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Reformulating the Classical Memristor Model

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ABSTRACT

In this article, we introduce an operator to replace the traditional memristance function in the classical memristor model. This approach transforms the nonlinear structure of the original model into a linear one. Under appropriate conditions, the operator can be further replaced by a fractional operator, maintaining the model's inherent linearity and expanding its applicability. Furthermore, we develop a fractional-order dynamical memristor model that extends the memory representation to capture more complex dynamics. Finally, we provide numerical examples to illustrate the theoretical results.

1 | Introduction

The word "memristor" is defined from the two terms "memory" and "resistor." A memristor is an electrical component that controls the flow of the electrical current in a circuit (resistor). The property of a memristor is that it will "remember" its state even if the power at the device will go off (memory). For some historical facts, see [1]. The memristor was firstly introduced by Leon Chua, see [2], as a component that relates electric charge and magnetic flux. In his studies, Chua generalized this concept, and identified the "memristance" function which is used to describe mathematically the non-linear dynamic relation between voltage and current including memory of past voltages/currents, see [3, 4]. Mathematically if the graph between voltage and current is a pinched hysteresis loop then this "proves" the memristor's existence, see [5, 6]. Other memristor models were also proposed, see [7, 8], but the previous papers mentioned introduced it mathematically.

Many researchers in the literature focus in approximating the behaviour of the memristor model, see [9–12], while other articles have introduced the concept of incorporating fractional derivatives to the model, see [13–15]. For some examples and applications, see [16–18].

Let $q(t),\ \Phi(t)$, be electric charge and magnetic flux, respectively. The derivative of one with respect to the other depends on the value of one or the other, and so each memristor is characterized by its memristance function describing the charge-dependent rate of change of flux with charge:

$$M(q) = \frac{d\Phi}{dq}.$$

Let $V(t),\ I(t)$ be voltage and current, respectively. Then

$$V(t) = \frac{d\Phi(t)}{dt}, \quad I(t) = \frac{dq(t)}{dt}.$$

Obviously,

$$\frac{V(t)}{I(t)} = \frac{\frac{d\Phi(t)}{dt}}{\frac{dq(t)}{dt}} = \frac{d\Phi}{dq} = M(q),$$

or, equivalently,

$$V = M(q)I \tag{1}$$

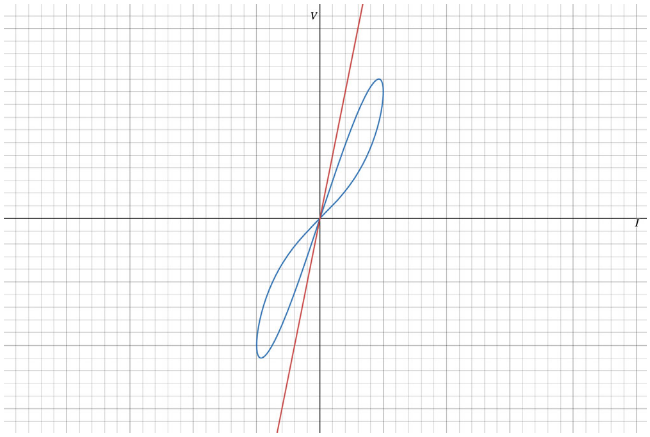

FIGURE 1 | V-I graphs: With "red" the case of $M(q(t)) \equiv R$, and with "blue" an example of a pinched hysteresis loop. [Colour figure can be viewed at wileyonlinelibrary.com]

If $M(q(t)) \equiv R$, R constant resistance, then $V = RI$, which is Ohm's law. In the case of a memristor, $M(q(t))$ is not constant and the graph V versus I is a pinched hysteresis loop, see Figure 1.

In this article, we initially reformulate the original memristor model by studying an operator which can replace the memristance function, and under certain conditions, is identically or numerically equivalent to a fractional-order operator. Additionally, by taking into account the memristor's memory, we not only further construct a fractional-order dynamical memristor model but also model the extension of its memory capabilities.

2 | Main Results

We begin the section with the following definition:

Definition 2.1. Let $k(t)$ be a function supported on $[0, +\infty)$, and let I be a function representing electric current. Then, with $\mathcal{M}_I$, we denote the following operator:

$$\mathcal{M}_I(k) = k * I \tag{2}$$

The symbol $*$ denotes the convolution of $k(t)$ and I, which is defined by

$$k * I = \int_0^t k(t-s) I(s)\, ds.$$

We refer to $\mathcal{M}_I(k)$ as a convolution-based operator, as it is defined through the convolution of the kernel $k(t)$ with the current $I(t)$. The operator $\mathcal{M}_I(k)$ satisfies linearity, causality, and memory structure properties. As mentioned in the introduction, in the case that a memristor exists, the graph V-I has the form of a pinched hysteresis loop, see Figure 1. Obviously, this graph is given by an equation of a curve in the form of $g(I, V) = 0$, or, equivalently, in the form of $V = f(I)$. Without loss of generality from (1), we can give to $M(q)I$ the form $f(I) = M(q)I$. We state the following theorem:

Theorem 2.1. *Consider the memristor model given by (1), where $f(I) = M(q)I$. If there exists a function $k(t)$ supported on $[0, +\infty)$, defined by*

$$k(t) = \frac{1}{M_0} \mathcal{L}^{-1}\left\{ \frac{F(\hat{I}(s))}{\hat{I}(s)} \right\},$$

then an equivalent model to (1) is given by

$$V = M_0 \mathcal{M}_I(k) \tag{3}$$

Here, M_0 is a constant normalization coefficient, $\mathcal{M}_I$ is the operator defined in (2), $\mathcal{L}$ denotes the Laplace transform, $\hat{I}(s) = \mathcal{L}\{I(t)\}$ represents the Laplace transform of $I(t)$, and $F(\hat{I}(s)) = \mathcal{L}\{f(I(t))\}$ denotes the Laplace transform of $f(I(t))$.

Proof. Let $f(I) = M(q)I$. We are interested in a function k supported on $[0, +\infty)$ such that

$$f(I) = M_0 \mathcal{M}_I(k),$$

or, equivalently, by using (2),

$$f(I) = M_0 k * I.$$

By applying the Laplace transform $\mathcal{L}$, we get

$$\mathcal{L}\{f(I(t))\} = M_0 \mathcal{L}\{k\} \mathcal{L}\{I\},$$

or, equivalently,

$$F(\hat{I}(s)) = M_0 \mathcal{L}\{k\} \hat{I}(s),$$

or, equivalently,

$$\mathcal{L}\{k\} = \frac{1}{M_0} \frac{F(\hat{I}(s))}{\hat{I}(s)},$$

or, equivalently,

$$k(t) = \frac{1}{M_0} \mathcal{L}^{-1}\left\{ \frac{F(\hat{I}(s))}{\hat{I}(s)} \right\}.$$

Hence, if the inverse Laplace transform of $\frac{F(\hat{I}(s))}{\hat{I}(s)}$ exists, then $k(t)$ can be defined and given from the above expression. In this case, (1) can be written in the form of (3). The proof is completed. □

We note that although the reformulated model (3) appears linear with respect to the introduced operator, the memory kernel $k(t)$ still captures the nonlinear characteristics of the original memristance function $M(q)$. Thus, the essential memory behavior of the system is preserved through the structure of the convolution operator.

In the following, we introduce classical convolution-type fractional operators, which are naturally aligned with the structure of the memory operators defined earlier and are compatible with Laplace transform techniques. The Caputo (C) fractional derivative of a function $f(t)$ of order $0 < a \leq 1$ is originally defined as

$$\mathscr{D}_c^a f(t) = \frac{1}{\Gamma(1-a)} \int_0^t (t-\tau)^{-a} \frac{d}{d\tau} f(\tau)\, d\tau,$$

where $\Gamma(\cdot)$ represents the gamma function. Building upon the Caputo (C) fractional derivative, alternative fractional operators have been developed to address specific modeling challenges and offer unique perspectives on memory effects, see [19–21]. The Caputo–Fabrizio (CF) fractional operator:

$$\mathscr{D}_{cf}^{a} f(t) = \frac{B(a)}{1-a} \int_0^t e^{-\frac{a}{1-a}(t-\tau)} \frac{d}{d\tau} f(\tau)\, d\tau,$$

and the Atangana–Baleanu (AB) fractional operator:

$$\mathscr{D}_{ab}^{a} f(t) = \frac{B(a)}{1-a} \int_0^t E_a\left[-a\frac{(t-\tau)^a}{1-a}\right] \frac{d}{d\tau} f(\tau)\, d\tau$$

where $E_a(z)$ denotes the Mittag-Leffler function. In both cases, $B(a)$ is a normalization function.

It should be noted that the (CF) and (AB) operators involve non-singular kernels and therefore do not strictly satisfy the classical definition of a fractional derivative. Following the observations presented in [22], in this work, we treat such operators as general memory operators rather than true fractional derivatives. A very recently defined fractional operator with a sinusoidal kernel, referred to in [23], is the (DS) fractional operator:

$$\mathfrak{D}_{\sin}^{a} f(t) = \frac{N(a)}{1-a} \int_0^t \sin\left[\frac{a}{1-a}(t-s)\right] \frac{d}{ds} f(s)\, ds,$$

where $N(a)$ is a normalization function. In the remainder of this section, we will use the following notations. The first-order derivative of $q(t)$ will be denoted by $q'(t)$, that is, $\frac{d}{dt}q(t) = q'(t)$. The fractional derivative of order a of $q(t)$ will be denoted by $q^{(a)}(t)$, meaning $\frac{d^a}{dt^a}q(t) = q^{(a)}(t)$. Moreover, we use the notation $\mathfrak{D}^a f(t)$ to represent either a fractional derivative, such as the classical Caputo (C) derivative, or a general memory operator, such as the Caputo–Fabrizio (CF), Atangana–Baleanu (AB), or (DS) types. We proceed to prove the following proposition:

Proposition 2.1. *We consider the operator (2) and the memristor model (3). If there exists function k supported on $[0, +\infty)$ such that $\mathcal{M}_I(k) = \frac{\tilde{M}_0}{M_0}\mathfrak{D}^a q$, $\tilde{M}_0$ constant, then an equivalent to (3) memristor model is given by*

$$V = \tilde{M}_0 \mathfrak{D}^a q \tag{4}$$

Where $0 < a < 1$, and $\mathfrak{D}^a$ is:

- *the (C) fractional operator if and only if $k(t) = t^{-a}$. In this case, $\tilde{M}_0 = M_0\Gamma(1-a)$;*
- *the (CF) fractional operator if and only if $k(t) = e^{-\frac{a}{1-a}t}$. In this case, $\tilde{M}_0 = M_0\frac{1-a}{B(a)}$;*
- *the (AB) fractional operator if and only if $k(t) = \sum_{k=0}^{\infty}(-1)^k\left[\frac{a}{1-a}\right]^k\frac{t^{ak}}{\Gamma(1+ak)}$. In this case, $\tilde{M}_0 = M_0\frac{1-a}{B(a)}$;*
- *the (DS) fractional operator if and only if $k(t) = \sin\left(\frac{a}{1-a}t\right)$. In this case, $\tilde{M}_0 = M_0\frac{1-a}{N(a)}$.*

Proof. We consider the operator (2), the memristor model (3), and $0 < a < 1$. If $k(t) = t^{-a}$ then

$$\mathcal{M}_I(k) = t^{-a} * I,$$

or, equivalently,

$$\mathcal{M}_I(k) = \int_0^t (t-s)^{-a} I(s) ds = \int_0^t (t-s)^{-a} q'(s) ds,$$

or, equivalently,

$$\mathcal{M}_I(k) = \Gamma(1-a)\mathscr{D}_c^{a} q(t).$$

Where

$$\mathscr{D}_c^{a} q(t) := \frac{1}{\Gamma(1-a)} \int_0^t (t-s)^{-a} q'(s) ds$$

is the (C) fractional derivative of order a. Hence, (3) takes the form

$$V = M_0\Gamma(1-a)\mathscr{D}_c^{a} q(t),$$

whereby setting $\mathscr{D}_c^{a} q(t) := \mathfrak{D}^a q(t)$ and $\tilde{M}_0 = M_0\Gamma(1-a)$, we arrive at (4). If $k(t) = e^{-\frac{a}{1-a}t}$ then

$$\mathcal{M}_I(k) = e^{-\frac{a}{1-a}t} * I,$$

or, equivalently,

$$\mathcal{M}_I(k) = \int_0^t e^{-\frac{a}{1-a}(t-s)} I(s) ds = \int_0^t e^{-\frac{a}{1-a}(t-s)} q'(s) ds,$$

or, equivalently,

$$\mathcal{M}_I(k) = \frac{1-a}{B(a)} \mathscr{D}_{cf}^{a} q(t).$$

Where

$$\mathscr{D}_{cf}^{a} q(t) = \frac{B(a)}{1-a} \int_0^t e^{-\frac{a}{1-a}(t-s)} q'(s) ds$$

is the (CF) fractional derivative of order a. Hence, (3) takes the form

$$V = M_0 \frac{1-a}{B(a)} \mathscr{D}_{cf}^{a} q(t),$$

whereby setting $\mathscr{D}_{cf}^{a} q(t) := \mathfrak{D}^a q(t)$ and $\tilde{M}_0 = M_0\frac{1-a}{B(a)}$ we arrive at (4). If $k(t) = \sum_{k=0}^{\infty}(-1)^k\left[\frac{a}{1-a}\right]^k\frac{t^{ak}}{\Gamma(1+ak)}$ then

$$\mathcal{M}_I(k) = \sum_{k=0}^{\infty}(-1)^k\left[\frac{a}{1-a}\right]^k \frac{t^{ak}}{\Gamma(1+ak)} * I,$$

or, equivalently,

$$\mathcal{M}_I(k) = \int_0^t \sum_{k=0}^{\infty}(-1)^k\left[\frac{a}{1-a}\right]^k \frac{(t-s)^{ak}}{\Gamma(1+ak)} I(s) ds,$$

or, equivalently,

$$\mathcal{M}_I(k) = \int_0^t \sum_{k=0}^{\infty}(-1)^k\left[\frac{a}{1-a}\right]^k \frac{(t-s)^{ak}}{\Gamma(1+ak)} q'(s) ds,$$

or, equivalently,

$$\mathcal{M}_I(k) = \sin\left[\frac{a}{1-a}(t-s)\right] B(a)\mathscr{D}_{ab}^{a} q(t).$$

Where

$$\mathscr{D}^{a}_{ab}q(t) = \frac{1}{\Gamma(1-a)} \int_0^t \sum_{k=0}^{\infty}(-1)^k \left[\frac{a}{1-a}\right]^k \frac{(t-s)^{ak}}{\Gamma(1+ak)} q'(s)ds$$

is the (AB) fractional derivative of order a. Hence, (3) takes the form

$$V = M_0 \frac{1-a}{B(a)} \mathscr{D}^{a}_{ab}q(t),$$

whereby setting $\mathscr{D}^{a}_{ab}q(t) := \mathfrak{D}^a q(t)$ and $\tilde{M}_0 = M_0 \frac{1-a}{B(a)}$ we arrive at (4). If $k(t) = \sin\left(\frac{a}{1-a}t\right)$ then

$$\mathcal{M}_I(k) = \sin\left(\frac{a}{1-a}t\right) * I,$$

or, equivalently,

$$\begin{aligned}\mathcal{M}_I(k) &= \int_0^t \sin\left[\frac{a}{1-a}(t-s)\right] I(s)ds \\ &= \int_0^t \sin\left[\frac{a}{1-a}(t-s)\right] q'(s)ds,\end{aligned}$$

or, equivalently,

$$\mathcal{M}_I(k) = \frac{1-a}{N(a)} \mathfrak{D}^a_{\sin} q(t).$$

Where

$$\mathfrak{D}^a_{\sin} q(t) = \frac{N(a)}{1-a} \int_0^t \sin\left[\frac{a}{1-a}(t-s)\right] q'(s)ds$$

is the (DS) fractional derivative of order a. Hence, (3) takes the form

$$V = M_0 \frac{1-a}{N(a)} \mathfrak{D}^a_{\sin} q(t),$$

whereby setting $\mathfrak{D}^a_{\sin} q(t) := \mathfrak{D}^a q(t)$ and $\tilde{M}_0 = M_0 \frac{1-a}{N(a)}$ we arrive at (4). The proof is completed. □

Physically, it is important to remember that memory effects are naturally present in memristor behavior, as the voltage depends on the accumulated charge. In the traditional memristor model, this memory is described through a nonlinear dependence of the voltage on the instantaneous value of the charge. However, under the conditions established in Proposition 2.1, the model can be reformulated as a linear expression involving a fractional operator acting on the charge $q(t)$. This reformulation introduces a more explicit and structured modeling of memory effects: the present behavior depends on the entire past evolution of the charge with a gradually decaying influence. The fractional-order a allows for the modeling of different types of memory behaviors, from strong long-term memory (for smaller values of a) to near-instantaneous response (as a approaches one), providing a more flexible and realistic description of systems with memory.

In addition, the framework allows for the introduction of alternative operators beyond those explicitly considered here. To preserve the structure and analytical tractability of the model, such operators must satisfy essential properties such as linearity, causality, memory structure, and compatibility with Laplace transform techniques.

Having established the memory structure through the fractional modeling framework, we now proceed to generalize the model expression. Equation (4) presents a linear expression and can be generalized for $0 \le a \le 1$ to cover the following cases:

- If $a = 0$ then for $\tilde{M}_0 = \frac{1}{C}$, with C being Capacitance, (4) takes the form $V = \frac{1}{C}q$;
- If $a = 1$ then $\frac{d}{dt}q = I$, and for $\tilde{M}_0 = R$, with R being resistance, (4) takes the form $V = RI$;
- If $0 < a < 1$ then (4) has the form $V = \tilde{M}_0 \mathfrak{D}^a q$, and under the assumptions of Proposition 2.1, this configuration is considered a memristor model.

In general, one may observe that in the case where $\mathfrak{D}^a$ represents a true fractional derivative, such as the classical Caputo (C) derivative, the expression $V = c\frac{d^a}{dt^a}I$ leads to different scenarios depending on the value of a. Notably, these scenarios can vary depending on the use of fractional operators other than the ones used in Proposition 2.1:

- If $a = 0$, it yields $V = cI$, corresponding to resistance, where $c = R$ is constant, illustrating Ohm's law.
- For $a = 1$, we have $V = c\frac{d}{dt}I$, representing inductance with $c = L$ as a constant.
- When $a = -1$, the expression becomes $V = c\int_0^t I(s)ds$, indicating capacitance with $c = \frac{1}{C}$ as a constant.
- The case becomes particularly interesting when a is fractional, posing the question of what physical interpretation can be assigned to $\frac{d^a}{dt^a}I$, which denotes the fractional derivative of order a of I.

We now consider a fractional derivative based on the exponential function. Notably, Fourier's definition, articulated through the Fourier transform, serves as an illustrative example of such a fractional derivative, see [24]. For a fractional, this fractional derivative is defined as follows:

$$\frac{d^a}{dx^a}e^{\mu x} = \mu^a e^{\mu x}, \quad \text{where } \mu \text{ is a constant} \tag{5}$$

Any function that is expressible as a sum of exponential functions, such as cosine and sine functions, can be differentiated in this manner. Using the fractional derivative (5), and if for example $I = \sin(t)$, we deduce

- For $a = 0$, $V = R\sin(t)$, aligning with resistance where $c = R$ is a constant.
- If $a = 1$, this leads to $V = L\cos(t) = L\sin\left(t + \frac{\pi}{2}\right)$, pertaining to inductance with $c = L$ as a constant.
- With $a = -1$, we find $V = -\frac{1}{C}\cos(t) = \frac{1}{C}\sin\left(t - \frac{\pi}{2}\right)$, reflecting capacitance where $c = \frac{1}{C}$ is a constant.
- When a is fractional, then $V = c\sin\left(t + a\frac{\pi}{2}\right)$ describes a property related to phase shifts or phase behavior in an electrical component, indicating a niche or newly proposed concept within certain research areas or theoretical studies.

Next, we focus on the property of the memristor that includes memory of past voltages/currents, and construct an alternative fractional-order dynamical memristor model. We state the following Theorem:

Theorem 2.2. *A fractional-order dynamical memristor model is described by the equation:*

$$V = M(q)\frac{d^a}{dt^a}q \tag{6}$$

where $M(\cdot)$ is the memristance function and $\frac{d^a}{dt^a}q$ denotes a fractional derivative of q of order a, with $0 < a < 1$. This formulation captures additional memory characteristics compared with the traditional model presented in (1), enabling the modeling of more complex dynamical behaviors associated with fractional-order effects.

Proof. From (1), we have

$$q'(t) = H(q(t)),$$

where $H(q(t)) = \frac{1}{M(q)}V$. In order to simply explain why the proposed fractional-order model and its memory effect will relate to the memristor model, we will use the discrete form of the above equation:

$$q_{k+1} = H(q_k).$$

This is a first-order difference equation. The term q_{k+1} is only related to just a previous step in time, namely the term q_k. This means that we obtain the values of q_{k+1} by only absorbing information from just a previous step in time k, and not by considering all the "history" of changes at times $k-1,\ k-2,\ \dots,k_0$. Where k_0 the initial time step which can be assumed zero, that is, $k_0 = 0$. To incorporate the information from all previous time steps, we use the fractional nabla operator of order a with $0 < a < 1$, which, for a sequence c_k, is defined as

$$\nabla^{-a}c_k = \sum_{j=0}^{k} d_{k-j}c_j,$$

where $d_{k-j} = \frac{1}{\Gamma(a)}\frac{\Gamma(k-j+a)}{\Gamma(k-j+1)}$; see [25, 26]. As mentioned, one has to consider not only one time step to absorb information from the past but also the "history" of changes throughout the timeline $0, 1, \dots, k-1, k$. Hence, we incorporate to the first-order difference equation further delays as follows:

$$q_{k+1} = \sum_{j=0}^{k} d_{k-j}H(q_j)$$

Equivalently, we then have

$$\nabla^a q_{k+1} = H(q_k).$$

Where $0 < a < 1$ is the fractional order of the nabla discrete operator. Returning to the continuous time equation, and by using the previous discussion, we propose the following fractional-order version of the dynamic memristor model:

$$\frac{d^a}{dt^a}q = H(q),$$

or, equivalently,

$$q^{(a)}(t) = \frac{1}{M(q(t))}V(t),$$

or, equivalently,

$$V = M(q)q^{(a)},$$

which corresponds to (6). This completes the proof. □

It should be noted that the reintroduction of nonlinearity through the fractional operator in the present model relies on the specific memory and convolution properties of fractional derivatives. Extending this behavior to more general operators would require additional structural assumptions, such as the existence of a memory structure and compatibility with the dynamical system formulation.

3 | Numerical Examples

In this section, we provide a set of examples to illustrate the main theoretical results established in the previous section. We first consider the traditional nonlinear memristor model $V = M(q)I$. We then reformulate this model into an equivalent form involving a convolution-based memory operator, as presented in Theorem 2.1. Following this, we show, based on Proposition 2.1, how this convolution structure can be replaced by a suitable fractional operator under appropriate conditions. Finally, we construct a fractional-order dynamical memristor model incorporating nonlinear dependence, as described in Theorem 2.2, to capture enhanced memory characteristics. Let the current be defined by $I(t) = \sin(t)$, and the voltage be given by

$$V(t) = \sin(t) - \frac{1}{2}\sin(2t).$$

We first consider the memristor model (1). It is straightforward to verify that

$$q = \int_0^t \sin(s)ds = 1 - \cos(t),$$

and hence

$$V(t) = \sin(t) - \sin(t)\cos(t) = (1 - \cos(t))\sin(t),$$

or, equivalently,

$$V = qI, \quad \text{with} \quad M(q) = q,$$

which is the form of (1). If we give to the above expression the form $V = f(I)$, we get

$$V = I\left(1 \pm \sqrt{1 - I^2}\right).$$

The graph V-I, which is a pinched hysteresis loop, is shown in Figure 2.

3.1 | Example 1: Reformulation Via the Convolution Operator $\mathcal{M}_I(k)$

As a first example, we apply Theorem 2.1 to reformulate the traditional memristor model in terms of the convolution operator

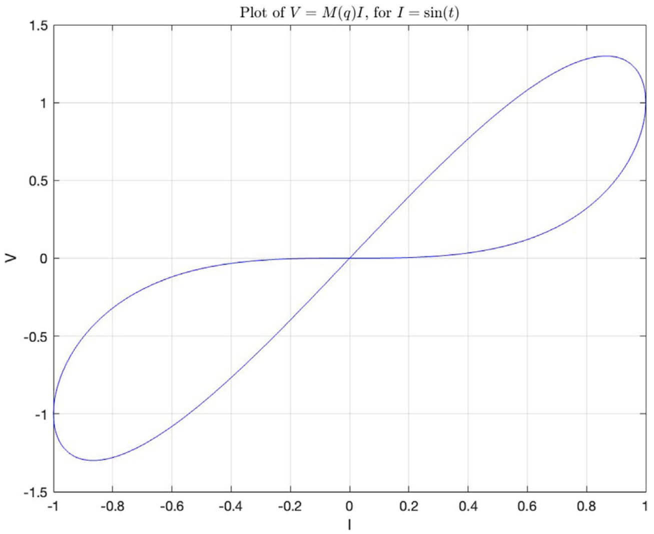

FIGURE 2 | V-I graph of the memristor model $V = M(q)I$ for $I = \sin(t)$. [Colour figure can be viewed at wileyonlinelibrary.com]

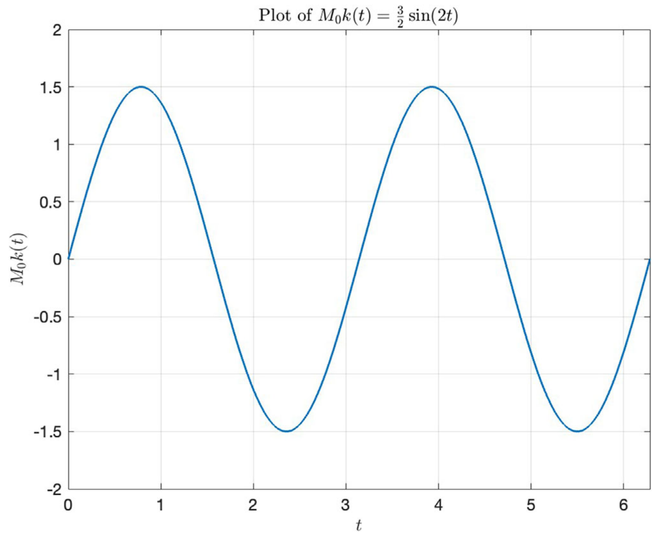

FIGURE 3 | Plot of $M_0k(t) = \frac{3}{2}\sin(2t)$ versus t. [Colour figure can be viewed at wileyonlinelibrary.com]

$\mathcal{M}_I(k)$ defined in (2). Our objective is to express the memristor model in the form given by (3). Specifically, we have

$$\sin(t) - \frac{1}{2}\sin(2t) = M_0k(t) * \sin(t),$$

or, equivalently, by applying the Laplace transform

$$\frac{1}{1+s^2} - \frac{1}{2}\frac{2}{4+s^2} = M_0\frac{1}{1+s^2}K(s),$$

where $K(s) = \mathcal{L}\{k(t)\}$ denotes the Laplace transform of $k(t)$. Equivalently,

$$M_0K(s) = 1 - \frac{1+s^2}{4+s^2},$$

or, equivalently,

$$M_0K(s) = 1 - \frac{4+s^2-3}{4+s^2},$$

or, equivalently,

$$M_0K(s) = 1 - 1 + \frac{3}{4+s^2},$$

or, equivalently,

$$K(s) = \frac{3}{2M_0}\frac{2}{4+s^2},$$

whereby applying the inverse Laplace transform, we get

$$k(t) = \frac{3}{2M_0}\sin(2t).$$

Hence,

$$\mathcal{M}_I(k) = k * I = \frac{3}{2M_0}\int_0^t \sin[2(t-s)]\sin(s)ds,$$

and the memristor model (3) is given by

$$V = \frac{3}{2}\mathcal{M}_I(\sin(2t)) \tag{7}$$

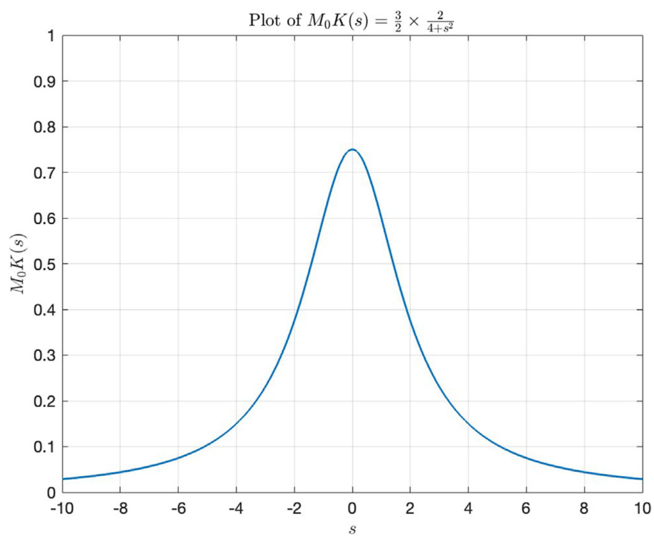

FIGURE 4 | Plot of $M_0K(s) = \frac{3}{2} \times \frac{2}{4+s^2}$ versus s. [Colour figure can be viewed at wileyonlinelibrary.com]

From the derivations above, by comparing the Laplace transforms, it follows that the normalization constant is $M_0 = \frac{3}{2}$. This value ensures consistency between the convolution model and the original expression of the voltage $V(t)$. To further illustrate the behavior of the model, we plot the function $M_0k(t)$, its Laplace transform $M_0K(s)$, and the operator $M_0\mathcal{M}_I(k)$, corresponding to the voltage $V(t)$. These plots provide additional insight into the structure of the memory kernel in both time and frequency domains, and clarify the action of the convolution operator.

The plot of $M_0k(t)$ as a function of t is shown in Figure 3, illustrating the time-domain behavior of the kernel. The corresponding Laplace transform $M_0K(s)$ is depicted in Figure 4, highlighting its frequency-domain characteristics. Finally, the resulting voltage $V(t) = M_0\mathcal{M}_I(k)$ is plotted in Figure 5, showing the effect of the convolution operator on the input current.

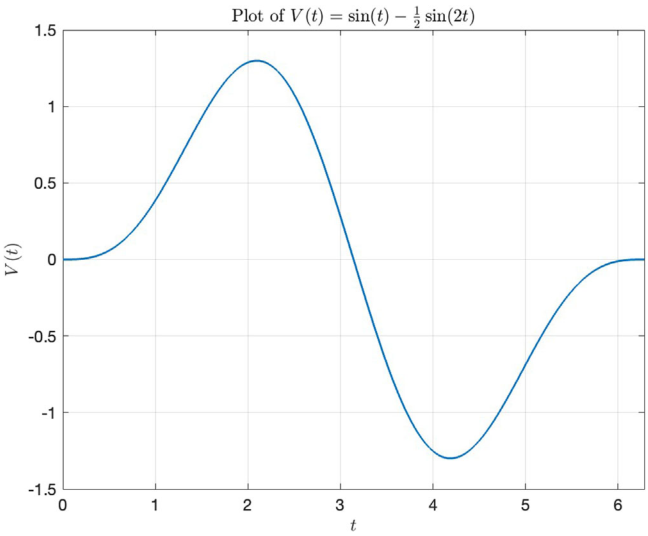

FIGURE 5 | Plot of $V(t) = \sin(t) - \frac{1}{2}\sin(2t)$ versus t. [Colour figure can be viewed at wileyonlinelibrary.com]

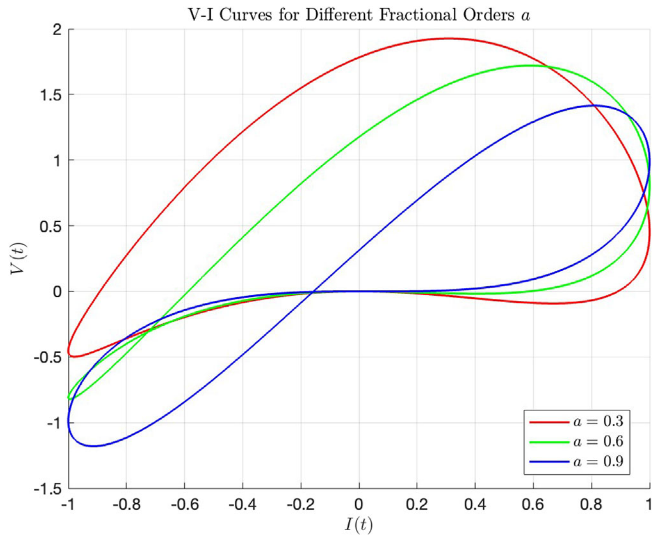

FIGURE 6 | V-I curves for different values of the fractional-order $a \in \{0.3, 0.6, 0.9\}$. [Colour figure can be viewed at wileyonlinelibrary.com]

3.2 | Example 2: Operator Replacement by a Fractional Operator

In this second example, we apply Proposition 2.1 to express the convolution operator $\mathcal{M}_I(k)$ in terms of a fractional operator. From the previous computations, we recall that

$$k(t) = \sin(2t).$$

We select the fractional-order $a = \frac{2}{3}$ and $N\left(\frac{2}{3}\right) = 1$. This choice allows the convolution operator to be replaced by a fractional operator associated with the sine kernel structure. Specifically, applying the (DS) fractional operator to the charge $q(t)$, we obtain an expression equivalent to (7):

$$V(t) = \frac{1}{2}\mathfrak{D}^{\frac{2}{3}}_{\sin} q(t).$$

Thus, the voltage $V(t)$ can be represented directly in terms of a fractional operator of the charge $q(t)$.

3.3 | Example 3: Construction of a Fractional Dynamical Memristor Model

We now proceed to the third example, where we construct a fractional-order dynamical memristor model that incorporates memory effects directly through a fractional derivative. Unlike the previous example, where a fractional operator was used primarily for reformulation, here the fractional-order behavior is embedded in the dynamical evolution of the system itself, as described by Theorem 2.2 and model Equation (6). We use the fractional derivative defined in (5), which gives

$$\frac{d^a}{dt^a}q(t) = -\cos\left(t + \frac{a\pi}{2}\right).$$

Hence, from Theorem 2.2 and (6), we have

$$V = (\cos(t) - 1)\cos\left(t + \frac{a\pi}{2}\right).$$

To illustrate the influence of the fractional-order a on the memory characteristics of the system, we compute and plot the corresponding V-I curves for representative values $a \in \{0.3, 0.6, 0.9\}$. The results are shown in Figure 6, where it is evident that the shape of the hysteresis loop varies significantly with a. This visualization highlights how the fractional order modifies the memory behavior: smaller values of a correspond to stronger memory effects, while larger values lead to a behavior closer to traditional models.

4 | Conclusions

In this article, we have proposed new mathematical memristor models. Initially, we introduced an operator and demonstrated that, under certain conditions, it could serve as a substitute for the memristance function in the original model. Moreover, we showed that in some instances, this operator could be replaced by a fractional-order operator. Further, we developed a fractional-order dynamical memristor model, incorporating the crucial aspect of memory, which accounts for the history of past voltages and currents. Examples were provided to substantiate our theoretical propositions. The mathematical structure of the developed models, based on convolution operators and fractional derivatives, also parallels the formulation of mechanical systems exhibiting memory effects, such as systems with hereditary properties or mechanical vibrations with damping. Extending the present framework to complex mechanical systems incorporating memory characteristics appears feasible and represents an interesting direction for future research.

Author Contributions

Ioannis Dassios: investigation, writing – original draft, methodology, validation, visualization, writing – review and editing, software, formal analysis, conceptualization. **Federico Milano:** conceptualization, funding acquisition, project administration, supervision, resources.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The authors have nothing to report.

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