

Midpoint-Radius Interval-based Method to Deal with Uncertainty in Power Flow Analysis

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Abstract

This paper presents a novel method based on midpoint-radius interval arithmetic to deal with uncertainties in the power flow problem. The proposed technique aims at finding a balance between accuracy and computational efficiency. It relies on an original decoupling of the interval power flow equations into midpoint and radius parts. This representation allows avoiding the factorisation of an interval Jacobian matrix. Moreover, the proposed formulation is combined with an optimisation problem in order to prevent overestimation of the interval solution while preserving uncertainty. The proposed technique proves to be more efficient than existing approaches based on interval and affine arithmetic and as accurate as the conventional Monte Carlo method.

Keywords: Uncertainty modelling, Power flow analysis, Interval arithmetic, Midpoint-radius

1. Introduction

1.1. Motivations

Uncertainty is inherent to any physical systems. This is particularly true for power systems, where uncertainty can have several causes, e.g., imprecise demand forecast, price variability, renewable energy generation, economic growth,

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industry placement, and line aging [1, 2]. Failing to properly account for uncertainties can, in some cases, lead to erroneous estimations or insecure operating conditions. Therefore, a reliable tool to handle several possible scenarios and combinations of scenarios is crucial to provide a clear understanding of the expected behaviour of the grid. This paper focuses on how to properly account for uncertainties in power flow (PF) analysis.

1.2. State of Art

In the literature, uncertainty in PF analysis has been handled mainly by two types of methods: probabilistic and interval-based.

The probabilistic approach relies on solving multiple instances of the PF problem for several (typically randomly generated) possible scenarios, and then aggregate results. The Monte Carlo method is the most common probabilistic approach. The Monte Carlo method is adequate for off-line analysis and is assumed to yield the “correct results”, provided that a sufficiently large amount of samples are considered [3]. However, the computational burden of the Monte Carlo method can be unsuitable for practical purposes, real-time analysis and preventive and/or corrective control actions [4]. For an extensive survey of probabilistic PF methods, the interested reader can refer to [3] and [5].

Interval-based methods rely on using intervals to model the system, according to a *possibility distribution* obtained from experience and historical data. Sentences such as “load between 0.5 and 1 pu” and “generation around 0.9 pu” can be easily translated into intervals. In [6], the interval Newton method is directly applied to a case of PF analysis with 5 buses, assuming small uncertainty in the nodal injected powers. In [1], the authors use *affine* arithmetic to keep track of correlation between inputs, a feature that is absent from traditional interval arithmetic. While interval arithmetic has a low computational burden with respect to probabilistic methods, the major drawback is its tendency to overestimate the intervals of the solution, especially if input parameters are characterised by wide intervals. Wide intervals can make the solution either of little practical interest or useless.

1.3. Contributions

The technique proposed in this paper deals with uncertainty in PF analysis and balances computational burden and accuracy of results. With this aim, we utilise a midpoint-radius representation of intervals and separate the solution
40 of the PF problem into a standard PF problem for the midpoint and an interval problem for the radius. The latter is solved through a carefully designed optimisation problem, where the constraints ensure that the solution reflects all the uncertainty of the input, while the objective function helps to prevent overestimation. The concept of linking the numerical results with uncertainty
45 in the input data is borrowed from backward error analysis [7].

The proposed technique enhances the one presented in [8] where the midpoint and radius problems were solved together, thus leading to a higher computational burden and lower accuracy than the technique proposed in this paper.

The proposed method is tested on the IEEE 57 bus test case system, proving
50 to yield results as accurate as the Monte Carlo method. A study of the computational burden of analogous methods is performed in order to show that the proposed method is competitive with state-of-the-art interval-based techniques and much more efficient than the Monte Carlo method.

1.4. Organisation

55 The remainder of the article is organised as follows. Section 2 reviews probabilistic and interval-based approaches to deal with uncertainty in PF analysis. Section 3 describes the proposed interval method for PF analysis. Section 4 presents a study of the computational complexity of various methods for PF analysis with uncertainty. Section 5 presents a case-study including a comparison
60 in terms of accuracy with the Monte Carlo method. Finally, Section 6 duly draws conclusions and outlines future work.

2. Uncertainty in PF analysis

This section reviews two main approaches used for dealing with uncertainty in PF analysis. These are the probabilistic approach and the interval-based

65 approach.

2.1. Probabilistic approach

The probabilistic approach models uncertainty as random variables with a certain probability distribution. The above relies on statistical data to obtain the probability distribution of the inputs. A *probabilistic PF* model is defined
70 by extending the PF equations to random variables. The equations are solved to obtain the distribution of the unknown variables. The solution method can be numerical, such as Monte Carlo method, or analytic. In the following, we focus exclusively on the Monte Carlo method as it is the most commonly used and is considered to be the most accurate approach [3].

75 *Monte Carlo method.* This is a numerical method to approximate the distribution of an unknown random variable. The method relies on the law of large numbers and sampling and consists of the following steps:

1. Create a number of scenarios by taking samples of known random variables.
- 80 2. For every scenario, compute a sample of the unknown variables using a deterministic model, e.g., simulation.
3. Aggregate the results into some relevant parameters.

Algorithm 1 illustrates the Monte Carlo method for probabilistic PF analysis. The algorithm computes the sample mean vector of bus voltages $\bar{\mathbf{v}}$, given the
85 distribution functions of bus power injections $\mathbf{F}_{\mathbf{S}}(\cdot)$. The function $\text{rng}(\cdot)$ returns a random number in the interval $[0, 1]$, and is used for sampling purposes.

Typically, a high number of samples is needed to achieve accurate results. For this reason, the method can become cumbersome if applied to real-world problems involved in the operation of power systems. Common applications are
90 power system planning and reliability analysis [9], or as a validation tool to test other techniques. Several examples can be found of the latter, in which the results from Monte Carlo method are considered the “correct” ones, e.g., [10] and [11].

Algorithm 1 Monte Carlo method for probabilistic PF analysis.

Input: Number of samples, n_s . Distribution functions of bus power injections,

$F_S(\cdot)$. Deterministic PF equations, $\mathbf{s} = \mathbf{f}(\mathbf{v})$.

Output: Sample mean of bus voltages, $\bar{\mathbf{v}}$.

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1: for  $h$  in  $1, \dots, n_s$  do
2:    $r = \text{rng}()$ 
3:    $\mathbf{s}^{(h)} = F_S^{-1}(r)$  {Sampling}
4:    $\mathbf{v}^{(h)} = \mathbf{f}^{-1}(\mathbf{s}^{(h)})$  {Deterministic PF}
5: end for
6:  $\bar{\mathbf{v}} = \frac{1}{n} \sum_h \mathbf{v}^{(h)}$  {Sample mean}

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2.2. Interval-based approach

95 The interval-based approach models uncertainty as intervals without a probability distribution. This allows using expert knowledge in the definition of input intervals, in case statistical data is lacking. An *interval PF* model is defined by extending the PF equations to interval variables. The equations are solved in order to compute interval bus voltages. These methods are self-validated, as
100 interval operations respect the property of isotonicity [12].

Interval-based Newton method. This is a method for bounding the zeros of a differentiable function $\mathbf{f}(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Given an initial interval guess $[\mathbf{x}]^{(0)}$, the method computes a series $[\mathbf{x}]^{(k)}$, $k = 1, 2, \dots$, such that,

$$\mathbf{x} \in [\mathbf{x}]^{(0)}, \mathbf{f}(\mathbf{x}) = \mathbf{0} \implies \mathbf{x} \in [\mathbf{x}]^{(k)} \subset [\mathbf{x}]^{(k-1)} \subset \dots \subset [\mathbf{x}]^{(0)}. \quad (1)$$

The method relies on the mean value theorem, which is applied with vectors in the current interval. This leads to a new interval that include all the zeros. However, the new interval might be overlapping the current one. Therefore, both are intersected in order to compute the next interval in the series. Given the current interval $[\mathbf{x}]^{(k)}$, the mean value theorem states that,

$$\mathbf{f}(\mathbf{x}) \in \mathbf{f}(\mathbf{y}) + \mathbf{J}_f \Big|_{\mathbf{x} \in [\mathbf{x}]^{(k)}} (\mathbf{x} - \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in [\mathbf{x}]^{(k)}, \quad (2)$$

where \mathbf{J}_f is the Jacobian matrix of $\mathbf{f}(\cdot)$. Note that the Jacobian matrix is evaluated on the whole interval $[\mathbf{x}]^{(k)}$. Thus, the result is an interval matrix. Enforcing $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, and choosing \mathbf{y} as the midpoint of $[\mathbf{x}]^{(k)}$ (denoted by $\tilde{\mathbf{x}}^{(k)}$), yields the following expression.

$$\mathbf{x} \in \tilde{\mathbf{x}}^{(k)} - \left(\mathbf{J}_f \Big|_{\mathbf{x} \in [\mathbf{x}]^{(k)}} \right)^{-1} \mathbf{f}(\tilde{\mathbf{x}}^{(k)}), \quad (3)$$

which leads to the interval Newton iteration,

$$[\mathbf{x}]^{(k+1)} = [\mathbf{x}]^{(k)} \cap \left(\tilde{\mathbf{x}}^{(k)} - \left(\mathbf{J}_f \Big|_{\mathbf{x} \in [\mathbf{x}]^{(k)}} \right)^{-1} \mathbf{f}(\tilde{\mathbf{x}}^{(k)}) \right). \quad (4)$$

The interval Newton method is applied to interval PF analysis in the same way as the Newton-Raphson (NR) method is applied to deterministic PF analysis. This idea was introduced in [6].

Note that the interval Newton iteration requires inverting an interval matrix
 105 (or, alternatively, solving a system of interval linear equations). Typical factorisation techniques include interval Gaussian elimination and the Krawczyk's method [13]. However, depending on the structure of the matrix, these methods can either not converge, take too long or simply deliver an impractical result [14].

Affine arithmetic method. A novel method for interval PF analysis has been proposed in [1]. The method relies on affine arithmetic, a refinement of interval arithmetic which translates intervals into *affine forms* as follows:

$$[x] = \tilde{x} + x_1\epsilon_1 + x_2\epsilon_2 + \dots + x_n\epsilon_n, \quad (5)$$

where \tilde{x} is the midpoint of $[x]$, the x_i 's are known partial deviations, and the
 110 ϵ_i 's are noise symbols representing sources of uncertainty. By definition, $x_i \in \mathbb{R}$ and $\epsilon_i \in [-1, 1]$. In affine arithmetic, the same noise symbol may appear in different expressions thus referring to the same source of uncertainty. Therefore, affine arithmetic is able to circumvent the so-called *dependency problem*, unlike traditional interval arithmetic (see Appendix 7).

In [1], the interval PF equations are re-written using affine forms for bus voltage magnitudes and phases, so that the problem is reduced to the interval

computation of the noise symbols through solving a linear system:

$$\mathbf{A} [\mathbf{x}] = [\mathbf{y}] - [\mathbf{b}], \quad (6)$$

115 where \mathbf{A} is a real matrix of known partial deviations, $[\mathbf{x}]$ is an interval vector of unknown noise symbols, $[\mathbf{y}]$ is an interval vector of specified nodal power injections, and $[\mathbf{b}]$ is an interval vector of known noise symbols introduced by the affine arithmetic computational process. Note that knowing the noise symbols allows one to compute intervals for all the dependent variables by using the affine
 120 forms. The solution of (6) is obtained as the solution of two linear optimisation problems, one for the lower bounds and one for the upper bounds.

The method is tested on the IEEE-57 bus test case system, and the resulting intervals are found to be fairly good approximations of the ones computed with the Monte Carlo method, with a little overestimation introduced by the affine
 125 arithmetic formulation.

Range arithmetic method. Yet another approach has been recently presented in [10], by the same authors of [1], this time using range arithmetic. Once again, the method is tested on the IEEE-57 bus test case system and results are a little more accurate than with affine arithmetic.

130 3. Proposed method for interval PF analysis

The proposed method for interval PF analysis relies on the midpoint-radius representation, which allows the PF equations to be solved separately for the midpoint and for the radius. For an introduction to midpoint-radius interval arithmetic, see Appendix 7.

In order to introduce our method, let us first recall the crisp PF equations as follows:

$$\mathbf{s}_i = \mathbf{v}_i \sum_{k \in \mathcal{N}} (Y_{ik} \mathbf{v}_k)^*, \quad \forall i \in \mathcal{N}, \quad (7)$$

where \mathbf{s}_i is the complex power injection at bus i , \mathbf{v}_i is the complex voltage at bus i , Y_{ik} is the element of the admittance matrix linking buses i and k , and \mathcal{N}

is the set of system buses. Let us also recall the definitions of complex power and voltage, as follows:

$$\mathbf{s} = p + jq, \quad (8)$$

$$\mathbf{v} = v\angle\theta, \quad (9)$$

135 where p is the real power injection, q is the reactive power injection, v is the voltage magnitude, θ is the voltage phase angle, and j is the imaginary unit.

Equation (7) is translated in interval form as follows:

$$[\mathbf{s}_i] = [\mathbf{v}_i] \sum_{k \in \mathcal{N}} (Y_{ik} [\mathbf{v}_k])^*, \quad \forall i \in \mathcal{N}, \quad (10)$$

where $[\mathbf{s}_i]$ and $[\mathbf{v}_i]$ are now complex intervals. The proposed method assumes that there is no uncertainty associated with line parameters, so that the element of the admittance matrix Y_{ik} is crisp. Note that this is a common assumption
140 in similar approaches [10].

Using the midpoint-radius representation, $[\mathbf{s}]$ and $[\mathbf{v}]$ are expressed in the following form:

$$[\mathbf{s}] = \langle \check{\mathbf{s}}, \rho s \rangle, \quad (11)$$

$$[\mathbf{v}] = \langle \check{\mathbf{v}}, \rho v \rangle, \quad (12)$$

where

$$\check{\mathbf{s}} = \check{p} + j\check{q}, \quad (13)$$

$$\check{\mathbf{v}} = \check{v}\angle\check{\theta}, \quad (14)$$

and $\rho s, \rho v \in \mathbb{R}$.

Note that the above definitions imply that both $[\mathbf{s}]$ and $[\mathbf{v}]$ define *discs* in the complex plane. In this way, the uncertainty associated with both real and reactive power is jointly defined by the power radius ρs . Similarly, the
145 uncertainty associated with both voltage magnitude and phase angle is jointly defined by the voltage radius ρv (see [15] for a complete description of complex intervals in the midpoint-radius format).

At this point we adapt traditional notions of crisp PF analysis to interval PF, resulting in the following assumptions:

- 150 • the real power injection and voltage magnitude midpoints, \check{p} and \check{v} , are specified on all generator buses;
- the real and reactive power injection midpoints, \check{p} and \check{q} , are specified on all load buses;
- the voltage magnitude and phase angle midpoints, \check{v} and $\check{\theta}$, are specified
155 on the slack bus.

In addition, we introduce the following novel assumptions specifically regarding interval PF:

- the power injection radius, ρs , is specified on all system buses;
- the voltage radius, ρv , is specified on all generator buses.

160 The above set of assumptions is summarised in Table 1.

Table 1: Known and unknown variables for each type of bus in the proposed interval PF analysis method.

| Bus type | Knowns | Unknowns |
|-----------|---|-------------------------------------|
| Slack | $\check{v}, \check{\theta}, \rho s, \rho v$ | \check{p}, \check{q} |
| Generator | $\check{p}, \check{v}, \rho s, \rho v$ | $\check{\theta}, \check{q}$ |
| Load | $\check{p}, \check{q}, \rho s$ | $\check{v}, \check{\theta}, \rho v$ |

By applying the rules of midpoint-radius interval arithmetic (see Appendix 7), equation (10) is split in midpoint and radius parts. This yields the following system of equations:

$$\check{s}_i = \check{v}_i \sum_{k \in \mathcal{N}} (Y_{ik} \check{v}_k)^*, \quad \forall i \in \mathcal{N}, \quad (15a)$$

$$\rho s_i = (|\check{v}_i| + \rho v_i) \sum_{k \in \mathcal{N}} |Y_{ik}| \rho v_k + \rho v_i \left| \sum_{k \in \mathcal{N}} Y_{ik} \check{v}_k \right|, \quad \forall i \in \mathcal{N}. \quad (15b)$$

Note that equation (15a) represents a crisp PF problem on the midpoints of all variables. Solving this problem, e.g., by using the NR method, yields \tilde{v}_i , the midpoint of the interval solution. In order to compute the radius ρv_i on load buses, we propose the following multi-objective optimisation problem:

$$\min \quad \rho v_i, \quad \forall i \in \mathcal{N} \quad (16a)$$

$$\text{s.t.} \quad (|\tilde{v}_i| + \rho v_i) \sum_{k \in \mathcal{N}} |Y_{ik}| \rho v_k + \rho v_i \left| \sum_{k \in \mathcal{N}} Y_{ik} \tilde{v}_k \right| \geq \rho s_i^{SP}, \quad \forall i \in \mathcal{N}, \quad (16b)$$

$$\rho v_i \geq \rho v_i^{SP}, \quad \forall i \in \mathcal{N}_{PV} \quad (16c)$$

$$\rho v_i \geq 0, \quad \forall i \in \mathcal{N}_{PQ} \quad (16d)$$

where \mathcal{N}_{PV} is the subset of generator buses, \mathcal{N}_{PQ} is the subset of load buses, ρs_i^{SP} is the power injection radius specified at bus $i \in \mathcal{N}$, and ρv_i^{SP} is the voltage radius specified at bus $i \in \mathcal{N}_{PV}$.

The problem presented above is the core of our method for interval PF analysis, as follows. Starting from the bottom, constraint (16d) states that the voltage radius on load buses must be non-negative, following a basic assumption in interval algebra. Constraint (16c) states the voltage radius on generator buses must be greater than or equal to the specified, so that the interval PF solution effectively takes into account all the uncertainty on the bus voltages. Similarly, constraint (16b) states that the computed power injection radius on all system buses must be greater than or equal to the specified, so that the solution effectively takes into account all the uncertainty on the power injections. Note that the left-hand side of (16b) is directly taken from (15b). Finally, the objective function (16a) intends to prevent overestimation of the solution intervals through minimizing the radius.

We can find similarities and differences between the proposed method (16) and the one from [1]. For example, the non-negativity of the voltage radius in our method seems to have the same effect as the bounds on the noise symbols in the method from [1]. A clear difference is that in the proposed method, the intervals are expressed in the midpoint-radius format, whereas in [1] they are expressed in the lower-upper format. This allows us to solve one optimisation

problem rather than two. Another big difference lies in the representation of complex intervals, as disks in the proposed method, and rectangles in the method from [1].

185 4. Complexity analysis

This section studies the time complexity of the proposed method as well as the Monte Carlo, classic interval arithmetic and affine arithmetic methods. In the remainder of the section, n_b denotes the number of buses in the power system.

190 In the Monte Carlo method, the running time is given by the number of samples multiplied by the running time of each instance of PF analysis. For illustration, let us assume that individual instances are solved through the NR method. At each iteration of the NR method, the LU factorisation of the Jacobian matrix dominates all other calculations with a running time proportional
195 to the time of multiplying two matrices, i.e., $O(n^{2.376})$, as shown in recent studies [16]. Under the very weak assumption that the number of iterations needed to converge is much lower than the problem size, we obtain a time complexity of $O(n_s \cdot n^{2.376})$, where n_s is the number of samples. Generally speaking, n_s should be chosen in accordance with the standard deviation of results in order
200 to achieve a target level of confidence. However, n_s is often determined empirically by looking at the standard deviation of results for increasing number of samples. When the standard deviation ceases to change, it means that enough samples have been taken. For instance, in the case study presented in Section 5, the “ideal” number of samples for a network of 57 buses is determined to be
205 about 5,000 (i.e., two orders of magnitude larger than n_b). In the Monte Carlo method, n is the size of the PF problem, which is about $2n_b$. For example, if we assume $n_s \approx 100 \cdot n_b$, the running time of the Monte Carlo method can be estimated as $5.19 \cdot n_s \cdot n_b^{2.376} \approx 519 \cdot n_b^{3.376}$.

Regarding the classic interval-based Newton method, calculation time is
210 dominated by solving an interval linear system. This can be done with the

algorithm described in [17], which has an asymptotic complexity of $O(n^3)$ and a running time of $5n^3$. In this case, n is the number of unknowns of the PF analysis, i.e., $2n_b$. Hence, the running time of the interval-based Newton method is $40 \cdot n_b^3$.

215 The affine arithmetic PF method proposed in [1] requires to solve two multi-objective optimisation problems. For illustration, let us assume that the interior point algorithm is used to solve these problems [18]. This algorithm has a time complexity of $O(n^{3.5})$, where n is the number of decision variables. In the affine arithmetic PF method described in [1], the number of decision variables is equal
 220 to the size of the PF problem, i.e., $2n_b$. Thus, the running time of the two multi-objective optimization problems can be estimated as $2 \cdot (2n_b)^{3.5} \approx 22.6 \cdot n_b^{3.5}$.

Finally, for the proposed method based on interval analysis, we need to solve one multi-objective optimisation problem with n_b decision variables. Assuming once again that the interior point algorithm is used, our method has an asymptotic complexity of $O(n^{3.5})$ and an estimated running time of $n_b^{3.5}$. To this time,
 225 one has to add the time required to solve the crisp PF problem for midpoint values of the intervals, which is $n_b^{2.376}$.

It is worth noticing that both our method and the affine arithmetic PF method described in [1] have the same asymptotic time complexity. However,
 230 we expect our method to be faster than the affine arithmetic approach, for the following reasons:

- Our method only needs to solve one standard PF problem and one optimisation problem, whereas the method in [1] requires to solve two optimisation problems.
- 235 • The number of decision variables in [1] is strictly higher than the number of decision variables in our problem (twice as large, in the worst case).

Table 2 summarises the above discussion. The Monte Carlo method is the slowest one as, typically, one has that $n_s \gg n_b$. The interval-based Newton method is the fastest one but its convergence cannot be always guaranteed.
 240 This fact consistently limit applications of the interval-based approach to the

PF problem. Finally, the affine arithmetic method and the one proposed in this paper are comparable, with our method being slightly faster.

Table 2: Asymptotic time complexity of different methods for PF analysis with inclusion of uncertainty.

| Method | Asymptotic complexity | Running time |
|-------------------|--------------------------|------------------------------------|
| Monte Carlo | $O(n_s \cdot n^{2.376})$ | $5.19 \cdot n_s \cdot n_b^{2.376}$ |
| Interval Newton | $O(n^3)$ | $40 \cdot n_b^3$ |
| Affine arithmetic | $O(n^{3.5})$ | $22.6 \cdot n_b^{3.5}$ |
| Proposed method | $O(n^{3.5})$ | $n_b^{2.376} + n_b^{3.5}$ |

n : problem size, n_b : number of buses, n_s : number of Monte Carlo samples.

5. Case study

This section studies the accuracy of the proposed method in comparison
 245 with Monte Carlo and (whenever possible) other approaches from literature. Experiments are carried on the IEEE-57 and IEEE-118 bus test systems [19], where uncertainty is introduced by assuming that generator and load bus power injections, and generator bus voltages, oscillate within an interval. The mid-point of these intervals is specified as the value from the original case definition,
 250 whereas the radius is specified so as to reflect an arbitrary level of uncertainty, as discussed below.

The proposed method is implemented in the Julia programming language [20], using the package for mathematical programming JuMP [21] and the interior-point solver Ipopt [22]. The multiple objective in (16a) is implemented
 255 using a *min-max* approach, where the maximum voltage radius is minimized. The Monte Carlo method is implemented using Dome [23] and assuming that all inputs follow a uniform distribution over the complex disks defined in Section 3.

5.1. IEEE-57 bus test system

Figure 1 shows the one-line diagram of the IEEE-57 bus system, which has 7 generators, 42 loads and 80 branches including transmission lines and transformers. On this system, three uncertainty scenarios are considered as presented in Table 3. Note that scenario (a) corresponds to the case analyzed in [1] and [10], where the uncertain variables are assumed to vary over rectangles in the complex plane. For this scenario, results of the Monte Carlo, affine arithmetic and range arithmetic methods are obtained from [10]. Furthermore, the midpoint of the interval PF solution in the proposed method is computed as the midpoint of the Monte Carlo solution from [10]. For scenarios (b) and (c), the midpoint of the interval solution and the Monte Carlo solution are computed using Dome [23].

Table 3: Uncertainty scenarios and specified radius (as percentage of the midpoint) for the IEEE-57 bus system case study.

| Scenario | Uncertainty level | Power injection radius | Voltage radius |
|----------|-------------------|------------------------|----------------|
| (a) | Medium | 20% | 0% |
| (b) | Low | 10% | 1% |
| (c) | High | 50% | 2% |

Figure 2(a) shows the bus voltage magnitude intervals obtained with the proposed method, the affine arithmetic method, the range arithmetic method and the Monte Carlo method for the medium uncertainty scenario (a). We observe that the intervals computed by the proposed method are very close to the ones by Monte Carlo at the vast majority of buses, whereas with the affine and range arithmetic methods there seems to be a consistent overestimation. Nonetheless, the proposed method is unable to match the 0% oscillation specified for the voltage magnitude at generator buses (namely, buses 2, 3, 6, 8, 9 and 12), whereas this feature is guaranteed by default in the affine and range arithmetic methods. All in all, the performance of the proposed method in this scenario is

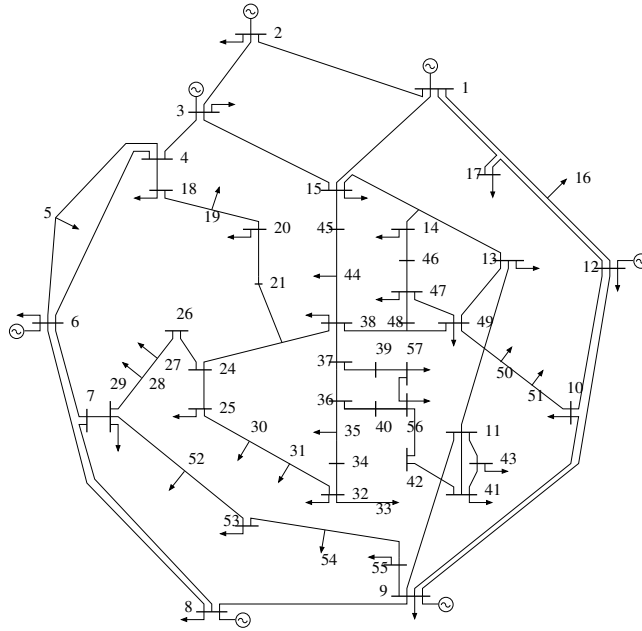
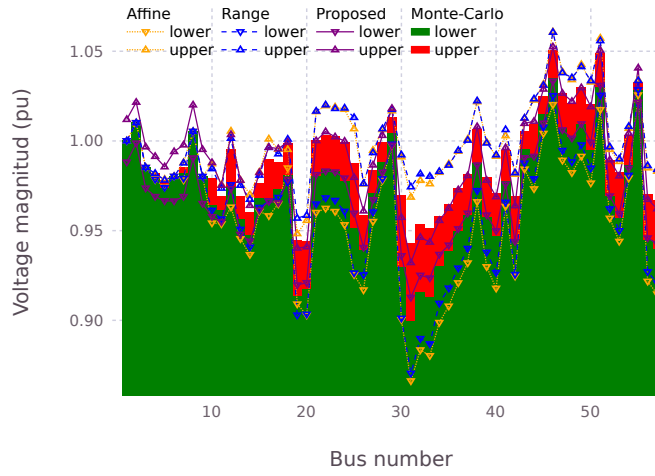


Figure 1: One-line diagram of the IEEE 57 bus test case system.

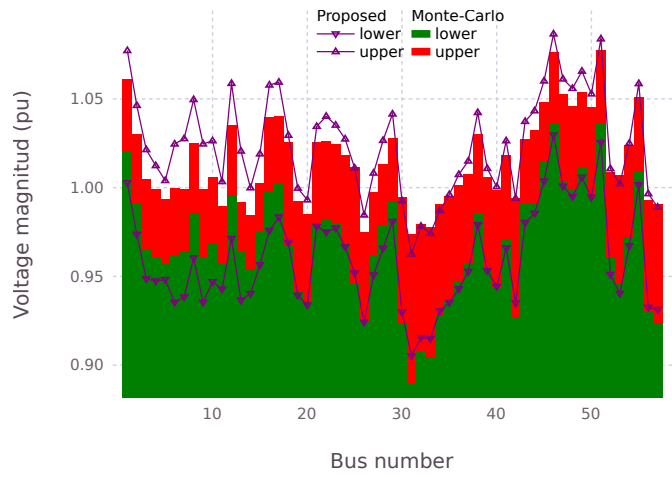


(a)

Figure 2: Bus voltage magnitude intervals obtained with different methods on the IEEE-57 bus test system. (a): medium uncertainty scenario (see Table 3).

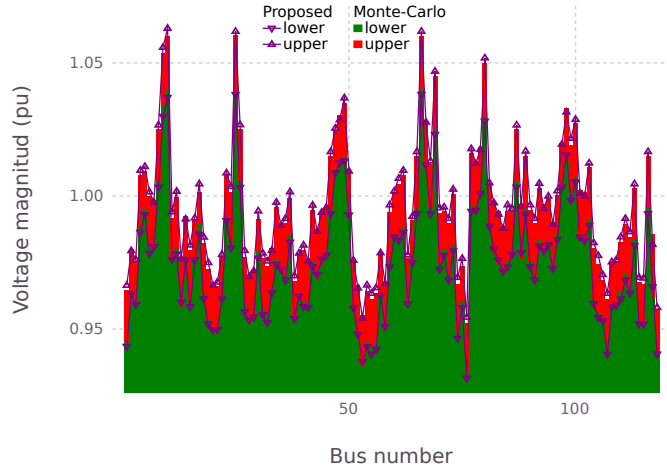


(b)



(c)

Figure 2: Bus voltage magnitude intervals obtained with different methods on the IEEE-57 bus test system. (b): low uncertainty scenario, (c): high uncertainty scenario (see Table 3).



(a)

Figure 3: Bus voltage magnitude intervals obtained with different methods on the IEEE-118 bus test system for a scenario of medium uncertainty, i.e., 20% oscillation on the generator and load bus power injections, and 1% oscillation on the generator bus voltages.

quite acceptable.

Figures 2(b) and 2(c) show the bus voltage magnitude intervals obtained with the proposed method and the Monte Carlo method for the low uncertainty scenario (b) and high uncertainty scenario (c), respectively. Note that the intervals computed by the proposed method for scenario (b) are very accurate. However, in scenario (c) there is a clear overestimation of the interval results at buses of low identifier. By looking at Fig. 1, we observe that buses with lower identifier are typically closer to generators, which in practice may help keeping the voltage magnitude within controlled bounds. Unfortunately, the proposed method is unable to capture this peculiarity which in turn becomes clear by running the heavy Monte Carlo method.

5.2. IEEE-118 bus test system

The IEEE-118 bus test system has 54 generators, 91 loads and 186 branches including transmission lines and transformers. On this system we consider only

295 one scenario of medium uncertainty, i.e., 20% oscillation on the generator and
load bus power injections, and 1% oscillation on the generator bus voltages.
Figure 3 shows the bus voltage magnitude intervals obtained with the proposed
method and the Monte Carlo method for this scenario. Note that the difference
between the two is very small, which demonstrates the accuracy and scalability
300 of the proposed approach.

6. Conclusion and future works

The paper proposes a novel method for PF analysis allowing to capture un-
certainties through an interval-based approach. The proposed solution proves
to be as efficient as but less demanding than methods previously proposed in the
305 literature, such as the interval-based Newton and affine arithmetic approaches.
The proposed method can be as accurate as the Monte Carlo method in situa-
tions of moderate uncertainty, with a consistently lower computational burden.
Indeed, a case study based on the IEEE-57 and IEEE-118 bus systems indicates
that, if the uncertainty is below a given threshold (namely, 20%), the results
310 obtained with the proposed method are very close to those provided by Monte
Carlo.

Future works will focus on enhancing the proposed optimisation problem
(16) in order to:

- a) properly deal with high uncertainty levels;
- 315 b) incorporate stability considerations and reactive power limits.

In addition, tests on larger networks, both academic and non-academic, will
also be performed so as to keep investigating the accuracy and scalability of the
proposed method.

7. Appendix

320 7.1. Midpoint-radius interval arithmetic

Intervals are convex sets of real numbers. They can be represented in sev-
eral formats, e.g., lower bound-upper bound, midpoint-radius, and even lower

bound-diameter. This article considers the midpoint-radius representation.

DEFINITION 1 (Midpoint-radius representation) Let $[x]$ be an interval defined by

$$[x] = \{x \in \mathbb{R} : |x - \check{x}| \leq \rho x\},$$

where $\check{x}, \rho x \in \mathbb{R}, \rho x \geq 0$. Then $[x]$ is noted $\langle \check{x}, \rho x \rangle$.

325 7.2. Interval operations and rounding

Interval operations are defined so as to respect the basic property of *inclusion isotonocity*, defined below.

DEFINITION 2 (Inclusion isotonicity) Let \circ be a basic arithmetic operation, i.e., $\circ \in \{+, -, \cdot, /\}$, $[x]$ and $[y]$ intervals. If

$$x \circ y \subseteq [x] \circ [y], \forall x \in [x], \forall y \in [y],$$

then \circ is said to be inclusion isotone.

Inclusion isotonicity ensures that no possible values are “left behind” when
330 performing interval operations. If floating-point numbers are used, the above means handling rounding errors inherent to floating-point arithmetic.

The IEEE-754 Standard for floating-point computation allows for self-validated interval arithmetic implementations by defining the following three rounding attributes: rounding upwards (towards infinity), rounding downwards (towards
335 minus infinity), and rounding to nearest [24]. The IEEE-754 Standard also defines the relative rounding error and the smallest representable (unnormalised) floating-point number, which are particularly useful when the midpoint-radius representation is used [15, 25].

Midpoint-radius interval arithmetic requires rounding to nearest when computing the midpoint, and rounding upwards when computing the radius. In
340 addition, the error in computing the midpoint is to the radius in order to satisfy inclusion isotonicity. Midpoint-radius interval operations are defined as follows.

DEFINITION 3 (Midpoint-radius interval operations) Let $[x] = \langle \tilde{x}, \rho x \rangle$ and $[y] = \langle \tilde{y}, \rho y \rangle$. Let $\square(\cdot)$ and $\triangle(\cdot)$ be the rounding attributes to nearest and towards plus infinity, respectively. Let ϵ be the relative rounding error, $\epsilon' = \frac{1}{2}\epsilon$ and η be the smallest representable (unnormalised) floating-point positive number. Then,

$$[x] \pm [y] = \langle z, \triangle(\epsilon'|z| + \rho x + \rho y) \rangle, \quad z = \square(\tilde{x} \pm \tilde{y}), \quad (17a)$$

$$[x] \cdot [y] = \langle z, \triangle(\eta + \epsilon'|z| + (|\tilde{x}| + \rho x)\rho y + |\tilde{y}|\rho x) \rangle, \quad z = \square(\tilde{x}\tilde{y}), \quad (17b)$$

$$\frac{1}{[y]} = \left\langle z, \triangle \left(\eta + \epsilon'|z| + \frac{-\rho y}{|\tilde{y}(\rho y - |\tilde{y}|)}} \right) \right\rangle, \quad z = \square\left(\frac{1}{\tilde{y}}\right), \quad 0 \notin [y]. \quad (17c)$$

It is important to note that both equations (17b) and (17c) introduce a bounded overestimation into the result. In the case of the multiplication, the
 345 overestimation is bounded by a factor of 1.5 [15].

The impact of rounding on performance depends on the computing architecture. For example, for most CPU architectures, the rounding attribute is a processor “mode”, and thus changes in rounding mode cause the entire instruction pipeline to be flushed.

350 7.3. The dependency problem

The so-called dependency problem causes overestimation in interval computations where the same variable occurs more than once. Since all the instances of the same variable are taken independently by the rules of interval arithmetic, the radius of the result might be expanded to unrealistic values. For example,
 355 consider the function, $f(x) = x^2 - 1$, where $x \in [-1, 1]$. Then, $f(x) \in [-1, 0]$. However, using interval arithmetic, $f([-1, 1]) = [-1, 1] \cdot [-1, 1] - 1 = [-2, 0]$.

This can become a major issue to the application of interval arithmetic to real-world problems, especially to non-linear ones. Accordingly, specific measures have to be taken to preserve the relevance of the results.

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