{4}$$

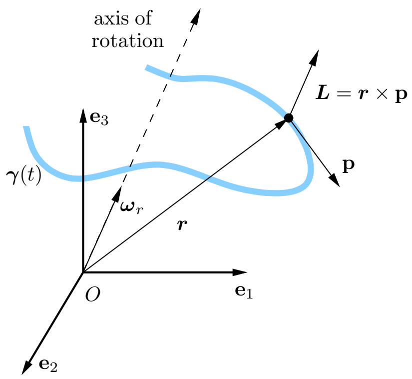


Fig. 1: Representation of momentum $\mathbf{p}$ and angular momentum $\boldsymbol{L}$ for a point particle of mass m and position vector $\boldsymbol{r}$. The particle rotates with angular velocity vector $\boldsymbol{\omega}_r$ with respect to the origin O. By construction, $\boldsymbol{L} \parallel \boldsymbol{\omega_r}$.

We can write the angular momentum as follows:

$$\begin{aligned} \boldsymbol{L} = \boldsymbol{r} \times \mathbf{p} &= m\,\boldsymbol{r} \times \boldsymbol{u} = m|\boldsymbol{r}|^2\,\boldsymbol{\omega}_r \\ &= I\,\boldsymbol{\omega}_r\,, \end{aligned} \tag{5}$$

where $I = m\,|\boldsymbol{r}|^2$ is the moment of inertia with respect to the origin O of the coordinates where the vector $\boldsymbol{r}$ is defined.

The *momentum density* ℓ is defined as the scalar quantity:

$$\ell = \boldsymbol{r} \cdot \mathbf{p} = m\,\boldsymbol{r} \cdot \boldsymbol{u} = \frac{1}{2} I'\,, \tag{6}$$

and, introducing the *radial speed* as:

$$\varrho_r = \frac{\boldsymbol{r} \cdot \boldsymbol{u}}{|\boldsymbol{r}|^2} = \frac{|\boldsymbol{r}|'}{|\boldsymbol{r}|}\,, \tag{7}$$

the momentum density can be written as:

$$\ell = I\,\varrho_r\,. \tag{8}$$

Putting together the momentum density and the angular momentum, one can define the following multivector:

$$\hat{L} = \ell + \boldsymbol{L} = I(\varrho_r + \boldsymbol{\omega}_r)\,,$$

or, equivalently:

$$\hat{L} = I\,\hat{\varOmega}_{\boldsymbol{r}}\,, \tag{9}$$

where $\hat{\varOmega}_{\boldsymbol{r}}$ is the *geometric frequency* operator proposed in [21]. In the remainder of this work, we utilize the geometric frequency as an operator that, when applied to a smooth vector $\boldsymbol{a}(t)$, returns the following expression:

$$\hat{\varOmega}_{\boldsymbol{a}} = \varrho_a + \boldsymbol{\omega}_a = \frac{\boldsymbol{a} \cdot \boldsymbol{a}'}{|\boldsymbol{a}|^2} + \frac{\boldsymbol{a} \times \boldsymbol{a}'}{|\boldsymbol{a}|^2}\,. \tag{10}$$

B. Energy Multivector

The time derivative of the angular momentum gives:

$$\boldsymbol{L}' = \boldsymbol{u} \times \mathbf{p} + \boldsymbol{r} \times \mathbf{p}' = \boldsymbol{r} \times \boldsymbol{f} = \boldsymbol{N}\,, \tag{11}$$

where $\boldsymbol{u} \times \mathbf{p} = \mathbf{0}$ because the momentum is parallel to the velocity and $\boldsymbol{N}$ is the resultant torque applied to the particle.

The time derivative of the momentum density satisfies the identity:

$$\begin{aligned} \ell' = \frac{1}{2} I'' &= \boldsymbol{u} \cdot \mathbf{p} + \boldsymbol{r} \cdot \mathbf{p}' \\ &= m|\boldsymbol{u}|^2 + \boldsymbol{r} \cdot \boldsymbol{f} \\ &= 2T - U\,, \end{aligned} \tag{12}$$

where

$$\begin{aligned} T &= \frac{1}{2} m|\boldsymbol{u}|^2 = \frac{1}{2} I(\varrho_r^2 + |\boldsymbol{\omega}_r|^2) \\ &= \frac{1}{2} I|\hat{\varOmega}_{\boldsymbol{r}}|^2 = \frac{1}{2}\frac{|\hat{L}|^2}{I}\,, \end{aligned} \tag{13}$$

is the kinetic energy of the rotating mass; and U is the potential energy from (1). This form appears also from the expressions of the potential energy given in (2) and (3) for the electrical and magnetic domains, respectively.

We define the *energy multivector* $\hat{E}$ as the time derivative $\hat{L}'$ of the momentum multivector $\hat{L}$:

$$\begin{aligned} \hat{E} = \hat{L}' &= \ell' + \boldsymbol{L}' \\ &= 2T - U + \boldsymbol{N}\,. \end{aligned} \tag{14}$$

All terms that appear in (14) have the units of an energy, although the torque $\boldsymbol{N}$ is conventionally expressed in Nm.[1]

C. Power Multivector

Differentiating ℓ' and $\boldsymbol{L}'$ with respect to time, one obtains:

$$\begin{aligned} \ell'' = 2T' - U' &= \boldsymbol{u}' \cdot \mathbf{p} + \boldsymbol{u} \cdot \mathbf{p}' + \boldsymbol{u} \cdot \mathbf{p}' + \boldsymbol{r} \cdot \mathbf{p}'' \\ &= m\,(3\,\boldsymbol{u} \cdot \boldsymbol{u}' + \boldsymbol{r} \cdot \boldsymbol{u}'')\,, \end{aligned} \tag{15}$$

where we have used the identity $\mathbf{p} = m\boldsymbol{u}$, and

$$\begin{aligned} \boldsymbol{L}'' = \boldsymbol{N}' &= \boldsymbol{u} \times \mathbf{p}' + \boldsymbol{r} \times \mathbf{p}'' \\ &= m(\boldsymbol{u} \times \boldsymbol{u}' + \boldsymbol{r} \times \boldsymbol{u}'')\,. \end{aligned} \tag{16}$$

Both ℓ'' and $\boldsymbol{L}''$ have the dimensions of power.[2] The latter term, i.e. the time derivative of the torque, is sometimes called *rotatum*. We define as *instantaneous active power* the term:

$$\boxed{p \equiv \boldsymbol{u} \cdot \boldsymbol{f}} \tag{17}$$

where we have used Newton's 2nd law $\boldsymbol{f} = \mathbf{p}'$. Note that:

$$p = \boldsymbol{u} \cdot m\,\boldsymbol{u}' = \frac{d}{dt}\left(\frac{1}{2} m\,\boldsymbol{u} \cdot \boldsymbol{u}\right) = T'\,, \tag{18}$$

and hence:

$$\ell'' = p + T' - U' = p + \mathcal{L}'\,, \tag{19}$$

where $\mathcal{L}$ is the Lagrangian of the system.

Substituting in (18) the expression of $\boldsymbol{u}'$ given in (93) (see Appendices B and A), namely:

$$\boldsymbol{u}' = \varrho_u\,\boldsymbol{u} + \boldsymbol{\omega}_u \times \boldsymbol{u} = \hat{\varOmega}_{\boldsymbol{u}} \otimes \boldsymbol{u}\,,$$

the instantaneous active power and, hence, the time derivative of the kinetic energy, can be written as:

$$p = T' = m\,\boldsymbol{u} \cdot \boldsymbol{u}' = m\,|\boldsymbol{u}|^2 \varrho_u = 2\,T\,\varrho_u\,. \tag{20}$$

Also, since $\boldsymbol{u} = \hat{\varOmega}_{\boldsymbol{r}} \otimes \boldsymbol{r}$, the acceleration can be written as:

$$\boldsymbol{u}' = \hat{\varOmega}_{\boldsymbol{u}} \otimes (\hat{\varOmega}_{\boldsymbol{r}} \otimes \boldsymbol{r})\,,$$

[1] Alternatively, one can interpret the torque as a *rotating energy*, not to be confused with the kinetic energy, which is due to the rotation of the particle and is a scalar.

[2] We note that as early as in [4], Steinmetz noted that the reactive power is linked to the torque of a rotating machine. However, Steinmetz links the torque itself, not its time derivative to the reactive power and does not consider generalized quantities nor the angular momentum and momentum density.

or, equivalently:

$$\begin{aligned}\boldsymbol{u}' &= \frac{d}{dt}(\varrho_r\,\boldsymbol{r} + \boldsymbol{\omega}_r \times \boldsymbol{r}) \\ &= \varrho_r'\,\boldsymbol{r} + \varrho_r\,\boldsymbol{u} + \boldsymbol{\omega}_r' \times \boldsymbol{r} + \boldsymbol{\omega}_r \times \boldsymbol{u} \\ &= \alpha_r \boldsymbol{r} + 2\boldsymbol{\omega}_r \times \boldsymbol{u}_\parallel + \boldsymbol{\omega}_r' \times \boldsymbol{r} + \boldsymbol{\omega}_r \times (\boldsymbol{\omega}_r \times \boldsymbol{r})\,, \\ &= \beta_r \boldsymbol{u}_\parallel + 2\boldsymbol{\omega}_r \times \boldsymbol{u}_\parallel + \boldsymbol{\omega}_r' \times \boldsymbol{r} + \boldsymbol{\omega}_r \times (\boldsymbol{\omega}_r \times \boldsymbol{r})\,,\end{aligned} \tag{21}$$

where we have defined:

$$\begin{aligned}\boldsymbol{u}_\parallel &= \varrho_r \boldsymbol{r}\,, \\ \alpha_r &= \frac{|\boldsymbol{r}|''}{|\boldsymbol{r}|} = \varrho_r' + \varrho_r^2\,, \\ \beta_r &= \frac{\alpha_r}{\varrho_r} = \frac{|\boldsymbol{r}|''}{|\boldsymbol{r}|'} = 2\varrho_r + \frac{\boldsymbol{r}\cdot\boldsymbol{u}'}{\boldsymbol{r}\cdot\boldsymbol{u}}\,,\end{aligned}$$

and we have utilized the following identities:

$$\begin{aligned}\varrho_r' \boldsymbol{r} &= (\alpha_r - \varrho_r^2)\boldsymbol{r} = (\beta_r \varrho_r - \varrho_r^2)\boldsymbol{r}\,, \\ \varrho_r\,\boldsymbol{u} &= \varrho_r^2\,\boldsymbol{r} + \varrho_r\,\boldsymbol{\omega}_r \times \boldsymbol{r} = \varrho_r^2\,\boldsymbol{r} + \boldsymbol{\omega}_r \times (\varrho_r\,\boldsymbol{r})\,, \\ \boldsymbol{\omega}_r \times \boldsymbol{u} &= \boldsymbol{\omega}_r \times (\varrho_r\,\boldsymbol{r} + \boldsymbol{\omega}_r \times \boldsymbol{r})\,.\end{aligned}$$

Equation (21) represents the well-known Coriolis' theorem for the accelerations in relative non-inertial coordinates, which, we recall, are the vectors $(\mathrm{T}, \mathrm{N}, \mathrm{B})$ of the Frenet frame. The right-hand side terms of the third line of (21) are, respectively, the relative time derivative of the velocity in Frenet-frame coordinates; the Coriolis' acceleration; the Euler's acceleration; and the centrifugal acceleration. Separating and identifying these terms is relevant to characterize the various components of the active and reactive power. This point is comprehensively discussed in the examples given in Section V.

The time derivative of the Lagrangian $\mathcal{L}'$ includes the term:

$$\boldsymbol{r}\cdot\mathbf{p}'' = \boldsymbol{r}\cdot\boldsymbol{f}' = m\,\boldsymbol{r}\cdot\boldsymbol{u}''\,, \tag{22}$$

which depends on the second time derivative of the velocity. Using (92), (93), the latter can be expressed in $(\mathrm{T}, \mathrm{N}, \mathrm{B})$ as:

$$\begin{aligned}\boldsymbol{u}'' = &(\alpha_u - \omega_\kappa^2)|\boldsymbol{u}|\,\mathrm{T} + \\ &(\omega_\kappa' + 2\omega_\kappa \varrho_u)|\boldsymbol{u}|\,\mathrm{N} + \\ &(\omega_\tau \omega_\kappa)|\boldsymbol{u}|\,\mathrm{B}\,,\end{aligned} \tag{23}$$

where:

$$\alpha_u = \frac{|\boldsymbol{u}|''}{|\boldsymbol{u}|} = \varrho_u' + \varrho_u^2\,. \tag{24}$$

Note also that the terms $2\omega_\kappa \varrho_u |\boldsymbol{u}|\,\mathrm{N}$, $-\omega_\kappa^2 |\boldsymbol{u}|\,\mathrm{T} + \omega_\tau\omega_\kappa |\boldsymbol{u}|\,\mathrm{B}$ and $\omega_\kappa' |\boldsymbol{u}|\,\mathrm{N}$ result from the Coriolis, centrifugal and Euler, respectively, accelerations cross-multiplied by the velocity vector $\boldsymbol{u} = |\boldsymbol{u}|\mathrm{T}$. Then, (22) can be rewritten as:

$$\begin{aligned}\boldsymbol{r}\cdot\boldsymbol{f}' = &\,m\,(\alpha_u - \omega_\kappa^2)\,\boldsymbol{r}\cdot\boldsymbol{u} + \\ &m\,(\omega_\kappa' + 2\omega_\kappa \varrho_u)\,|\boldsymbol{u}|\,\boldsymbol{r}\cdot\mathrm{N} + \\ &m\,(\omega_\tau \omega_\kappa)|\boldsymbol{u}|\,\boldsymbol{r}\cdot\mathrm{B}\,.\end{aligned} \tag{25}$$

Given that $\mathrm{N} = -\mathrm{T} \times \mathrm{B}$ and $\mathrm{B} = \mathrm{T} \times \mathrm{N}$ and applying the following identity for the scalar triple product (97) — see in Appendix C —, one has:

$$\begin{aligned}m\,|\boldsymbol{u}|\,\boldsymbol{r}\cdot\mathrm{N} &= -m\,\mathrm{B}\cdot(\boldsymbol{r}\times\boldsymbol{u}) = -\boldsymbol{L}\cdot\mathrm{B} = -I\,\boldsymbol{\omega}_r\cdot\mathrm{B}\,, \\ m\,|\boldsymbol{u}|\,\boldsymbol{r}\cdot\mathrm{B} &= m\,\mathrm{N}\cdot(\boldsymbol{r}\times\boldsymbol{u}) = \boldsymbol{L}\cdot\mathrm{N} = I\,\boldsymbol{\omega}_r\cdot\mathrm{N}\,.\end{aligned}$$

Finally, substituting the expression of $m\,\boldsymbol{r}\cdot\boldsymbol{u}$ from (8), equation (25) can be written as:

$$\begin{aligned}\boldsymbol{r}\cdot\boldsymbol{f}' = &\,I\,(\alpha_u - \omega_\kappa^2)\,\varrho_r + \\ &I\,(\omega_\tau\omega_\kappa)\,\boldsymbol{\omega}_r\cdot\mathrm{N} - \\ &I\,(\omega_\kappa' + 2\omega_\kappa\varrho_u)\,\boldsymbol{\omega}_r\cdot\mathrm{B}\,.\end{aligned} \tag{26}$$

We define as *instantaneous reactive power* pseudo-vector the cross product of the velocity by the force, namely:

$$\boxed{\boldsymbol{Q} \equiv \boldsymbol{u}\times\boldsymbol{f}} \tag{27}$$

Expression (27) appears in a variety of works based on geometric algebra, e.g., [10], [12], [15], [16], [28], although all these works use voltage and current instead of velocity and force. Most importantly, in these references, the reactive power is defined without providing the physical rationale of its expression. Equation (16) shows that $\boldsymbol{Q}$ is in effect a component of the time derivative of the torque. More specifically, recalling that $\boldsymbol{f} = m\boldsymbol{u}'$ and the expression of the vector $\boldsymbol{\omega}_u$ given in (92) in Appendix B, (27) can be equivalently written as:

$$\begin{aligned}\boldsymbol{Q} &= m\,\boldsymbol{u}\times\boldsymbol{u}' = m\,|\boldsymbol{u}|^2\boldsymbol{\omega}_u \\ &= 2T\,\boldsymbol{\omega}_u = 2T\,\omega_\kappa\,\mathrm{B} \\ &= I\,|\boldsymbol{\omega}_r|^2\,\omega_\kappa\,\mathrm{B}\,,\end{aligned} \tag{28}$$

that is, *the instantaneous reactive power vector lays along the direction of the binormal and its magnitude is given by twice the kinetic energy by the azimuthal frequency*.

Finally, we define the cross product of the position and the *yank*, i.e., the time derivative of the force, as the *rotatum residual* pseudo-vector, namely:

$$\boldsymbol{R} \equiv \boldsymbol{r}\times\boldsymbol{f}'\,. \tag{29}$$

Observing that $\boldsymbol{f}' = m\boldsymbol{u}''$ and substituting the expression of $\boldsymbol{u}''$ from (23) into (29) and recalling that $\boldsymbol{u} = u\,\mathrm{T}$, we obtain:

$$\begin{aligned}\boldsymbol{R} = &\,m\,(\alpha_u - \omega_\kappa^2)\,|\boldsymbol{u}|\,\boldsymbol{r}\times\mathrm{T} + \\ &m\,(\omega_\kappa' + 2\omega_\kappa\varrho_u)\,|\boldsymbol{u}|\,\boldsymbol{r}\times\mathrm{N} + \\ &m\,(\omega_\tau\omega_\kappa)\,|\boldsymbol{u}|\,\boldsymbol{r}\times\mathrm{B}\,.\end{aligned} \tag{30}$$

Then, substituting $\mathrm{N} = -\mathrm{T}\times\mathrm{B}$ and $\mathrm{B} = \mathrm{T}\times\mathrm{N}$, and using the Lagrange and Jacobi identities (98) and (99), respectively, given in Appendix C, one has:

$$\begin{aligned}|\boldsymbol{u}|\,\boldsymbol{r}\times\mathrm{T} &= \boldsymbol{r}\times\boldsymbol{u} \\ &= |\boldsymbol{r}|^2\boldsymbol{\omega}_r\,, \\ |\boldsymbol{u}|\,\boldsymbol{r}\times\mathrm{N} &= -\boldsymbol{r}\times(\boldsymbol{u}\times\mathrm{B}) \\ &= \boldsymbol{u}\times(\mathrm{B}\times\boldsymbol{r}) + \mathrm{B}\times(\boldsymbol{r}\times\boldsymbol{u}) \\ &= (\boldsymbol{r}\cdot\boldsymbol{u})\,\mathrm{B} - |\boldsymbol{r}|^2\boldsymbol{\omega}_r\times\mathrm{B} \\ &= |\boldsymbol{r}|^2(\varrho_r\,\mathrm{B} - \boldsymbol{\omega}_r\times\mathrm{B})\,, \\ |\boldsymbol{u}|\,\boldsymbol{r}\times\mathrm{B} &= \boldsymbol{r}\times(\boldsymbol{u}\times\mathrm{N}) \\ &= -\boldsymbol{u}\times(\mathrm{N}\times\boldsymbol{r}) - \mathrm{N}\times(\boldsymbol{r}\times\boldsymbol{u}) \\ &= -(\boldsymbol{r}\cdot\boldsymbol{u})\,\mathrm{N} + |\boldsymbol{r}|^2\boldsymbol{\omega}_r\times\mathrm{N} \\ &= -|\boldsymbol{r}|^2(\varrho_r\,\mathrm{N} - \boldsymbol{\omega}_r\times\mathrm{N})\,,\end{aligned}$$

where we have used the fact that $\boldsymbol{u}$ is perpendicular to N and B by construction of the Frenet frame. Substituting the expressions above into (30) yields:

$$\begin{aligned}\boldsymbol{R} = &\, I\,(\alpha_u - \omega_\kappa^2)\,\boldsymbol{\omega}_r - \\ &\, I\,(\omega_\kappa\omega_\tau)\,(\varrho_r\,\mathbf{N} - \boldsymbol{\omega}_r \times \mathbf{N}) + \\ &\, I\,(\omega_\kappa' + 2\omega_\kappa\varrho_u)\,(\varrho_r\,\mathbf{B} - \boldsymbol{\omega}_r \times \mathbf{B})\,.\end{aligned} \tag{31}$$

We define the *instantaneous power multivector* as follows:

$$\begin{aligned}\hat{W} \equiv \hat{E}' &= \ell'' + \boldsymbol{L}'' \\ &= 2T' - U' + \boldsymbol{N}' \\ &= p + \boldsymbol{Q} + \mathcal{L}' + \boldsymbol{R} \\ &= \hat{S} + \hat{R}\,,\end{aligned} \tag{32}$$

which is composed of the sum of two multivectors, $\hat{S} = p + \boldsymbol{Q}$ and $\hat{R} = \mathcal{L}' + \boldsymbol{R}$. Specifically, from (20) and (28), we obtain:

$$\hat{S} = 2T\,(\varrho_u + \boldsymbol{\omega}_u)\,,$$

or, equivalently:

$$\boxed{\boxed{\hat{S} = 2T\,\hat{\Omega}_{\boldsymbol{u}}}} \tag{33}$$

or, equivalently, using (13):

$$\hat{S} = I\,|\hat{\Omega}_{\boldsymbol{r}}|^2\,\hat{\Omega}_{\boldsymbol{u}} = \frac{|\hat{L}|^2}{I}\,\hat{\Omega}_{\boldsymbol{u}}\,, \tag{34}$$

and

$$\begin{aligned}\hat{R} = &\, I\,(\alpha_u - \omega_\kappa^2)\,\hat{\Omega}_{\boldsymbol{r}} - \\ &\, I\,(\omega_\kappa\omega_\tau)\,\mathbf{N} \otimes \hat{\Omega}_{\boldsymbol{r}} + \\ &\, I\,(\omega_\kappa' + 2\omega_\kappa\varrho_u)\,\mathbf{B} \otimes \hat{\Omega}_{\boldsymbol{r}}\,.\end{aligned} \tag{35}$$

Equation (33) is the main result of this paper: it represents the sought link among the internal stored energy, the geometric frequency and the instantaneous power. Also the multivector $\hat{R}$ is defined for the first time in this work and is composed of the rate of change of the Lagrangian and the rotatum residual.

D. Stationary Conditions

Based on the discussion above, we observe that $\hat{E} = 0$ is the condition for which the momentum density and the angular momentum of a system are conserved. These conditions are not only consistent with the notion that, in stationary conditions, active and reactive powers are conserved, but provides also a physical reason for such a conservation in terms of the well-known conservation laws of classical mechanics.

If $\hat{E} = \text{const.}$, then the system is in stationary conditions and the energy balance is kept with constant $2T - U$ and $\boldsymbol{N}$. Clearly, T and U do not need to be constant *per se*. From $\hat{E} = \text{const.}$, it also descends that $\hat{W} = 0$. This, of course, does not mean that the power, active or reactive, is null. Rather, it implies $\hat{S} = -\hat{R}$ and, hence, the following identities:

$$\begin{aligned}p &= -\mathcal{L}'\,, \\ \boldsymbol{Q} &= -\boldsymbol{R}\,.\end{aligned} \tag{36}$$

For example, in a single-phase AC circuit in stationary conditions, the instantaneous active power of any of its elements is not constant, as it is well-known, however, $2T' - U'$ must be.

E. Electrical and Magnetic Domains

We can now express the generalized momentum, energy and power multivectors in terms of electrical and magnetic quantities. With this aim, we use the correspondences among position, velocity, momenta and forces given in Table I.

For the electrical domain, we obtain:

$$\begin{aligned}\hat{L}_e &= \boldsymbol{\varphi} \cdot \boldsymbol{q} + \boldsymbol{\varphi} \times \boldsymbol{q} = I_e\,\hat{\Omega}_{\boldsymbol{\varphi}}\,, \\ \hat{E}_e &= 2T_e - U_e + \boldsymbol{N}_e\,, \\ \hat{W}_e &= 2T_e\,\hat{\Omega}_{\boldsymbol{v}} + \hat{R}_e = p_e + \mathcal{L}_e' + \boldsymbol{Q}_e + \boldsymbol{R}_e\,,\end{aligned} \tag{37}$$

where $I_e = C\,|\boldsymbol{\varphi}|^2$ and T_e, U_e are defined in (2), and:

$$\begin{aligned}\boldsymbol{N}_e &= \boldsymbol{\varphi} \times \boldsymbol{\imath}\,, \qquad & p_e &= \boldsymbol{v} \cdot \boldsymbol{\imath}\,, \\ \boldsymbol{Q}_e &= \boldsymbol{v} \times \boldsymbol{\imath}\,, \qquad & \boldsymbol{R}_e &= \boldsymbol{\varphi} \times \boldsymbol{\imath}'\,.\end{aligned} \tag{38}$$

The force, in the electrical domain corresponding to the current, can be expressed in terms of the Coriolis theorem — see (21):

$$\begin{aligned}\boldsymbol{\imath} = C\boldsymbol{v}' = &\, C\,\beta_\varphi \boldsymbol{v}_\parallel + 2C\,\boldsymbol{\omega}_\varphi \times \boldsymbol{v}_\parallel + \\ &\, C\,\boldsymbol{\omega}_\varphi' \times \boldsymbol{\varphi} + C\,\boldsymbol{\omega}_\varphi \times (\boldsymbol{\omega}_\varphi \times \boldsymbol{\varphi})\,,\end{aligned} \tag{39}$$

where the right-hand side terms, express, in order, the relative, Coriolis, Euler, and centrifugal components of the current.

For the magnetic domain, we obtain:

$$\begin{aligned}\hat{L}_m &= \boldsymbol{q} \cdot \boldsymbol{\varphi} + \boldsymbol{q} \times \boldsymbol{\varphi} = I_m\,\hat{\Omega}_{\boldsymbol{q}}\,, \\ \hat{E}_m &= 2T_m - U_m + \boldsymbol{N}_m\,, \\ \hat{W}_m &= 2T_m\,\hat{\Omega}_{\boldsymbol{\imath}} + \hat{R}_m = p_m + \mathcal{L}_m' + \boldsymbol{Q}_m + \boldsymbol{R}_m\,,\end{aligned} \tag{40}$$

where $I_m = L\,|\boldsymbol{q}|^2$ and T_m, U_m are defined in (3) and:

$$\begin{aligned}\boldsymbol{N}_m &= \boldsymbol{q} \times \boldsymbol{v}\,, \qquad & p_m &= \boldsymbol{\imath} \cdot \boldsymbol{v}\,, \\ \boldsymbol{Q}_m &= \boldsymbol{\imath} \times \boldsymbol{v}\,, \qquad & \boldsymbol{R}_m &= \boldsymbol{q} \times \boldsymbol{v}'\,.\end{aligned} \tag{41}$$

The force in the magnetic domain corresponds to the voltage and can be also expressed in terms of the Coriolis theorem:

$$\begin{aligned}\boldsymbol{v} = L\,\boldsymbol{\imath}' = &\, L\,\beta_q \boldsymbol{\imath}_\parallel + 2L\,\boldsymbol{\omega}_q \times \boldsymbol{\imath}_\parallel + \\ &\, L\,\boldsymbol{\omega}_q' \times \boldsymbol{q} + L\,\boldsymbol{\omega}_q \times (\boldsymbol{\omega}_q \times \boldsymbol{q})\,,\end{aligned} \tag{42}$$

where the right-hand side terms, express, in order, the relative, Coriolis, Euler, and centrifugal components of the voltage.

It is relevant to note that from the expressions of the reactive power in the electrical domain, one obtains:

$$\boldsymbol{Q}_e = \boldsymbol{v} \times \boldsymbol{\imath} = -\boldsymbol{\imath} \times \boldsymbol{v}\,, \tag{43}$$

which has thus opposite sign with respect to the expression of $\boldsymbol{Q}_m$ for the magnetic domain, as it is the usual convention. Interestingly, with the proposed approach, the opposite sign is obtained as a natural consequence of the different meaning of $\boldsymbol{v}$ and $\boldsymbol{\imath}$ in the electrical and magnetic domains, that is, velocity/force and force/velocity, respectively.

F. Inclusion of Losses

In the formulation provided in [26], losses are either voltage-controlled conductances or current-controlled resistances, depending on whether these elements are in parallel with a capacitance or in series with an inductance. Similarly, we can account for the cases illustrated in Fig. 2.

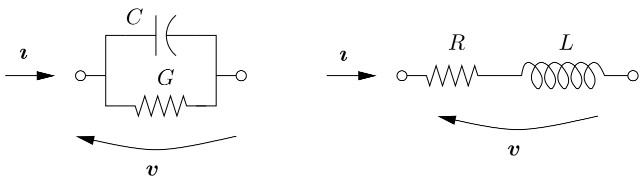


Fig. 2: Left: $\boldsymbol{v}$-controlled conductance; Right: $\boldsymbol{\imath}$-controlled resistance.

The definitions of p and $\boldsymbol{Q}$ given in (17) and (27), respectively, still apply as long as the generalized force is given by $\boldsymbol{f}_e = \boldsymbol{\imath} - G\,\boldsymbol{v}$ for the capacitance and $\boldsymbol{f}_m = \boldsymbol{v} - R\,\boldsymbol{\imath}$ for the inductance. The terms $G\,\boldsymbol{v}$ and $R\,\boldsymbol{\imath}$, which are generalized non-conservative forces, only affect the active power, as expected, as $\boldsymbol{v} \times G\,\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{\imath} \times R\,\boldsymbol{\imath} = \boldsymbol{0}$.

We note also that nonlinear circuit elements, in the same vein as discussed in [26], can be captured as long as C, L and R are represented with adequate nonlinear functions. The analysis of nonlinear components is, however, beyond the scope of this paper and will be considered in future work.

G. Moment of Inertia as an Operator

We have so far considered only point particles for which the moment of inertia is a scalar, namely $I = m|\boldsymbol{r}|^2$. We can easily generalize this notation assuming a system composed of a set of discrete masses.[3] All definitions and developments presented remain the same, except for the fact that the moment of inertia becomes a symmetrical tensor. Thus, instead of one quantity, one has to define the six elements of the tensor. The inertia tensor can be also conveniently thought as an operator that returns a pseudovector, as follows:

$$\boldsymbol{I}_{\boldsymbol{\omega}_r} = \sum_h m_h\,\boldsymbol{r} \times (\boldsymbol{\omega}_r \times \boldsymbol{r})\,. \tag{44}$$

The formulation in terms of inertia operator $\boldsymbol{I}_{\boldsymbol{\omega}_r}$ is useful in the study of unbalanced three-phase AC systems. Using (44), the angular momentum and kinetic energy can be written as:

$$\boldsymbol{L} = \boldsymbol{I}_{\boldsymbol{\omega}_r}\,, \qquad T = \frac{1}{2}\boldsymbol{\omega}_r \cdot \boldsymbol{I}_{\boldsymbol{\omega}_r}\,. \tag{45}$$

Similarly, one can define all other quantities previously described in the paper. In particular, we observe that (33), that is, the proposed expression of the instantaneous power, holds and, if the kinetic energy is expressed in terms of the inertia operator, it can be rewritten as:

$$\boxed{\hat{S} = (\boldsymbol{\omega}_r \cdot \boldsymbol{I}_{\boldsymbol{\omega}_r})\,\hat{\Omega}_{\boldsymbol{u}}} \tag{46}$$

IV. Link to Geometric Invariants

In this section, we derive the analytical expressions of $\boldsymbol{\omega}_r$ and T in terms of the geometric invariants, such as curvature and torsion provided by the Frenet apparatus. With this aim, we express the instantaneous power multivector discussed in the previous section using the relative coordinates defined by the moving Frenet frame.

[3] The most general definition of the inertia operator is based on a continuum but, for simplicity, we only consider the discrete case in this work.

Let us consider the space curve $\boldsymbol{\gamma}$ represented in Fig. 3. This curve has coordinates $\boldsymbol{r}$ with respect to the origin O of fixed Cartesian axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The change of coordinates on the moving Frenet frame is given by:

$$\boldsymbol{\xi} = \mathbf{F}\,\boldsymbol{r}\,, \tag{47}$$

where

$$\mathbf{F} = [\mathrm{\mathbf{T}}, \mathrm{\mathbf{N}}, \mathrm{\mathbf{B}}]^{\intercal}\,, \tag{48}$$

i.e., the rows of $\mathbf{F}$ are the tangent, normal and binormal vectors of the Frenet frame (see Appendix B). Then, $\boldsymbol{r}$ is rewritten as:

$$\boldsymbol{r} = \mathbf{F}^{-1}\,\boldsymbol{\xi} = \mathbf{F}^{\intercal}\,\boldsymbol{\xi}\,. \tag{49}$$

Figure 3 illustrates the coordinate transformation. The position of point P along the curve $\boldsymbol{\gamma}(t)$ is represented by the vector $\boldsymbol{r}$, which describes the distance $\overline{OP}$ in the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ coordinates. The basis $(\mathrm{\mathbf{T}}, \mathrm{\mathbf{N}}, \mathrm{\mathbf{B}})$ of the Frenet frame which is determined based on the curve $\boldsymbol{\gamma}(t)$ can be also utilized as coordinates to define the segment $\overline{OP}$.

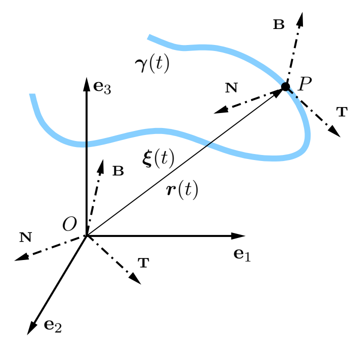


Fig. 3: Coordinates of a space curve $\boldsymbol{\gamma}(t)$ in absolute and Frenet coordinates.

We are interested in expressing the velocity $\boldsymbol{u}$ and acceleration $\boldsymbol{u}'$ as functions of $\boldsymbol{\xi}$, the velocity $\boldsymbol{\xi}'$ and acceleration $\boldsymbol{\xi}''$. Since $\boldsymbol{\xi}$ is defined on a non-inertial frame, namely the moving Frenet frame, its motion is relative. One has thus to take into account in the motion also the coordinates $(\mathrm{\mathbf{T}}, \mathrm{\mathbf{N}}, \mathrm{\mathbf{B}})$. First note that, by construction, $\mathbf{F}$ is orthonormal, that is, its transpose is equal to its inverse ($\mathbf{F}^{\intercal} = \mathbf{F}^{-1}$) and its determinant is $\det(\mathbf{F}) = 1$. Then, from Cartan's theory on moving frames [29], the following result holds:

$$\boldsymbol{\Omega}_{\mathrm{d}} = \mathbf{F}\,(\mathbf{F}')^{\intercal} = -\mathbf{F}'\mathbf{F}^{\intercal} = \begin{bmatrix} 0 & -\omega_\kappa & 0 \\ \omega_\kappa & 0 & -\omega_\tau \\ 0 & \omega_\tau & 0 \end{bmatrix}, \tag{50}$$

and ω_τ and ω_κ are the torsional and azimuthal frequencies [22], [23]. Equation (50) can be written as:

$$\mathbf{F}' = -\boldsymbol{\Omega}_{\mathrm{d}}\,\mathbf{F}\,, \tag{51}$$

which is the matrix form of the well-known Frenet-Serret equations (91) given in Appendix B. The skew-symmetric matrix $\boldsymbol{\Omega}_{\mathrm{d}}$ can be written as:

$$\boldsymbol{\Omega}_{\mathrm{d}} = \boldsymbol{\omega}_{\mathrm{d}} \times\,, \tag{52}$$

where $\boldsymbol{\omega}_{\mathrm{d}}$ is the *Darboux vector*:

$$\boldsymbol{\omega}_{\mathrm{d}} = \omega_\tau\,\mathrm{\mathbf{T}} + \omega_\kappa\,\mathrm{\mathbf{B}}\,. \tag{53}$$

We can now express the velocity and acceleration using the coordinates $(\mathrm{T},\mathrm{N},\mathrm{B})$. The velocity can be written as:

$$\boldsymbol{u} = \mathbf{F}^{\intercal}\boldsymbol{\xi}' + (\mathbf{F}')^{\intercal}\boldsymbol{\xi}\,, \tag{54}$$

and, multiplying by $\mathbf{F}$ both sides:

$$\boldsymbol{\nu} = \mathbf{F}\,\boldsymbol{u} = \mathbf{F}\mathbf{F}^{\intercal}\boldsymbol{\xi}' + \mathbf{F}(\mathbf{F}')^{\intercal}\boldsymbol{\xi}\,, \tag{55}$$

or, equivalently,

$$\boldsymbol{\nu} = \boldsymbol{\xi}' + \boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi}\,. \tag{56}$$

The acceleration is given by:

$$\begin{aligned}\boldsymbol{\alpha} &= \mathbf{F}\mathbf{x}'' \\ &= \boldsymbol{\xi}'' + 2\boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi}' + \boldsymbol{\omega}_{\mathrm{d}}' \times \boldsymbol{\xi} + \boldsymbol{\omega}_{\mathrm{d}} \times (\boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi})\,.\end{aligned} \tag{57}$$

In fact, from (55), one has:

$$\begin{aligned}\boldsymbol{\nu}' &= \mathbf{F}'\boldsymbol{u} + \mathbf{F}\boldsymbol{u}' \\ &= \mathbf{F}'\mathbf{F}^{\intercal}(\boldsymbol{\xi}' + \boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi}) + \boldsymbol{\alpha} \\ &= -\boldsymbol{\omega}_{\mathrm{d}} \times (\boldsymbol{\xi}' + \boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi}) + \boldsymbol{\alpha} \\ &= -\boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi}' - \boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi} + \boldsymbol{\alpha}\,,\end{aligned} \tag{58}$$

where we have defined $\boldsymbol{\alpha} = \boldsymbol{F}\boldsymbol{u}'$ and, from (56):

$$\boldsymbol{\nu}' = \boldsymbol{\xi}'' + \boldsymbol{\omega}_{\mathrm{d}}' \times \boldsymbol{\xi} + \boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi}\,. \tag{59}$$

Moreover, using the Lagrange formula for the triple cross product (98), equation (57) can be equivalently written as:

$$\boldsymbol{\alpha} = \boldsymbol{\xi}'' + 2\boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi}' + \boldsymbol{\omega}_{\mathrm{d}}' \times \boldsymbol{\xi} + (\boldsymbol{\xi} \cdot \boldsymbol{\omega}_{\mathrm{d}})\,\boldsymbol{\omega}_{\mathrm{d}} - \omega_{\mathrm{d}}^2\,\boldsymbol{\xi}\,, \tag{60}$$

where $\omega_{\mathrm{d}}^2 = \omega_\kappa^2 + \omega_\tau^2$ is the magnitude of the Darboux vector and is referred to as *Lancret curvature* for unit-speed curves [30]. Similarly to (21), equation (60) is yet another version of the Coriolis theorem. This version features a rotation vector $\boldsymbol{\omega}_{\mathrm{d}}$ that defines the intrinsic rotation of the curve $\boldsymbol{r}$ and is defined by the curvature and torsion of the curve itself.

A. Momentum Density

In this section, we provide the expressions of the momentum density and angular momentum using the relative coordinates on the moving Frenet frame. This operation allows decomposing the energy and instantaneous power into terms that have specific meaning, similar to the terms into which the acceleration is decomposed in (60). The goal is to be able to identify unequivocally and based on a precise physical interpretation the active and reactive powers in various operating conditions.

We begin with the momentum density ℓ, the magnitude of which, as any scalar, is unaltered in relative coordinates. In fact, defining the momentum in $(\mathrm{T},\mathrm{N},\mathrm{B})$ coordinates:

$$\boldsymbol{\pi} = \mathbf{F}\mathbf{p} = m\,\boldsymbol{\nu}\,, \tag{61}$$

the momentum density becomes:

$$\begin{aligned}\ell &= \boldsymbol{r} \cdot \mathbf{p} = m\,\boldsymbol{r} \cdot \boldsymbol{u} \\ &= (\mathbf{F}^{\intercal}\boldsymbol{\xi}) \cdot (\mathbf{F}^{\intercal}\boldsymbol{\pi}) = (\mathbf{F}^{\intercal}\boldsymbol{\xi})^{\intercal}(\mathbf{F}^{\intercal}\boldsymbol{\pi}) \\ &= \boldsymbol{\xi}^{\intercal}\mathbf{F}\mathbf{F}^{\intercal}\boldsymbol{\pi} = \boldsymbol{\xi}^{\intercal}\boldsymbol{\pi} = \boldsymbol{\xi} \cdot \boldsymbol{\pi} \\ &= m\,\boldsymbol{\xi} \cdot \boldsymbol{\nu} = m\,\boldsymbol{\xi} \cdot \boldsymbol{\xi}'\,,\end{aligned} \tag{62}$$

where the last expression has been obtained using (56) and $\boldsymbol{\xi} \perp (\boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi})$. Then, note that:

$$\varrho_r = \frac{\boldsymbol{r} \cdot \boldsymbol{u}}{\boldsymbol{r} \cdot \boldsymbol{r}} = \frac{\boldsymbol{\xi} \cdot \boldsymbol{\xi}'}{\boldsymbol{\xi} \cdot \boldsymbol{\xi}}\,, \tag{63}$$

and that:

$$|\boldsymbol{r}|^2 = \boldsymbol{r}^{\intercal}\boldsymbol{r} = (\mathbf{F}^{\intercal}\boldsymbol{\xi})^{\intercal}(\mathbf{F}^{\intercal}\boldsymbol{\xi}) = \boldsymbol{\xi}^{\intercal}\mathbf{F}\mathbf{F}^{\intercal}\boldsymbol{\xi} = \boldsymbol{\xi}^{\intercal}\boldsymbol{\xi} = |\boldsymbol{\xi}|^2\,,$$

where we have used that $\mathbf{F}$ is orthonormal and, hence, $\mathbf{F}\mathbf{F}^{\intercal}$ is the identity matrix.[4] The moment of inertia is:

$$I = m\,\boldsymbol{r} \cdot \boldsymbol{r} = m\,|\boldsymbol{r}|^2 = m\,|\boldsymbol{\xi}|^2\,. \tag{64}$$

From the latter three equations descends that the momentum density is $\ell = I\,\varrho_r$, which is the same expression as (8). In conclusion, since the Frenet frame imposes only a rotation of the coordinates, all scalar quantities have the same value in absolute and relative coordinates.

B. Angular Momentum

The angular momentum in the Frenet frame becomes:

$$\begin{aligned}\boldsymbol{\Lambda} &= \mathbf{F}\boldsymbol{L} = \mathbf{F}(\boldsymbol{r} \times \mathbf{p}) \\ &= (\mathbf{F}\boldsymbol{r}) \times (\mathbf{F}\mathbf{p}) = m\,(\mathbf{F}\boldsymbol{r}) \times (\mathbf{F}\boldsymbol{u}) \\ &= m\,\boldsymbol{\xi} \times \boldsymbol{\nu} = m\,\boldsymbol{\xi} \times (\boldsymbol{\xi}' + \boldsymbol{\omega}_{\mathrm{d}} \times \boldsymbol{\xi}) \\ &= m\,\boldsymbol{\xi} \times \boldsymbol{\xi}' + m\,|\boldsymbol{\xi}|^2\boldsymbol{\omega}_{\mathrm{d}} - m\,(\boldsymbol{\omega}_{\mathrm{d}} \cdot \boldsymbol{\xi})\,\boldsymbol{\xi}\,.\end{aligned} \tag{65}$$

Note that the identity between first and second line of (65) holds because $\mathbf{F}$ is orthonormal and because of the property of the cross product (95) given in Appendix C. Dividing and multiplying the first and third terms of the last expression in (65) by $|\boldsymbol{\xi}|^2$ leads to:

$$\boldsymbol{\Lambda} = I\,[\boldsymbol{\omega}_\xi + \boldsymbol{\omega}_{\mathrm{d}} - \boldsymbol{\nu}_{\mathrm{d}}]\,, \tag{66}$$

where we have defined the relative angular velocity $\boldsymbol{\omega}_\xi$ as:

$$\boldsymbol{\omega}_\xi = \frac{\boldsymbol{\xi} \times \boldsymbol{\xi}'}{\boldsymbol{\xi} \cdot \boldsymbol{\xi}}\,, \tag{67}$$

and the component of the relative position along the axis of rotation given by the Darboux vector $\boldsymbol{\nu}_{\mathrm{d}}$ as:

$$\boldsymbol{\nu}_{\mathrm{d}} = \frac{\boldsymbol{\omega}_{\mathrm{d}} \cdot \boldsymbol{\xi}}{\boldsymbol{\xi} \cdot \boldsymbol{\xi}}\,\boldsymbol{\xi}\,. \tag{68}$$

From (66), we obtain the expression of the orbital angular velocity in the Frenet frame:

$$\boxed{\mathbf{F}\,\boldsymbol{\omega}_r = \boldsymbol{\omega}_\xi + \boldsymbol{\omega}_{\mathrm{d}} - \boldsymbol{\nu}_{\mathrm{d}}} \tag{69}$$

that shows that the orbital angular velocity is, in general, different from the Darboux vector and coincides with it — except for the coordinate change due to the Frenet frame — if and only if $\boldsymbol{\xi} \parallel \boldsymbol{\xi}'$ and $\boldsymbol{\omega}_{\mathrm{d}} \perp \boldsymbol{\xi}$. Moreover, in the first example discussed in Section V, we show that $\boldsymbol{\omega}_\xi = \boldsymbol{\nu}_{\mathrm{d}} = \mathbf{0}$, and hence $\boldsymbol{\omega}_r = \boldsymbol{\omega}_{\mathrm{d}}$, hold in balanced stationary conditions.

[4] This result is consistent as the coordinate transformation from $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to $(\mathrm{T},\mathrm{N},\mathrm{B})$ consists only in a rotation and, thus, preserves lengths.

C. Kinetic Energy

The kinetic energy can be written as follows:

$$\begin{aligned} T &= \frac{1}{2} I|\boldsymbol{\omega}_r|^2 = \frac{1}{2} m|\boldsymbol{u}|^2 = \frac{1}{2} m|\boldsymbol{\nu}|^2 \\ &= \frac{1}{2} m\{|\boldsymbol{\xi}'|^2 + |\boldsymbol{\xi}|^2\omega_{\rm d}^2 - (\boldsymbol{\omega}_{\rm d}\cdot\boldsymbol{\xi})^2 + 2\boldsymbol{\xi}'\cdot(\boldsymbol{\omega}_{\rm d}\times\boldsymbol{\xi})\} \\ &= \frac{1}{2} I\,(\omega_{\rm d}^2 - |\boldsymbol{\nu}_{\rm d}|^2) + \frac{1}{2} m\,\{|\boldsymbol{\xi}'|^2 + 2\boldsymbol{\omega}_{\rm d}\cdot(\boldsymbol{\xi}\times\boldsymbol{\xi}')\} \\ &= \frac{1}{2} I\,(\omega_{\rm d}^2 - |\boldsymbol{\nu}_{\rm d}|^2 + 2\boldsymbol{\omega}_{\rm d}\cdot\boldsymbol{\omega}_\xi) + \frac{1}{2} m\,|\boldsymbol{\xi}'|^2\,, \end{aligned} \tag{70}$$

where we have utilized (97) and (100) — see Appendix C. Using (67) and (100) again, one has:

$$\boldsymbol{\omega}_\xi\cdot\boldsymbol{\omega}_\xi = \frac{|\boldsymbol{\xi}|^2|\boldsymbol{\xi}'|^2 - (\boldsymbol{\xi}\cdot\boldsymbol{\xi}')^2}{|\boldsymbol{\xi}|^4}\,, \tag{71}$$

hence, (70) can be rewritten as:

$$T = \frac{1}{2} I\,(\omega_{\rm d}^2 - |\boldsymbol{\nu}_{\rm d}|^2 + 2\boldsymbol{\omega}_{\rm d}\cdot\boldsymbol{\omega}_\xi + |\boldsymbol{\omega}_\xi|^2 + \varrho_r^2)\,, \tag{72}$$

or, equivalently:

$$\boxed{T = \frac{1}{2} I(|\boldsymbol{\omega}_\xi + \boldsymbol{\omega}_{\rm d} - \boldsymbol{\nu}_{\rm d}|^2 + \varrho_r^2)} \tag{73}$$

The latter equation is, as expected, equivalent to the expression of the kinetic energy given in (13) and allows expressing the active power in (33) as a function of geometric invariants, i.e., ϱ_r, $\boldsymbol{\omega}_{\rm d}$ and $\hat{\Omega}_{\boldsymbol{u}}$, as well as of the relative angular and linear velocities $\boldsymbol{\omega}_\xi$ and $\boldsymbol{\nu}_{\rm d}$, respectively.

V. Examples

This section illustrates theoretical results of this work through examples on three-phase AC systems. These consider a variety of cases, including balanced, unbalanced, non-sinusoisal (with harmonics) and non-stationary (in transient conditions). In all examples, we assume that voltages/currents are curves in three dimensions on Cartesian coordinates:

$$\begin{aligned} \mathbf{e}_1 &= (1,0,0)\,, \\ \mathbf{e}_2 &= (0,1,0)\,, \\ \mathbf{e}_3 &= (0,0,1)\,. \end{aligned}$$

A. Stationary Balanced Sinusoidal Case

We start by considering a stationary balanced sinusoidal three-phase voltage with constant angular frequency ω_o and constant magnitude. The voltage vector is:

$$\boldsymbol{v} = v_a\,\mathbf{e}_1 + v_b\,\mathbf{e}_2 + v_c\mathbf{e}_3\,, \tag{74}$$

where

$$\begin{aligned} v_a &= V\cos(\omega_o t)\,, \\ v_b &= V\cos(\omega_o t - 2\pi/3)\,, \\ v_c &= V\cos(\omega_o t + 2\pi/3)\,. \end{aligned}$$

This case is analogous to a mechanical system where a point particle follows a circular trajectory (see Fig. 4).

The magnetic flux, i.e., the primitive of $\boldsymbol{v}$, is given by:

$$\boldsymbol{\varphi} = \frac{V}{\omega_o}[\sin(\omega_o t)\,\mathbf{e}_1 + \sin(\omega_o t - \frac{2\pi}{3})\,\mathbf{e}_2 + \sin(\omega_o t + \frac{2\pi}{3})\,\mathbf{e}_3]\,,$$

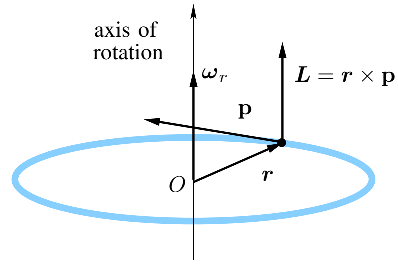


Fig. 4: A point particle that rotates with constant radius and angular speed is analogous to a stationary balanced sinusoidal 3-phase system.

and hence:

$$\boldsymbol{\varphi} = -\frac{v}{\omega_o}\,\mathbf{N}\,, \quad \boldsymbol{v} = v\,\mathbf{T}\,, \quad \boldsymbol{\omega}_v = \omega_o\,\mathbf{B}\,.$$

Then, applying the formulas given in Appendix B, we obtain:

$$v = \sqrt{\tfrac{3}{2}}V\,, \qquad \omega_\kappa = \omega_o\,, \qquad \omega_\tau = 0\,,$$

and $\varrho_\varphi = \varrho_v = 0$, as the magnitudes of the flux and the voltage are constant. Moreover, from (69), and observing that $\boldsymbol{\omega}_\xi = \boldsymbol{\nu}_{\rm d} = \mathbf{0}$, the following identity holds:

$$\boldsymbol{\omega}_v = \boldsymbol{\omega}_\varphi\,, \tag{75}$$

where, in this case, $\mathbf{F}\,\boldsymbol{\omega}_\varphi = \boldsymbol{\omega}_\varphi$, as $\boldsymbol{\omega}_\varphi$ is parallel to $\mathbf{B}$.

Assume a balanced three-phase capacitor (with capacitance C per phase) with voltage vector $\boldsymbol{v}$. Since $\rho_\varphi = 0$, the capacitor is subject to a current vector:

$$\boldsymbol{\imath} = C\,\boldsymbol{v}' = C\,\boldsymbol{\omega}_v\times\boldsymbol{v} = \omega_o v\,\mathbf{N}\,,$$

or, equivalently, using (75) and the identity $\boldsymbol{v} = \boldsymbol{\omega}_\varphi\times\boldsymbol{\varphi}$:

$$\boldsymbol{\imath} = C\,\boldsymbol{\omega}_\varphi\times(\boldsymbol{\omega}_\varphi\times\boldsymbol{\varphi})\,,$$

which, using the correspondences of Table I, is a generalized centrifugal force. In other words, the only non-null component of $\boldsymbol{v}'$ in (21) is the centrifugal acceleration. The relative, Coriolis and Euler components in (39) are zero in this case. Considering a numerical example with $\omega_o = 100\pi$ rad/s, $V = 20$ kV, $C = 10$ μF, the current $\boldsymbol{\imath}$, or, more precisely, the phase currents $\imath_a$, $\imath_b$, $\imath_c$, where:

$$\boldsymbol{\imath} = \imath_a\,\mathbf{e}_1 + \imath_b\,\mathbf{e}_2 + \imath_c\mathbf{e}_3\,,$$

are shown in Fig. 5.

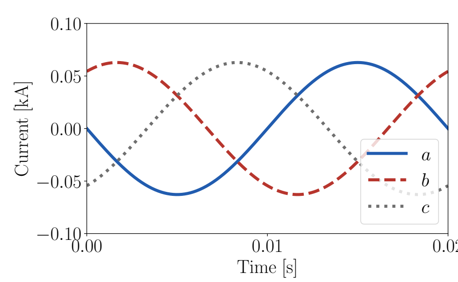


Fig. 5: A 3-phase capacitor with stationary balanced sinusoidal AC voltage is always subject to a current that is centrifugal: $\boldsymbol{\imath} = C\boldsymbol{\omega}_\varphi\times(\boldsymbol{\omega}_\varphi\times\boldsymbol{\varphi})$. The relative, Coriolis, and Euler components are zero.

The generalized momentum is defined as:

$$\boldsymbol{L} = \boldsymbol{\varphi}\times C\boldsymbol{v} = C\frac{v^2}{\omega_o}\,\mathbf{B} = C\frac{v^2}{\omega_o^2}\,\boldsymbol{\omega}_u = I_e\,\boldsymbol{\omega}_u\,,$$

where I_e is the generalized momentum of inertia of the condenser and $\boldsymbol{\omega}_{\rm d} = \boldsymbol{\omega}_u$ because $\omega_\tau = 0$. Then, observing that $\boldsymbol{L}' = \boldsymbol{N} = \boldsymbol{0}$ and hence $\boldsymbol{L}'' = \boldsymbol{N}' = \boldsymbol{0}$, the instantaneous reactive power of the capacitor becomes:

$$\boldsymbol{Q} = -\boldsymbol{\varphi} \times \boldsymbol{\imath}' = \boldsymbol{v} \times \boldsymbol{\imath} = C v^2 \, \boldsymbol{\omega} = I_e \, \omega_o^2 \, \boldsymbol{\omega} = 2 T_e \, \boldsymbol{\omega} \, ,$$

where T_e is the capacitor's stored kinetic energy, as defined in (2). In turn, in balanced stationary conditions, the capacitor's reactive power is due to the centrifugal force $\boldsymbol{\imath} = C\boldsymbol{v}'$.

In the numerical example considered, the instantaneous reactive power pseudovector is:

$$\boldsymbol{Q} = 1.088 \, \mathbf{e}_1 + 1.088 \, \mathbf{e}_2 + 1.088 \, \mathbf{e}_3 \quad \text{MVAr} \, .$$

Finally, we note that, as expected, the instantaneous active power of the condenser is null, in fact:

$$p = \boldsymbol{v} \cdot \boldsymbol{\imath} = \boldsymbol{v} \cdot C\boldsymbol{v}' = 0 \, ,$$

as $\boldsymbol{v} \perp \boldsymbol{v}'$. This is also consistent with the common knowledge that the inner product is $\boldsymbol{u} \cdot \boldsymbol{f} = 0$ for a mass rotating at constant angular velocity along a circle of constant radius, as the centrifugal force is parallel to the position vector $\boldsymbol{r}$ and, hence, perpendicular to the velocity (see Fig. 4).

B. Stationary Unbalanced Sinusoidal Case

Let us now consider that the same three-phase capacitor has an unbalanced voltage. In this example, the voltage vector is given by (74), where:

$$\begin{aligned} v_a &= V_a \cos(\omega_o t) \, , \\ v_b &= V_b \cos(\omega_o t - 2\pi/3) \, , \\ v_c &= V_c \cos(\omega_o t + 1.6\pi/3) \, , \end{aligned}$$

with $V_a = 20$ kV, $V_b = 19$ kV, $V_c = 23$ kV.

The relative, Coriolis, Euler, and centrifugal components of the current applied to the capacitor in this case are illustrated in Fig. 6. We observe that:

$$2C\boldsymbol{\omega}_\varphi \times \boldsymbol{v}_\parallel = -C\boldsymbol{\omega}_\varphi' \times \boldsymbol{\varphi} \, ,$$

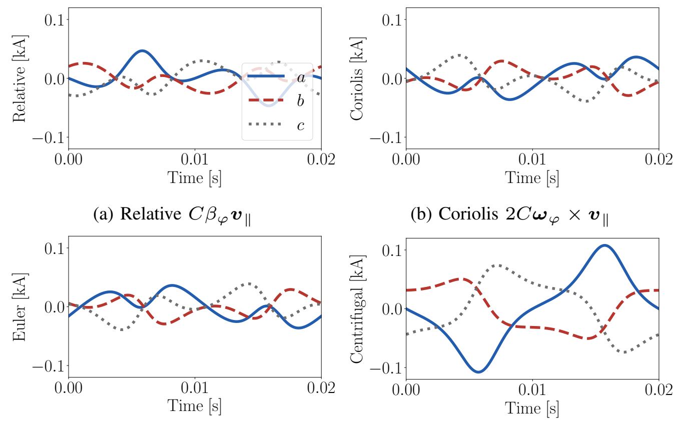
that is, for an unbalanced voltage, the Coriolis and Euler components are opposite and when summed up cancel out each other. Hence, the "force" that the capacitor is subject to is the current vector:

$$\boldsymbol{\imath} = C\beta_\varphi \boldsymbol{v}_\parallel + C\boldsymbol{\omega}_\varphi \times (\boldsymbol{\omega}_\varphi \times \boldsymbol{\varphi}) \, ,$$

that is, the sum of the relative and centrifugal component. The phase currents applied to the capacitor are shown in Fig. 7.

The instantaneous active power and the components of the instantaneous reactive power pseudovector are illustrated in Fig. 8. As expected, the active power has a period that is half of the period of the voltage/current. For the reactive power pseudovector we have that:

$$\boldsymbol{Q} = 0.807 \, \mathbf{e}_1 + 1.437 \, \mathbf{e}_2 + 1.034 \, \mathbf{e}_3 \quad \text{MVAr} \, .$$



(a) Relative $C\beta_\varphi \boldsymbol{v}_\parallel$ (b) Coriolis $2C\boldsymbol{\omega}_\varphi \times \boldsymbol{v}_\parallel$

(c) Euler $C\boldsymbol{\omega}_\varphi' \times \boldsymbol{\varphi}$ (d) Centrifugal $C\boldsymbol{\omega}_\varphi \times (\boldsymbol{\omega}_\varphi \times \boldsymbol{\varphi})$

Fig. 6: 3-phase capacitor with stationary unbalanced sinusoidal AC voltage: Relative, Coriolis, Euler and centrifugal components of current. The Coriolis and Euler components are opposite to each other.

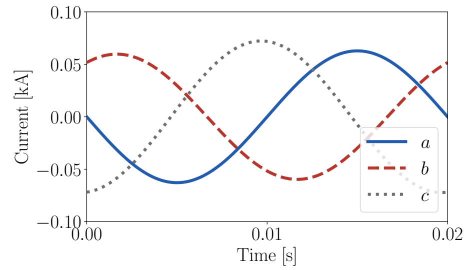
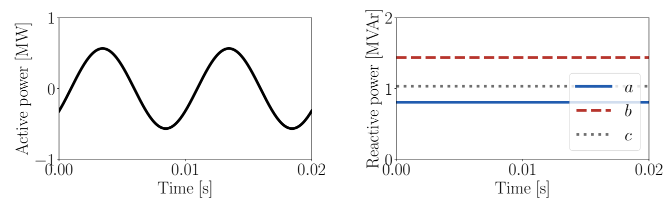


Fig. 7: A 3-phase capacitor with stationary unbalanced sinusoidal AC voltage is subject to the sum of a relative and a centrifugal current: $\boldsymbol{\imath} = C\beta_\varphi \boldsymbol{v}_\parallel + C\boldsymbol{\omega}_\varphi \times (\boldsymbol{\omega}_\varphi \times \boldsymbol{\varphi})$. The Coriolis and Euler components cancel out each other.



(a) Active power p (b) Components of reactive power $\boldsymbol{Q}$

Fig. 8: 3-phase capacitor in stationary unbalanced case: instantaneous powers.

C. Stationary Balanced Non-Sinusoidal Case

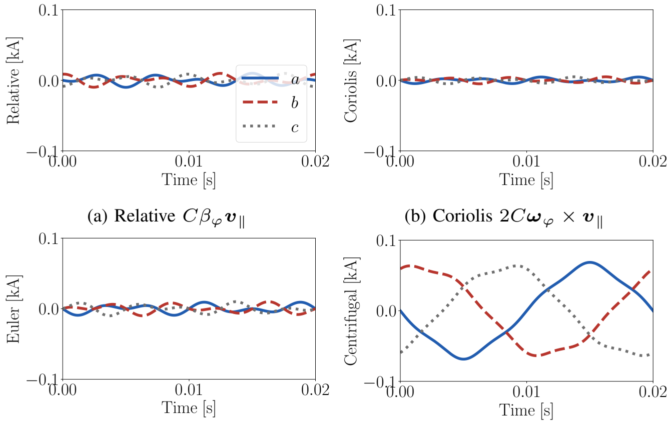
Let us consider again that the capacitor's voltage is stationary but that now includes a fifth harmonic component, as follows:

$$\begin{aligned} v_a &= V[\cos(\omega_o t) + 0.05\cos(5\omega_o t)] \, , \\ v_b &= V[\cos(\omega_o t - 2\pi/3) + 0.05\cos(5\omega_o t - 2\pi/3)] \, , \\ v_c &= V[\cos(\omega_o t + 2\pi/3) + 0.05\cos(5\omega_o t + 2\pi/3)] \, , \end{aligned}$$

where $V = 20$ kV. Moreover, to see the effect of losses, we also consider that a conductance (with value G per phase) is connected in parallel to the capacitor. The current $\boldsymbol{\imath}$ applied to the parallel combination of capacitor and conductance is then:

$$\boldsymbol{\imath} = \boldsymbol{\imath}_C + G\,\boldsymbol{v} \, .$$

The relative, Coriolis, Euler, and centrifugal components of the current $\boldsymbol{\imath}_C$ applied to the capacitor are illustrated in Fig. 9. All components are observed to be non-zero and vary in time. Assuming that $G = 0.01$ S, the current $\boldsymbol{\imath}$ is shown in Fig. 10.



(a) Relative $C\beta_\varphi \boldsymbol{v}_\|$ (b) Coriolis $2C\boldsymbol{\omega}_\varphi \times \boldsymbol{v}_\|$

(c) Euler $C\boldsymbol{\omega}'_\varphi \times \boldsymbol{\varphi}$ (d) Centrifugal $C\boldsymbol{\omega}_\varphi \times (\boldsymbol{\omega}_\varphi \times \boldsymbol{\varphi})$

Fig. 9: 3-phase capacitor with stationary balanced non-sinusoidal AC voltage: Relative, Coriolis, Euler and centrifugal components of applied current $\boldsymbol{\imath}_C$.

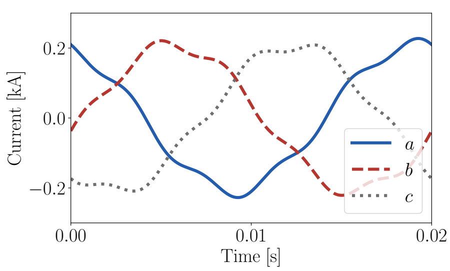


Fig. 10: Stationary balanced non-sinusoidal case: Current $\boldsymbol{\imath}$ applied to the combination of capacitor and conductance.

Let us assume now that a three-phase inductor with $L = 0.02$ H per phase and a three-phase resistance with $R = 8\ \Omega$ per phase, are connected in series with the parallel combination of the capacitor and conductance, as shown in Fig. 11.

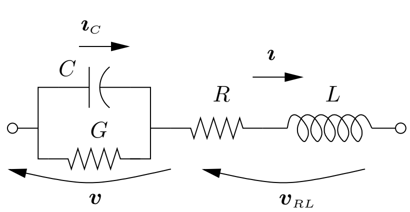


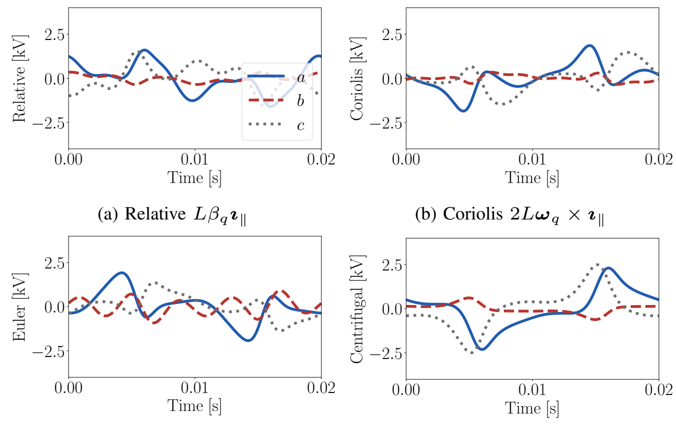
Fig. 11: Resistance (R) and inductor (L) connected in series to the parallel of a capacitor (C) and a conductance (G).

The voltage vector $\boldsymbol{v}_{RL}$ applied to the series of the inductor and resistance is:

$$\boldsymbol{v}_{RL} = \boldsymbol{v}_L + R\,\boldsymbol{\imath} = \boldsymbol{v}_L + R\,(\boldsymbol{\imath}_C + G\boldsymbol{v})\,.$$

The relative, Coriolis, Euler and centrifugal components of the voltage $\boldsymbol{v}_L$ applied to the inductor are plotted in Fig. 12. The figure shows that the high levels of harmonic distortion considered in this example result in all four force components having a more or less equal contribution to the voltage $\boldsymbol{v}_L$. The profile of the voltage $\boldsymbol{v}_{RL}$ is shown in Fig. 13.

Finally, the total instantaneous active and reactive powers



(a) Relative $L\beta_q \boldsymbol{\imath}_\|$ (b) Coriolis $2L\boldsymbol{\omega}_q \times \boldsymbol{\imath}_\|$

(c) Euler $L\boldsymbol{\omega}'_q \times \boldsymbol{q}$ (d) Centrifugal $L\boldsymbol{\omega}_q \times (\boldsymbol{\omega}_q \times \boldsymbol{q})$

Fig. 12: 3-phase inductor with stationary balanced non-sinusoidal AC current $\boldsymbol{\imath}$: Relative, Coriolis, Euler and centrifugal components of applied voltage $\boldsymbol{v}_L$.

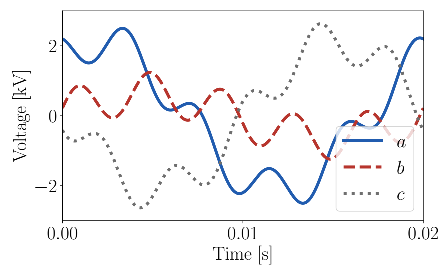
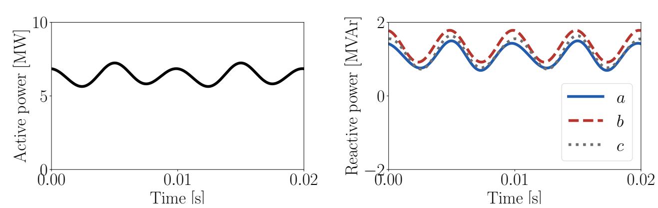


Fig. 13: Stationary balanced non-sinusoidal case: Voltage $\boldsymbol{v}_{RL}$ applied to the series of inductor and resistance.

of the circuit considered in Fig. 11 are:

$$p = G|\boldsymbol{v}|^2 + R|\boldsymbol{\imath}|^2\,,$$
$$\boldsymbol{Q} = \boldsymbol{v} \times \boldsymbol{\imath}_C + \boldsymbol{\imath} \times \boldsymbol{v}_{RL}\,.$$

The profiles of p and of the components of $\boldsymbol{Q}$ for the numerical example examined are presented in Fig. 14.



(a) Active power p (b) Components of reactive power $\boldsymbol{Q}$

Fig. 14: Stationary non-sinusoidal case: instantaneous powers for circuit shown in Fig 11.

D. Non-Stationary Case

In this example, we consider again the same three-phase capacitor studied above, which however is now assumed to be subject to a non-stationary voltage vector, with:

$$v_a = V\cos\theta(t)\,,$$
$$v_b = V\cos(\theta(t) - 2\pi/3)\,,$$
$$v_c = V\cos(\theta(t) + 2\pi/3)\,,$$

where $\theta(t) = \omega_o t - 0.04\omega_o e^{-0.3t}(1.66\cos\frac{\pi}{10}t + 1.59\sin\frac{\pi}{10}t)$. The derivative of $\theta(t)$ is:

$$\theta'(t) = \omega_o - \omega_o 0.04 e^{-0.3t}\sin(0.1\pi t)\,,$$

which emulates the profile of a typical underfrequency response following a negative active power imbalance in a power system (see also Fig. 15).

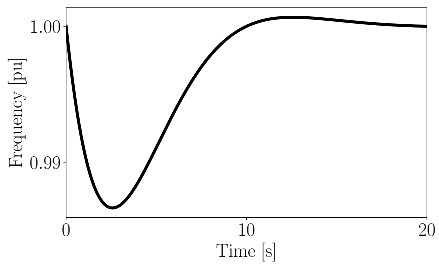
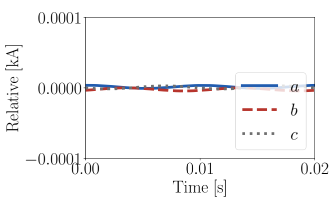
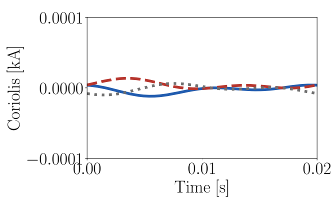


Fig. 15: Non-stationary case: frequency $\theta'(t)$ (in pu).

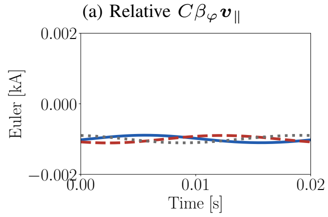
Figure 16 shows the first period of the relative, Coriolis, Euler, and centrifugal components of the current applied to the capacitor. In this case, the centrifugal component largely determines the profile of the capacitor's current, with the contribution of the relative, Coriolis components being non-zero but practically negligible. On the other hand, the Euler component contributes a small yet noticeable negative current.



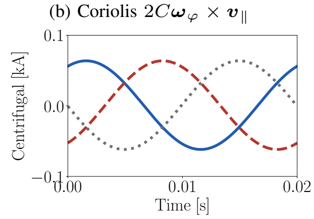
(a) Relative $C\beta_\varphi \boldsymbol{v}_\parallel$



(b) Coriolis $2C\boldsymbol{\omega}_\varphi \times \boldsymbol{v}_\parallel$



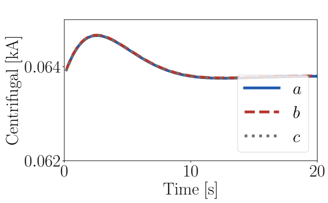
(c) Euler $C\boldsymbol{\omega}'_\varphi \times \boldsymbol{\varphi}$



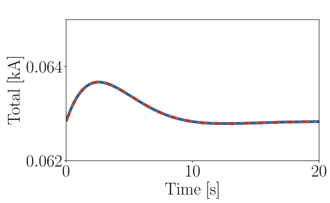
(d) Centrifugal $C\boldsymbol{\omega}_\varphi \times (\boldsymbol{\omega}_\varphi \times \boldsymbol{\varphi})$

Fig. 16: 3-phase capacitor with non-stationary balanced AC voltage: Relative, Coriolis, Euler and centrifugal components of current.

The upper envelopes of the centrifugal and total capacitor current for the duration of the frequency transient are illustrated in Fig. 17. It is seen that the total current has a small offset of about -1 A with respect to the centrifugal, which is again due to the effect of the Euler current.



(a) Centrifugal



(b) Total

Fig. 17: 3-phase capacitor with non-stationary AC voltage: upper envelopes of centrifugal and total current. The offset is due to the effect of a small negative Euler current.

Finally, for the instantaneous active power we have that $p = 0$, as expected. On the other hand, following from the time-varying frequency examined, the components of $\boldsymbol{Q}$ go through a transient, as illustrated in Fig. 18.

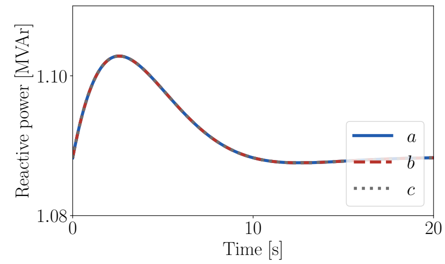


Fig. 18: Non-stationary case: components of capacitor reactive power $\boldsymbol{Q}$.

VI. Conclusions

In this work, we propose a novel definition of instantaneous power. This definition is based on generalized Lagrangian coordinates and well-known concepts borrowed from classical mechanics and basic differential geometry. The proposed expression (33) for the instantaneous power reveals new insights on its physical meaning as it is shown to be the product of kinetic energy and a geometric frequency operator. We also provide a rigorous mathematical framework to this interpretation and show the instantaneous power's dependence on the geometric invariants provided by the Frenet frame apparatus such as curvature, and torsion. The proposed approach also allows decomposing active and reactive powers into various components, each with a precise physical meaning based on apparent forces. For the stationary sinusoidal case, for example, we show that the reactive power is exclusively due to a centrifugal acceleration. Other terms, such as Coriolis and Euler accelerations, appear in unbalanced and non-sinusoidal conditions. The proposed approach has two main advantages compared with currently available definitions of the instantaneous power: (i) it allows a better and, as based on a mechanical analogy, intuitive understanding of what reactive power is; and (ii) it provides a mathematical framework that can help improve the performance and design better controllers for unbalanced and non-sinusoidal systems. This appears particularly important in modern power systems that are dominated by power electronic converters. In future work, we aim at extending the proposed approach to nonlinear circuit components as well as exploiting its features to improve power system dynamic performance.

Appendix A
Algebra of Multivectors

Multivectors extend the concept of complex numbers, quaternions and vectors to a collection of quantities that include scalars, vectors, bivectors (or pseudo-vectors), trivectors, etc. (see, for example, the first chapter in [31]). Since in this work we do not use more than three dimensions, it suffices to consider multivectors composed exclusively of scalars and vectors. Moreover, pseudo-vectors can be operated as vectors through Hodge duality. In turn, the multivectors considered in this paper are equivalent to Hamiltonian quaternions:

$$\hat{X} = \lambda + \boldsymbol{X}\,, \tag{76}$$

where λ is a scalar and $\boldsymbol{X}$ is a vector or pseudo-vector. Then, the *geometric product* is equivalent to the Hamiltonian product of quaternions and can be written using only the inner and cross products, as follows:

$$\begin{aligned}\hat{X} \otimes \hat{Y} &= (\lambda + \boldsymbol{X}) \otimes (\mu + \boldsymbol{Y}) \\ &= (\lambda\,\mu - \boldsymbol{X} \cdot \boldsymbol{Y}) + (\lambda\,\boldsymbol{Y} + \mu\,\boldsymbol{X} + \boldsymbol{X} \times \boldsymbol{Y})\,.\end{aligned} \tag{77}$$

Same rules apply if one of any of the two multivectors includes a vector instead of a pseudo-vector, e.g., $\hat{X} = \lambda + \boldsymbol{x}$. The conjugate of a multivector is defined as:

$$\hat{X}^* = \lambda - \boldsymbol{X}\,. \tag{78}$$

Then, one has:

$$(\hat{X} \otimes \hat{Y})^* = \hat{Y}^* \otimes \hat{X}^*\,, \tag{79}$$

and

$$\hat{X} \otimes \hat{X}^* = |\hat{X}|^2 = \lambda^2 + |\boldsymbol{X}|^2\,. \tag{80}$$

Appendix B
Frenet Frame of Space Curves

Let us consider a space curve $\boldsymbol{x} : [0, +\infty) \to \mathbb{R}^3$ with $\boldsymbol{x} = (x_1, x_2, x_3)$. Where $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$, is the set of parametric equations for the curve. Equivalently:

$$\boldsymbol{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3\,, \tag{81}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis. The arc length s of the curve is defined as:

$$s = \int_0^t \sqrt{\boldsymbol{u}(r) \cdot \boldsymbol{u}(r)}\, dr + s_0\,, \tag{82}$$

from which one obtains the expression:

$$s' = \frac{ds}{dt} = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = |\boldsymbol{u}|\,, \tag{83}$$

where

$$\boldsymbol{u} = \frac{d}{dt}(x_1\,\mathbf{e}_1) + \frac{d}{dt}(x_2\,\mathbf{e}_2) + \frac{d}{dt}(x_3\,\mathbf{e}_3)\,, \tag{84}$$

and $\cdot$ denotes the inner product of two vectors, which in three dimensions, for $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3)$, becomes:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3\,. \tag{85}$$

The arc length s is an invariant of the curve. It is relevant to observe that, according to the chain rule, the derivative of $\boldsymbol{x}$ with respect to s can be written as:

$$\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{ds} = \frac{d\boldsymbol{x}}{dt}\frac{dt}{ds} = \frac{\boldsymbol{u}}{s'} = \frac{\boldsymbol{u}}{|\boldsymbol{u}|}\,. \tag{86}$$

The vector $\dot{\boldsymbol{x}}$ has magnitude 1 and is tangent to the curve $\boldsymbol{x}$.

The Frenet frame is defined by the tangent vector T, the normal vector N and the binormal vector B, as follows:

$$\mathrm{T} = \dot{\boldsymbol{x}}\,, \qquad \mathrm{N} = \frac{\ddot{\boldsymbol{x}}}{|\ddot{\boldsymbol{x}}|}\,, \qquad \mathrm{B} = \mathrm{T} \times \mathrm{N}\,, \tag{87}$$

where $\times$ represents the cross product, which in three dimensions can be written as the determinant of a matrix, as follows:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\,. \tag{88}$$

In [22], the azimuthal (ω_κ) and torsional (ω_τ) frequencies are defined as follows:

$$\omega_\kappa = \frac{|\boldsymbol{u} \times \boldsymbol{u}'|}{|\boldsymbol{u}|^2}\,, \tag{89}$$

$$\omega_\tau = \frac{\boldsymbol{u} \cdot (\boldsymbol{u}' \times \boldsymbol{u}'')}{\omega_\kappa^2\,|\boldsymbol{u}|^3}\,. \tag{90}$$

These frequencies link the Frenet frame with its time derivatives, as follows:

$$\begin{aligned}\mathrm{T}' &= \omega_\kappa \mathrm{N}\,, \\ \mathrm{N}' &= -\omega_\kappa \mathrm{T} + \omega_\tau \mathrm{B}\,, \\ \mathrm{B}' &= -\omega_\tau \mathrm{N}\,,\end{aligned} \tag{91}$$

which are the well-known Frenet-Serret equations.

Finally, (87) can be equivalently expressed as:

$$\begin{aligned}\boldsymbol{u} &= |\boldsymbol{u}|\,\mathrm{T}\,, \\ \boldsymbol{n}_u &= \boldsymbol{u}' - \varrho_u\,\boldsymbol{u} = \sqrt{|\boldsymbol{u}'|^2 - (|\boldsymbol{u}|')^2}\,\mathrm{N}\,, \\ \boldsymbol{\omega}_u &= \boldsymbol{u} \times \boldsymbol{n}_u = \frac{\boldsymbol{u} \times \boldsymbol{u}'}{|\boldsymbol{u}|^2} = \omega_\kappa\,\mathrm{B}\,,\end{aligned} \tag{92}$$

where $\varrho_u = |\boldsymbol{u}|'/|\boldsymbol{u}|$ is the *radial frequency* defined in [22]. Noting that $\boldsymbol{n}_u = \boldsymbol{\omega}_u \times \boldsymbol{u}$, one can obtain the following expression for the first time derivative of $\boldsymbol{u}$ [22]:

$$\boldsymbol{u}' = \varrho_u\,\boldsymbol{u} + \boldsymbol{\omega}_u \times \boldsymbol{u}\,. \tag{93}$$

Appendix C
Vector Product Identities

The following identities for triple and quadruple vector products are relevant for the derivations presented in the work.

- The dot (inner) product is commutative:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}\,. \tag{94}$$

- The cross product is anticommutative:

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}\,. \tag{95}$$

- If $\mathbf{R}$ is a matrix that satisfies $\det(\mathbf{R}) = 1$, then:

$$(\mathbf{R}\,\boldsymbol{a}) \times (\mathbf{R}\,\boldsymbol{b}) = \mathbf{R}\,(\boldsymbol{a} \times \boldsymbol{b})\,. \tag{96}$$

- Scalar triple vector product circular shift:

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a})\,. \tag{97}$$

- Lagrange identity:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}\,. \tag{98}$$

- Jacobi identity:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \mathbf{0}\,. \tag{99}$$

- Scalar quadruple product:

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})\,. \tag{100}$$

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