Participation factors for singular systems of differential equations

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Abstract: In this article, we provide a method to measure the participation of system eigenvalues in system states, and vice versa, for a class of singular linear systems of differential equations. This method deals with eigenvalue multiplicities and covers all cases by taking into account both the algebraic and geometric multiplicity of the eigenvalues of the system matrix pencil. A Möbius transform is applied to determine the relative contributions associated with the infinite eigenvalue that appears because of the singularity of the system. The paper is a generalization of the conventional participation analysis, which provides a measure for the coupling between the states and the eigenvalues of systems of ordinary differential equations with distinct eigenvalues. Numerical examples are given including a classical DC circuit and a 2-bus power system dynamic model.

Keywords : participation factor, singularity, dynamical system, Möbius transform, differential equations

1 Introduction

Participation factors were firstly introduced by Perez-Arriaga *et al.* in [24] to carry out modal analysis of a linear time-invariant dynamic system of ordinary differential equations. A participation factor is known to represent the sensitivity of an eigenvalue to variations of an element of the state matrix [23]. It has been also viewed as modal energy in the sense described by MacFarlane [16]. Although participation factors have been defined and widely employed as a tool for small-signal stability analysis of a dynamic system, the participation factor is also an important case of residue analysis [15], which is of major importance during the design of linear control systems. It has been also utilized in the application of model equivalencing techniques [3]. Recent studies have also tackled the participation analysis of nonlinear systems [22], [28]. The participation factors of a system are typically collected to form a matrix, which is known as the system participation matrix.

Application of appropriate initial conditions to the time response of a linear time invariant dynamic system of differential equations allows to determine a measure that expresses the relative activity of a state in the structure of an eigenvalue and *vice versa*. This measure is termed participation factor.

Definition 1.1. Consider a linear system of ordinary differential equations in the form:

$$Y' = AY \ ,$$

where $Y \in \mathbb{R}^{m\times 1}$, are the state variables, and $A \in \mathbb{R}^{m\times m}$, is the state matrix. Let s_i be an eigenvalue of A (or more precisely of $sI_m - A$, where I_m is the $m\times m$ identity matrix) and all the eigenvalues be distinct, i.e., $s_i \neq s_j$, $i \neq j$, and $i, j = 1, 2, \dots, m$. Let also v_i, w_i be the right and left eigenvectors associated with s_i, respectively. If Y_k is the k-th state of the system, the participation factor is defined as:

$$p_{k,i} = w_{i,k} v_{k,i} \ ,$$

where $v_{k,i}$ is the k-th row element of v_i and $w_{i,k}$ is the k-th column element of w_i.

The participation factor $p_{k,i}$ basically expresses the relative contribution of Y_k in the structure of the eigenvalue s_i, and *vice versa*, but has also various other interpretations.

From Definition 1.1 we can see that the main assumptions of classical modal participation analysis are the following:

- All eigenvalues are distinct.
- The system is modelled as a set of ordinary differential equations, i.e., all eigenvalues are finite.

However, dynamic system models often introduce multiple eigenvalues. In addition, dynamic systems can be modeled through singular systems of differential equations [19], which include eigenvalues at infinity.

In general, singular systems of linear differential/difference equations are inherent in many physical, engineering, mechanical, and financial models. Having in mind such applications, for instance in finance, we provide the well-known input-output Leontief model and its several important extensions, see [1], [4], [6]. Singular systems also appear in control theory, see [2], in macroeconomics, see [11], circuit theory, see [30], and in the modeling of power systems, see [17], [18], [20]. There is also a large number of applications of a special case of singular systems of differential equations called differential–algebraic equations.

We consider the following system:

$$EY'(t) = AY(t) \ , \tag{1}$$

where $E, A \in \mathbb{R}^{r\times m}$, $Y : [0, +\infty] \to \mathbb{R}^{m\times 1}$. The matrices E and A can be non-square ($r \neq m$) or square ($r = m$) with E singular, i.e., det(E)=0. With Y' we denote the first order derivative of $Y(t)$. The pencil $sE - A$ is then used to study this system. A matrix pencil is a family of matrices $sE - A$, parametrized by a complex number s, see [13], [14].

Definition 1.2. Given $E, A \in \mathbb{C}^{r\times m}$, and an arbitrary $s \in \mathbb{C}$, the matrix pencil $sE - A$ is called:

1. Regular when $r = m$ and $\det(sE - A) \not\equiv 0$;
2. Singular when $r \neq m$, or $r = m$ and $\det(sE - A) \equiv 0$.

To simply understand the concept of the pencil, in system (1) when A is square and $E = I_m$, where I_m is the identity matrix, the zeros of the function $\det(sE - A)$ are the eigenvalues of A. Consequently, the problem of finding the non-trivial solutions of the equation

$$sEX = AX \ ,$$

is called the generalized eigenvalue problem, see [26]. Although the generalized eigenvalue problem looks like a simple generalization of the usual eigenvalue problem it exhibits some important differences. Firstly, it is possible for E to be singular in which case the problem has infinite eigenvalues. To see this write the generalized eigenvalue problem in the reciprocal form:

$$EX = s^{-1}AX \ .$$

If E is singular with a null vector X, then $EX = 0_{m,1}$, so that X is an eigenvector of the reciprocal problem corresponding to eigenvalue $s^{-1} = 0$; i.e., $s \to \infty$. A second non-trivial case is the determinant $\det(sE-A)$, when E, A are square matrices, to be identically zero, independent of s. And finally there is the case for both matrices E, A to be non-square (for $r \neq m$).

Remark 1.1. Given $E, A \in \mathbb{C}^{r\times m}$, and an arbitrary $s \in \mathbb{C}$, if pencil $sE - A$ is:

(a) Regular, since $\det(sE-A) \not\equiv 0$, there exists a matrix $\tilde{P} : \mathbb{C} \to \mathbb{R}^{m\times m}$ (which can be computed via the Gauss-Jordan Elimination Method, see [26]) such that:

$$\tilde{P}(s)(sE - A) = \tilde{A}(s).$$

Where $\tilde{A} : \mathbb{C} \to \mathbb{R}^{m\times m}$ is a diagonal matrix with non-zero elements;

(b) Singular and $r > m$, then there exists a matrix $\tilde{P} : \mathbb{C} \to \mathbb{R}^{r\times r}$ (which can be computed via the Gauss-Jordan Elimination Method) such that

$$\tilde{P}(s)(sE - A) = \begin{bmatrix} \tilde{A}(s) \\ 0_{r_1,m} \end{bmatrix}, \quad \text{with} \quad \tilde{P}(s) = \begin{bmatrix} \tilde{P}_1(s) \\ \tilde{P}_2(s) \end{bmatrix}. \tag{2}$$

Where $\tilde{A} : \mathbb{C} \to \mathbb{R}^{m_1\times m}$, with $m_1 + r_1 = r$, is a matrix such that if $[\tilde{a}_{ij}]_{1\leq i\leq m_1}^{1\leq j\leq m}$ are its elements, for $i = j$ all elements are non-zero and for $i \neq j$ all elements are zero and $\tilde{P}_1(s) \in \mathbb{R}^{m_1\times r}$, $\tilde{P}_2(s) \in \mathbb{R}^{r_1\times r}$.

Throughout the paper, with 0_{ij} we will denote the zero matrix of i rows and j columns, with ^{T} the transpose tensor, and with I_m the identity matrix $m \times m$. Finally, let $B_{n_1} \in \mathbb{C}^{n_1\times n_1}$, $B_{n_2} \in \mathbb{C}^{n_2\times n_2}$, $\dots$, $B_{n_r} \in \mathbb{C}^{n_r\times n_r}$. With the direct sum

$$B_{n_1} \oplus B_{n_2} \oplus \cdots \oplus B_{n_r} \ ,$$

we will denote the block diagonal matrix:

$$blockdiag \begin{bmatrix} B_{n_1} & B_{n_2} & \dots & B_{n_r} \end{bmatrix} \ .$$

To the best of our knowledge the concept of participation factors has not been fully analyzed and exploited for singular systems of differential equations. The specific contributions of the paper are as follows:

- A generalization of the conventional formulation of the participation analysis problem for singular systems of differential equations. The formulation is provided for systems with either singular or regular matrix pencils.

- A new formulation, which allows to derive the participation factor for systems with multiple eigenvalues. The special case that the geometric multiplicity equals the algebraic one, as well as the case that the eigenvalues are distinct, are also derived.

- A methodology to determine the participation factors associated with infinite modes, which is derived by employing a special Möbius transformation.

2 Mathematical Background

Firstly, we will study the existence of solutions of system (1). We state the following Theorem:

Theorem 2.1. There exist solutions for (1) if and only if:

(a) The pencil of the system is regular; or

(b) The pencil of the system is singular with $r > m$ and

$$\tilde{P}_2(s)E = 0_{m_1,1}, \quad \text{and} \quad m_1 = m. \tag{3}$$

Where $\tilde{P}_2(s)$ is defined in (2).

Proof. Let $\mathcal{L}\{Y(t)\} = Z(s)$ be the Laplace transform of $Y(t)$ respectively. By applying the Laplace transform $\mathcal{L}$ into (1), we get:

$$E\mathcal{L}\{Y'(t)\} = A\mathcal{L}\{Y(t)\} \ ,$$

or, equivalently,

$$E(sZ(s) - Y_0) = AZ(s) \ .$$

Where $Y_0 = Y(0)$, i.e., the initial condition of (1). Since we assume that Y_0 is unknown we can use an unknown constant vector $C \in \mathbb{R}^{m\times 1}$ and give to the above expression the following form:

$$(sE - A)Z(s) = EC \ . \tag{4}$$

We have two cases. The first is (a) $r = m$ and $\det(sE - A)$ to be equal to a polynomial with order less than m (regular pencil). The second case is (b) $r \neq m$, or $r = m$ with $\det(sE - A) \equiv 0$, $\forall$ arbitrary $s \in \mathbb{C}$ (singular pencil).

In the case of (a), since the pencil is assumed regular, we have that $\det(sE-A) \not\equiv 0$. Then $Z(s)$ in (4) can be defined and consequently $Y(t)$ always exists and is given by $Y(t) = \mathcal{L}^{-1}\{(sE-A)^{-1}EC\}$. Hence in the case of a regular pencil, the solution of (1) always exists. In the case of (b), if $r < m$, in (4) there are at least $m-r$ unknown functions and m equations. Hence $Z(s)$ can not be defined uniquely.

If $r > m$ then there exists a matrix $\tilde{P} : \mathbb{C} \to \mathbb{R}^{r\times r}$ (which can be computed via the Gauss-Jordan Elimination Method) such that

$$\tilde{P}(s)(sE-A) = \begin{bmatrix} \tilde{A}(s) \\ 0_{r_1,m} \end{bmatrix}.$$

Where $\tilde{A} : \mathbb{C} \to \mathbb{R}^{m_1\times m}$, with $m_1 + r_1 = r$, is a matrix such that if $[\tilde{a}_{ij}]_{1\leq i\leq m_1}^{1\leq j\leq m}$ are its elements, for $i = j$ all elements are non-zero and for $i \neq j$ all elements are zero. Then by setting

$$\tilde{P}(s) = \begin{bmatrix} \tilde{P}_1(s) \\ \tilde{P}_2(s) \end{bmatrix},$$

where $\tilde{P}_1(s) \in \mathbb{R}^{m_1\times r}$, $\tilde{P}_2(s) \in \mathbb{R}^{r_1\times r}$, system (4) takes the form:

$$\begin{bmatrix} \tilde{A}(s) \\ 0_{r_1,m} \end{bmatrix} Z(s) = \begin{bmatrix} \tilde{P}_1(s) \\ \tilde{P}_2(s) \end{bmatrix} EC \ ,$$

from where we get

$$\tilde{A}(s)Z(s) = \tilde{P}_1(s)EC \ , \quad \text{and} \quad 0_{r_1,m}Z(s) = \tilde{P}_2(s)EC \ .$$

If $0_{r_1,m}Z(s) = \tilde{P}_2(s)EC$ holds, then $Z(s)$ can be defined in $\tilde{A}(s)Z(s) = \tilde{P}_1(s)EC$ if $m_1 = m$. Hence, $Z(s)$ in (4) can be defined and consequently $Y(t)$ always exists and is given by $Y(t) = \mathcal{L}^{-1}\{\tilde{A}(s)^{-1}EC\}$ if and only if (3) holds. In any other case we have more unknown functions than equations or no solutions.

If $r = m$ then there exists a matrix $\tilde{P} : \mathbb{C} \to \mathbb{R}^{r\times r}$ (which can be computed via the Gauss-Jordan Elimination Method) such that:

$$\tilde{P}(s)(sE-A) = \tilde{A}(s) \oplus 0_{r_2,m_2} \ ,$$

where $\tilde{A} : \mathbb{C} \to \mathbb{R}^{r_1\times m_1}$ with $r_1 \leq m_1$ (because we apply Gauss-Jordan Elimination Method at the rows). All elements of $\tilde{A}(s)$ are zero except the ones in the diagonal which are all non-zero elements. Also, $r_1 + r_2 = m_1 + m_2 = m$. Then system (3) can have solutions if and only if $r_2 = m_2 = 0$, i.e. $r_1 = m_1 = m$; In any other case, we have more unknown functions than equations or no solutions. But since we are in the case where $r = m$ and the pencil is singular, i.e., $\det(sE-A) \equiv 0$, this assumption can never hold. To sum up, there exists solution for the system if the pencil is regular or singular with $r > m$ and $\tilde{A}(s)$ $m \times m$ and $\tilde{P}_2(s)F = \tilde{P}_2(s)U(s) = 0_{m-r,1}$. The proof is completed.

In this article we are interested in two cases: (a) system (1) with regular pencil, (b) system (1) with singular pencil, $r > m$, and (3) to hold. In both cases we proved that there exist solutions.

For a *regular pencil*, see [13], [14], there exist non-singular matrices P, Q $\in \mathbb{C}^{m\times m}$ such that:

$$\begin{aligned} PEQ &= I_p \oplus H_q \ , \\ PAQ &= J_p \oplus I_q \ . \end{aligned} \tag{5}$$

Where

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \ , \ Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \ ,$$

with $P_1 \in \mathbb{C}^{p\times m}$, $P_2 \in \mathbb{C}^{q\times m}$ and $Q_p \in \mathbb{C}^{m\times p}$, $Q_q \in \mathbb{C}^{m\times q}$. Furthermore, $H_q \in \mathbb{C}^{q\times q}$ is a nilpotent matrix with index q_*, constructed by using the algebraic multiplicity of the infinite eigenvalue, and $J_p \in \mathbb{C}^{p\times p}$ is a Jordan matrix constructed by the finite eigenvalues of the pencil and their algebraic multiplicity. Where p is the sum of all algebraic multiplicities of the finite eigenvalues and q the algebraic multiplicity of the infinite. Consequently, $p + q = m$.

P_1 is a matrix with rows p linear independent (generalized) left eigenvectors of the p finite eigenvalues of $sE - A$; P_2 is a matrix with columns q linear independent (generalized) left eigenvectors of the infinite eigenvalue of $sE - A$ with algebraic multiplicity q; Q_p is a matrix with columns p linear independent (generalized) right eigenvectors of the p finite eigenvalues of $sE - A$; and Q_q is a matrix with columns q linear independent (generalized) right eigenvectors of the infinite eigenvalue of $sE - A$ with algebraic multiplicity q. By applying the above expressions into (1), we get the following eight equalities:

$$\begin{array}{ll} P_1AQ_p = J_p & P_1EQ_p = I_p \\ P_1AQ_q = 0_{p,q} & P_1EQ_q = 0_{p,q} \\ P_2AQ_p = 0_{q,p} & P_2EQ_p = 0_{q,p} \\ P_2AQ_q = I_q \ , & P_2EQ_q = H_q \ . \end{array}$$

The *singular pencil* with $r > m$ is characterized by the set of the finite–infinite eigenvalues, and the minimal row indices, see [5], [8], [14]. Let $\mathcal{N}_l$ be the left null space of a matrix respectively. Then the equations $V^T(s)(sE - A) = 0_{1,m}$ have solutions in $V(s)$, which are vectors in the rational vector space $\mathcal{N}_l(sE - A)$. The binary vectors $V^T(s)$ express dependence relationships among the rows of $sE - A$. Note that $V(s) \in \mathbb{C}^{r\times 1}$ are polynomial vectors. Let t=dim$[\mathcal{N}_l(sE - A)]$. It is known that $\mathcal{N}_l(sE - A)$ as rational vector spaces, are spanned by minimal polynomial bases of minimal degrees

$$\zeta_1 = \zeta_2 = ... = \zeta_h = 0 < \zeta_{h+1} \leq ... \leq \zeta_{h+k=t} \ ,$$

which is the set of *row minimal indices* of $sE - A$. This means there are t row minimal indices, but $t - h = k$ non-zero row minimal indices. We are interested only in the k non-zero minimal indices. To sum up, the invariants of a singular pencil with $r > m$ are the finite − infinite eigenvalues of the pencil and the minimal row indices as described above. Following the above given analysis, there exist non-singular matrices P, Q with $P \in \mathbb{C}^{r\times r}$, $Q \in \mathbb{C}^{m\times m}$, such that:

$$\begin{aligned} PEQ &= E_K = I_p \oplus H_q \oplus E_\zeta \ , \\ PAQ &= A_K = J_p \oplus I_q \oplus A_\zeta \ . \end{aligned} \tag{6}$$

The matrices P, Q can be written as:

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_p & Q_q & Q_\zeta \end{bmatrix},$$

with $P_1 \in \mathbb{C}^{p\times r}$, $P_2 \in \mathbb{C}^{q\times r}$, $P_3 \in \mathbb{C}^{\tilde{\zeta}_1\times r}$, $\tilde{\zeta}_1 = k + \sum_{i=1}^{k}[\zeta_{h+i}]$ and $Q_p \in \mathbb{C}^{m\times p}$, $Q_q \in \mathbb{C}^{m\times q}$, $Q_\zeta \in \mathbb{C}^{m\times\tilde{\zeta}_2}$ and $\tilde{\zeta}_2 = \sum_{i=1}^{k}[\zeta_{h+i}]$. Where J_p is the Jordan matrix for the finite eigenvalues, H_q a nilpotent matrix with index q_* which is actually the Jordan matrix of the zero eigenvalue of the pencil $sA - E$. The matrices E_ζ, A_ζ are defined as

$$E_\zeta = \begin{bmatrix} I_{\zeta_{h+1}} \\ 0_{1,\zeta_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} I_{\zeta_{h+2}} \\ 0_{1,\zeta_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} I_{\zeta_{h+k}} \\ 0_{1,\zeta_{h+k}} \end{bmatrix},$$

and

$$A_\zeta = \begin{bmatrix} 0_{1,\zeta_{h+1}} \\ I_{\zeta_{h+1}} \end{bmatrix} \oplus \begin{bmatrix} 0_{1,\zeta_{h+2}} \\ I_{\zeta_{h+2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0_{1,\zeta_{h+k}} \\ I_{\zeta_{h+k}} \end{bmatrix},$$

with $p + q + \sum_{i=1}^{k}[\zeta_{h+i}] + k = r$, $p + q + \sum_{i=1}^{k}[\zeta_{h+i}] = m$.

Proposition 2.1. We consider system (1) with a regular pencil, or a singular pencil with $r > m$ and that (3) holds. Let J_p be the Jordan matrix of the finite eigenvalues, and Q_p the matrix that contains all linear independent eigenvectors as defined in (5), (6). Then there exists a solution and is given by:

$$Y(t) = Q_p e^{J_p t} Z_p(0)\ , \tag{7}$$

where $Z_p(0) \in \mathbb{C}^{p\times p}$ is constant vector.

Proof. From Theorem 2.1, there exists a solution for (1) if and only if the pencil is regular, or singular with $r > m$ and (3) holds. If the pencil is regular, by substituting the transformation

$$Y(t) = QZ(t)$$

into (1), and by multiplying by P, we obtain:

$$PEQZ'(t) = PAQZ(t)\ .$$

Let Q_p, Q_q be the matrices that contain all eigenvectors of the finite, and infinite eigenvalues respectively. Then by setting

$$Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \end{bmatrix}, \quad Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix},$$

with $Z_{p_0}(t) \in \mathbb{C}^{p_0\times 1}$, $Z_p(t) \in \mathbb{C}^{p\times 1}$, $Z_q(t) \in \mathbb{C}^{q\times 1}$, we arrive easily at the following two subsystems of (1):

$$Z_p'(t) = J_p Z_p(t)\ ;$$

$$H_q Z_q'(t) = Z_q(t)\ .$$

The first subsystem has solution:

$$Z_p(t) = e^{J_p t} Z_p(0) \ .$$

For the second subsystem let q_* be the index of the nilpotent matrix H_q, i.e. $H_q^{q_*} = 0_{q,q}$. Then we obtain the following matrix equations:

$$\begin{array}{c} H_q Z_q^{'}(t) = Z_q(t) \\ H_q^2 Z_q^{''}(t) = H_q Z_q^{'}(t) \\ H_q^3 Z_q^{'''}(t) = H_q^2 Z_q^{''}(t) \\ H_q^4 Z_q^{(4)}(t) = H_q^3 Z_q^{'''}(t) \\ \vdots \\ H_q^{q_*-1} Z_q^{(q_*-1)}(t) = H_q^{q_*-2} Z_q^{(q_*-2)}(t) \\ H_q^{q_*} Z_q^{(q_*}(t) = H_q^{q_*-1} Z_q^{(q_*-1)}(t) \ . \end{array}$$

By taking the sum of the above equations we arrive easily at the solution:

$$Z_q(t) = 0_{q,1} \ .$$

By using the solutions of the two subsystems, we obtain:

$$Y(t) = QZ(t) = \begin{bmatrix} Q_p & Q_q \end{bmatrix} \begin{bmatrix} e^{J_p t} Z_p(0) \\ 0_{q,1} \end{bmatrix} ,$$

or, equivalently,

$$Y(t) = Q_p e^{J_p t} Z_p(0) \ .$$

If the pencil is singular with $r > m$ and (3) holds, then by substituting the transformation $Y(t) = QZ(t)$ into (1) we obtain:

$$EY'(t)QZ(t) = AQZ(t) + V(t) \ ,$$

whereby, multiplying by P, using (6) and setting $Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \\ Z_\zeta(t) \end{bmatrix}$, $Z_p(t) \in \mathbb{C}^{p\times 1}$, $Z_p(t) \in \mathbb{C}^{q\times 1}$ and $Z_\zeta(t) \in \mathbb{C}^{\tilde{\zeta}_2 \times 1}$, we arrive at the subsystems

$$Z_p'(t) = J_p Z_p(t) \ ,$$

$$H_q Z_q'(t) = Z_q(t) \ ,$$

and

$$E_\zeta Z_\zeta'(t) = A_\zeta Z_\zeta(t) \ .$$

The solutions of the first two subsystems are $Z_p(t) = e^{J_p t} Z_p(0)$ and $Z_q(t) = 0_{q,1}$, respectively. For the third subsystem, let

$$Z_\zeta(t) = \begin{bmatrix} Z_{\zeta_{h+1}}(t) \\ Z_{\zeta_{h+2}}(t) \\ \vdots \\ Z_{\zeta_{h+k}}(t) \end{bmatrix}, \quad \text{with} \quad Z_{\zeta_{h+i}}(t) = \begin{bmatrix} Z_{\zeta_{h+i},1}(t) \\ Z_{\zeta_{h+i},2}(t) \\ \vdots \\ Z_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix} ,$$

where $Z_{\zeta_{h+i}}(t) \in \mathbb{C}^{(\zeta_{h+i})\times 1}, \quad i = 1, 2, ..., k$. From the analysis in (6) by replacing into the subsystem we get:

$$\begin{bmatrix} I_{\zeta_{h+i}} \\ 0_{1,\zeta_{h+i}} \end{bmatrix} Z'_{\zeta_{h+i}}(t) = \begin{bmatrix} 0_{1,\zeta_{h+i}} \\ I_{\zeta_{h+i}} \end{bmatrix} Z_{\zeta_{h+i}}(t) \ ,$$

or, equivalently, by using the above expressions:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} Z'_{\zeta_{h+i},1}(t) \\ Z'_{\zeta_{h+i},2}(t) \\ \vdots \\ Z'_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} Z_{\zeta_{h+i},1}(t) \\ Z_{\zeta_{h+i},2}(t) \\ \vdots \\ Z_{\zeta_{h+i},\zeta_{h+i}}(t) \end{bmatrix},$$

or, equivalently,

$$\begin{gathered} Z'_{\zeta_{h+i},1}(t) = 0 \ , \\ Z'_{\zeta_{h+i},2}(t) = Z_{\zeta_{h+i},1}(t) \ , \\ \vdots \\ Z'_{\zeta_{h+i},\zeta_{h+i}}(t) = Z_{\zeta_{h+i},\zeta_{h+i}-1}(t) \ , \\ 0 = Z_{\zeta_{h+i},\zeta_{h+i}}(t) \ . \end{gathered}$$

We have a system of $\zeta_{h+i}+1$ differential equations and ζ_{h+i} unknowns. Starting from the last equation we get the solutions:

$$\begin{gathered} Z_{\zeta_{h+i},\zeta_{h+i}}(t) = 0 \ , \\ Z_{\zeta_{h+i},\zeta_{h+i}-1}(t) = 0 \ , \\ Z_{\zeta_{h+i},\zeta_{h+i}-2}(t) = 0 \ , \\ \vdots \\ Z_{\zeta_{h+i},1}(t) = 0 \ . \end{gathered}$$

Hence $Z_\zeta(t) = 0_{\tilde{\zeta}_2,1}$, and

$$Y(t) = QZ(t) = \begin{bmatrix} Q_p & Q_q & Q_\zeta \end{bmatrix} \begin{bmatrix} e^{J_p t} Z_p(0) \\ 0_{q,1} \\ 0_{\tilde{\zeta}_2,1} \end{bmatrix},$$

or, equivalently,

$$Y(t) = Q_p e^{J_p t} Z_p(0) \ .$$

The proof is completed.

3 Main Results

In this section we will present our main results. As written in the previous section there exists solution for (1) when the pencil is regular, or singular with $r > m$ and (3) holds. In both cases, from Proposition 2.1, the solution is given by (7), and is related to J_p, the Jordan matrix of the finite eigenvalues, and Q_p, the matrix that contains all linear independent right eigenvectors. Let:

- $\lambda_i \in \mathbb{C}$, $i = 1, 2, ..., \nu$, be finite eigenvalue, and p_i be the rank of the corresponding Jordan block, where $\sum_{i=1}^{\nu} p_i = p$.
- the infinite eigenvalue have algebraic multiplicity q.

Theorem 3.1. We consider system (1) with a regular pencil, or a singular pencil with $r > m$ and for which (3) holds. Let λ_i, $i = 1, 2, ..., \nu$, be a finite eigenvalue of the pencil, p_i be rank of corresponding Jordan block, $\sum_{i=1}^{\nu} p_i = p$, and $u_{i,j}$, $j = 1, 2, ..., p_i$ linear independent (including the generalised) eigenvectors. Then the general solution of (1) is given by:

$$Y(t) = \sum_{i=1}^{\nu} e^{\lambda_i t} \sum_{j=1}^{p_i} \big(\sum_{k=1}^{j} c_{i,j-(k-1)} t^{k-1} \big) u_{i,j} \ , \tag{8}$$

where $c_{i,j-(k-1)} \in \mathbb{C}$, constants.

Proof. From Proposition 2.1 the solution of system (1) is given by:

$$Y(t) = Q_p e^{J_p t} Z_p(0) \ .$$

The Jordan matrix has the form:

$$J_p := J_{p_1}(\lambda_1) \oplus \cdots \oplus J_{p_\nu}(\lambda_\nu) \ ,$$

where

$$J_{p_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & \dots & 0 & 0 \\ 0 & \lambda_i & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix} \in \mathbb{C}^{p_i \times p_i}, \quad i = 1, 2, ..., \nu.$$

In addition:

$$e^{J_p t} := e^{J_{p_1}(\lambda_1) t} \oplus \cdots \oplus e^{J_{p_\nu}(\lambda_\nu) t} \ ,$$

and

$$e^{J_{p_i}(\lambda_i)} = \begin{bmatrix} e^{\lambda_i t} & e^{\lambda_i t} t & e^{\lambda_i t} \frac{t^2}{2!} & \dots & e^{\lambda_i t} \frac{t_i^p}{p_i!} \\ 0 & e^{\lambda_i t} & e^{\lambda_i t} \frac{t^2}{2!} & \dots & e^{\lambda_i t} \frac{t^{p_i - 1}}{(p_i - 1)!} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\lambda_i t} & e^{\lambda_i t} \frac{t^2}{2!} \\ 0 & 0 & \dots & 0 & e^{\lambda_i t} \end{bmatrix} \in \mathbb{C}^{p_i \times p_i}, \quad i = 1, 2, ..., \nu.$$

The matrix Q_p has as columns the p linear independent (generalized) eigenvectors, and can be written in the form:

$$Q_p = \begin{bmatrix} u_{1,p_1} & \dots & u_{1,2} & u_{1,1} & \dots & u_{\nu,p_\nu} & \dots & u_{\nu,2} & u_{\nu,1} \end{bmatrix} \ ,$$

where $u_{i,j},\ j = 1, 2, ..., p_i$ linear independent eigenvectors of $\lambda_i,\ i = 1, 2, ..., \nu$. Finally, $Z_p(0)$ can be written as:

$$Z_p(0) = \left[\begin{array}{cccccccccc} c_{1,p_1} & \dots & c_{1,2} & c_{1,1} & \dots & c_{\nu,p_\nu} & \dots & c_{\nu,2} & c_{\nu,1} \end{array}\right]^T ,$$

where $c_{i,j} \in \mathbb{C},\ u_{i,j},\ i = 1, 2, ..., \nu,\ j = 1, 2, ..., p_i$, constants. If we replace the above expressions in the general solution we arrive at (8). The proof is completed.

Corollary 3.1. We consider system (1) with a regular pencil, or a singular pencil with $r > m$ and for which (3) holds. Let the finite eigenvalues be either distinct, or with algebraic multiplicity equal to geometric, i.e., $p_i = 1$ is the rank of corresponding Jordan block. Then, in Theorem 3.1, $\nu = p$, $u_i = u_{i,j}$, and the general solution of (1) can be written as:

$$Y(t) = \sum_{i=1}^{p} u_i e^{\lambda_i t} c_i \ , \tag{9}$$

where $c_i \in \mathbb{C}$, constants.

Based on the above results, we now provide a Theorem about the participation factors of system (1).

Theorem 3.2. We consider system (1) with a regular pencil, or a singular pencil with $r > m$ and for which (3) holds. Let $\lambda_i,\ i = 1, 2, ..., \nu$, be a finite eigenvalue of the pencil, p_i be rank of corresponding Jordan block, $\sum_{i=1}^{\nu} p_i = p$, and $w_{i,j}$, $u_{i,j}$, $j = 1, 2, ..., p_i$ left, right respectively linear independent (including the generalised) eigenvectors. Then:

(a) The solution of (1) with initial condition $Y(0)$ is given by:

$$Y(t) = \sum_{i=1}^{\nu} e^{\lambda_i t} \sum_{j=1}^{p_i} \big(\sum_{k=1}^{j} t^{k-1} w_{i,j-(k-1)} E Y(0) \big) u_{i,j} \ . \tag{10}$$

(b) Let $Y_\mu(t)$ be the μ-th element of $Y(t)$. Then the participation of the h-th eigenvalue, $h = 1, 2, ..., \nu$ in $Y_\mu(t)$, $\mu = 1, 2, ..., m$, is given by:

$$\pi_{h,\mu} = \sum_{j=1}^{p_h} \big(\sum_{k=1}^{j} t^{k-1} w_{h,j-(k-1)} E Y(0) \big) u_{h,j}^{(\mu)} \ , \quad \text{(Participation Factors)} \tag{11}$$

where $u_{h,j}^{(\mu)}$ is the μ-th element of the eigenvector $u_{h,j}$.

Proof. By using the transformation $Y(t) = QZ(t)$ from the proof in Proposition 2.1, we have $Y(t) = Q_p Z_p(t)$ or, equivalently,

$$Y = Q_p Z_p \ .$$

From (5) we have that $P_1EQ_p = I_p$. By multiplying the above expression by P_1E we have:

$$P_1EY = P_1EQ_pZ_p \ ,$$

or, equivalently,

$$Z_p = P_1EY \ .$$

Hence:

$$Z_p(0) = P_1EY(0) \ .$$

The matrix P_1 has as rows the p linear independent (generalized) left eigenvectors, and can be written in the form:

$$P_1 = \begin{bmatrix} w_{1,p_1} \\ \vdots \\ w_{1,2} \\ w_{1,1} \\ \vdots \\ w_{\nu,p_\nu} \\ \vdots \\ w_{\nu,2} \\ w_{\nu,1} \end{bmatrix} .$$

Where $w_{i,j}, \ j = 1,2,...,p_i$ linear independent left eigenvectors of $\lambda_i, \ i = 1,2,...,\nu$. By replacing the above expressions into the general solution given in Theorem 3.1, we arrive at (10). Let $Y_\mu(t)$ be the μ-th element of $Y(t)$. Then (10) takes the form:

$$Y_\mu(t) = \sum_{i=1}^{\nu} e^{\lambda_i t} \sum_{j=1}^{p_i} \big(\sum_{k=1}^{j} t^{k-1} w_{i,j-(k-1)} EY(0) \big) u_{i,j}^{(\mu)} \ .$$

Furthermore:

$$\frac{\partial Y_\mu(t)}{\partial e^{\lambda_h t}} = \sum_{j=1}^{p_h} \big(\sum_{k=1}^{j} t^{k-1} w_{h,j-(k-1)} EY(0) \big) u_{h,j}^{(\mu)} \ .$$

which are the participation factors, i.e., the participation of the h-th eigenvalue, $h = 1,2,...,\nu$, in $Y_\mu(t)$, $\mu = 1,2,...,m$. The proof is completed.

Corollary 3.2. We consider system (1) with a regular pencil, or a singular pencil with $r > m$ and that (3) holds. Let the finite eigenvalues be either distinct, or with algebraic multiplicity equal to geometric, i.e., $p_i = 1$ is the rank of corresponding Jordan block. Then in Theorem 3.2, in (10) we have $\nu = p$, $u_{i,j} = u_i$, and:

(a) The solution of (1) with initial condition $Y(0)$ is given by:

$$Y(t) = \sum_{i=1}^{p} w_i E Y(0) u_i e^{\lambda_i t}.$$

(b) Let $Y_\mu(t)$ be the μ-th element of $Y(t)$. Then the participation of the h-th eigenvalue, $h = 1, 2, ..., p$ in $Y_\mu(t)$, $\mu = 1, 2, ..., m$, is given by:

$$\pi_{h,\mu} = w_h E Y(0) u_h^{(\mu)}, \quad \text{(Participation Factors)} \tag{12}$$

where $u_h^{(\mu)}$ is the μ-th element of the eigenvector u_h.

Remark 3.1. The participation factors $\pi_{h,\mu}$, as defined in Theorem 3.2, and Corollary 3.2, are elements of the matrix Π with dimension $\nu \times m$ and is called Participation Matrix.

Remark 3.2. By applying a simple Möbius transform into (1), we arrive at the system $A\hat{Y}' = E\hat{Y}$ which is the dual system of (1). Let $Y_\mu(t)$ be the μ-th element of $Y(t)$, and $\hat{Y}_\mu(t)$ be the μ-th element of $\hat{Y}(t)$. Then the participation of the infinite eigenvalue of $sE - A$ in $Y_\mu(t)$, $\mu = 1, 2, ..., m$, is equal to the participation of the zero eigenvalue of $\hat{s}A - E$ in $\hat{Y}_\mu(t)$, $\mu = 1, 2, ..., m$. This is a direct result from the duality between (1) and its dual system, or, additionally, between their pencils $sE - A$, and $\hat{s}A - E$ respectively, see [19]. As a consequence through transformation $s \longrightarrow \frac{1}{\hat{s}}$:

- A zero eigenvalue of $sE - A$ is an infinite eigenvalue of $\hat{s}A - E$;
- A non-zero finite eigenvalue λ_i defines a non-zero finite eigenvalue $\frac{1}{\lambda_i}$ of $\hat{s}A - E$;
- An infinite eigenvalue of $sE - A$ is a zero eigenvalue of $\hat{s}A - E$.

Note that an eigenvector (left, or right) of the infinite eigenvalue of $sE - A$ is also an eigenvector of the zero eigenvalue of $\hat{s}A - E$.

4 Numerical Examples

In this section we may use (11) and (12) to define the participation factors for a singular system of differential equations. Note that, in classical modal participation analysis, the participation factors, i.e. the participation of the h-th eigenvalue, $h = 1, 2, ..., \nu$, in $Y_\mu(t)$, $\mu = 1, 2, ..., m$, are conventionally determined by specifying $Y_\mu(0) = 1$, and $Y_i(0) = 0$, $i \neq \mu$, see [24].

4.1 Numerical example 1

We consider system (1) with

$$E = \begin{bmatrix} 12 & -3 & 0 & 0 & 0 \\ 4 & 1 & -1 & 3 & 0 \\ 0 & -4 & -5 & 1 & 0 \\ 8 & 2 & -5 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} -17 & 8 & -2 & 5 & 3 \\ -7 & -3 & 3 & -8 & 1 \\ 13 & 9 & 9 & 3 & 1 \\ -12 & -7 & 13 & -22 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The pencil $sE - A$ has $\nu = 2$ finite eigenvalues $\lambda_1 = -2$, $\lambda_2 = -3$, of algebraic multiplicity $p_1 = 2$, $p_2 = 1$ and infinite eigenvalues λ_3, λ_4. The geometric multiplicity κ_i of the finite eigenvalue λ_i is found as the dimension of the null space of $\lambda_i E - A$. In our case, $\kappa_1 = 1$, $\kappa_2 = 1$. The right and left eigenvectors of $sE - A$ associated with the finite eigenvalue $\lambda_1 = -2$ are:

$$u_{1,1} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ u_{1,2} = \begin{bmatrix} 0.0049 \\ -3.282 \cdot 10^7 \\ -3.282 \cdot 10^7 \\ 0 \\ 0.0049 \end{bmatrix}, \ w_{1,1} = \begin{bmatrix} -0.2308 \\ -0.3846 \\ 0.0769 \\ 0 \\ 1 \end{bmatrix}^T, \ w_{1,2} = \begin{bmatrix} -0.1426 \\ -0.2376 \\ 0.0475 \\ 0 \\ 0.6178 \end{bmatrix}^T,$$

where $u_{1,2}$, $w_{1,2}$ are generalized eigenvectors determined from $(A - \lambda_1 E)u_{12} = Eu_{11}$ and $w_{12}(A - \lambda_1 E) = w_{11}E$ respectively. The right and left eigenvectors of $sE - A$ associated with the finite eigenvalue $\lambda_2 = -3$ are:

$$u_{2,1} = \begin{bmatrix} 0 \\ 1 \\ -0.5 \\ 0 \\ 0 \end{bmatrix}, \quad w_{2,1} = \begin{bmatrix} -0.3333 \\ 1 \\ 0.1111 \\ 0 \\ -0.1111 \end{bmatrix}^T.$$

The sensitivities $\pi_{\mu,h}$ are obtained from (11) as follows:

$$\pi_{\mu,h} = \sum_{j=1}^{p_h} \Big(\sum_{k=1}^{j} t^{k-1} w_{h,j-(k-1)} EY(0) \Big) u_{h,j}^{(\mu)} .$$

For λ_1 and λ_2 we have respectively:

$$\begin{aligned} \pi_{\mu,1} &= \sum_{j=1}^{2} \Big(\sum_{k=1}^{j} t^{k-1} w_{1,j-(k-1)} EY(0) \Big) u_{1,j}^{(\mu)} \\ &= w_{1,1}EY(0)u_{1,1}^{(\mu)} + \Big(\sum_{k=1}^{2} t^{k-1} w_{1,2-(k-1)} EY(0) \Big) u_{1,2}^{(\mu)} \\ &= w_{1,1}EY(0)u_{1,1}^{(\mu)} + w_{1,2}EY(0)u_{1,2}^{(\mu)} + t w_{1,1}EY(0)u_{1,2}^{(\mu)} , \end{aligned}$$

$$\pi_{\mu,2} = w_{2,1}EY(0)u_{2,1}^{(\mu)} .$$

Consider $Y_\mu(0) = 1$, and $Y_i(0) = 0$, $i \neq \mu$, which lead to the participation factors related to the system finite modes. We have the following:

- For $\pi_{1,h}$, we have $Y(0) = \begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}^T$. Hence,

$$\pi_{1,1} = 0.0130 + 0.0209t \ , \ \pi_{1,2} = 0 \ .$$

- For $\pi_{2,h}$, we have $Y(0) = \begin{bmatrix}0 & 1 & 0 & 0 & 0\end{bmatrix}^T$. Hence,

$$\pi_{2,1} = 0.3290 + 1.0839t \ , \ \pi_{2,2} = 0.6667 \ .$$

- For $\pi_{3,h}$, we have $Y(0) = \begin{bmatrix}0 & 0 & 1 & 0 & 0\end{bmatrix}^T$. Hence,

$$\pi_{3,1} = 0.6580 + 2.1678t \ , \ \pi_{3,2} = 0.3333 \ .$$

- For $\pi_{4,h}$, we have $Y(0) = \begin{bmatrix}0 & 0 & 0 & 1 & 0\end{bmatrix}^T$. Hence,

$$\pi_{4,1} = 0 \ , \ \pi_{4,2} = 0 \ .$$

- For $\pi_{5,h}$, we have $Y(0) = \begin{bmatrix}0 & 0 & 0 & 0 & 1\end{bmatrix}^T$. Hence,

$$\pi_{5,1} = 0 \ , \ \pi_{5,2} = 0 \ .$$

The results are summarized in Table 1, where we assumed $t \to 0$. Since E is a 5×5 matrix with rank equal to 3, there exist $5 - 3 = 2$ variables the participation of which to the system finite eigenvalues is zero. These variables are Y_4 and Y_5. In addition, Table 1 shows that Y_3 is dominant in λ_1, while Y_2 is dominant in λ_2.

Table 1: Participation factors associated to finite modes.

	λ_1	λ_2
Y_1	0.0130	0
Y_2	0.3290	0.6667
Y_3	0.6580	0.3333
Y_4	0	0
Y_5	0	0

4.2 Numerical example 2

Consider the problem of a DC voltage source feeding a DC motor that drives a fan. The air-stream created by the fan is assumed to push a hanging plate. The system is described by the following set of differential algebraic equations, which are assumed to be valid around a given operating point:

$$
\begin{aligned}
L\Delta i'(t) &= -k\Delta\omega(t) - R\Delta i(t) + \Delta e(t)\\
J\Delta\omega'(t) &= \Delta T(t) - \mu\Delta\omega(t)\\
I\Delta\theta''(t) &= \Delta F(t) - mg\Delta\theta(t) - \eta\Delta\theta'(t)\\
0 &= k\Delta i(t) - \Delta T(t)\\
0 &= v\Delta T(t) - \Delta F(t)\\
0 &= \Delta e(t)\ ,
\end{aligned}
$$

where for every variable Y_i, $\Delta Y_i = Y_i - Y_i(0)$; i is the circuit DC current; ω is the angular velocity of the fan; θ is the angle of the plate with respect to the vertical axis and θ' its rate of change; T is the motor mechanical torque; F is the force applied to the plate by the air stream; and e is the DC voltage input; For the system parameters we have $L = 1$ mH; $k = 1/\pi$ V $\cdot$ s; $R = 0.2\ \Omega$; $J = 0.013$ kg $\cdot$ m^2; $\mu = 10/(6\pi^2)$ kg $\cdot$ m^2/s; $I = 0.0137$ kg $\cdot$ m; $m = 0.2$ kg; $g = 9.81$ m/s^2; $\eta = 0.216$ kg $\cdot$ m/s; $v = 0.11$ m^{-1}.

If we define the vector $\Delta Y = \begin{bmatrix}\Delta i & \Delta\omega & \Delta\theta & \Delta\theta' & \Delta T & \Delta F & \Delta e\end{bmatrix}^T$, the coefficient matrix E and the matrix A are:

$$
E = \begin{bmatrix}
0.001 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0.013 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0.0137 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

,

$$
A = \begin{bmatrix}
-0.2 & -0.3183 & 0 & 0 & 0 & 0 & 1\\
0 & -0.1689 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & -1.962 & -0.216 & 0 & 1 & 0\\
0.3183 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0.11 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

The pencil $sE - A$ has $\nu = 4$ finite eigenvalues $\lambda_1 = -137.3$, $\lambda_2 = -75.6849$, $\lambda_3 = -7.8832 + 9.0037i$, $\lambda_4 = -7.8832 - 9.0037i$, of algebraic multiplicity $p_1 = p_2 = p_3 = p_4 = 1$, and infinite eigenvalues λ_5, λ_6, λ_7. The participation factors associated with the infinite eigenvalue do not emerge when the primal problem is considered. As discussed in Remark 3.2, such participation factors can be easily calculated if we make use of the dual transform:

$$
s = \frac{1}{z}\ .
$$

The dual system is:

$$
A\Delta Y'(t) = E\Delta Y(t)\ .
$$

The eigenvectors of $zA - E$ associated with the finite eigenvalue $\dfrac{1}{\lambda_1}$ are:

$$u_{1,1} = \begin{bmatrix} -1 \\ 0.1970 \\ -0.0002 \\ 0.0208 \\ -0.3183 \\ -0.0350 \\ 0 \end{bmatrix}, \quad w_{1,1} = \begin{bmatrix} 14.6932 \\ 2.8940 \\ 0 \\ 0 \\ 2.8940 \\ 0 \\ -14.6932 \end{bmatrix}^T.$$

The eigenvectors of $zA - E$ associated with the finite eigenvalue $\dfrac{1}{\lambda_2}$ are:

$$u_{2,1} = \begin{bmatrix} -1 \\ 0.3905 \\ -0.0005 \\ 0.0413 \\ -0.3183 \\ -0.0350 \\ 0 \end{bmatrix}, \quad w_{2,1} = \begin{bmatrix} -13.4432 \\ -5.2502 \\ 0 \\ 0 \\ -5.2502 \\ 0 \\ 13.4432 \end{bmatrix}^T.$$

The eigenvectors of $zA - E$ associated with the finite eigenvalue $\dfrac{1}{\lambda_3}$ are:

$$u_{3,1} = \begin{bmatrix} 0 \\ 0 \\ 0.0551 - 0.0619i \\ -0.9917 - 0.0083i \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_{3,1} = \begin{bmatrix} (-9.8646 - 5.0416i) \cdot 10^{14} \\ (2.1886 - 1.4414i) \cdot 10^{15} \\ (-3.9631 - 2.0168i) \cdot 10^{15} \\ (-2.5179 + 1.0083i) \cdot 10^{16} \\ (-5.8112 - 3.3219i) \cdot 10^{14} \\ (-2.5179 + 1.0083i) \cdot 10^{16} \\ (9.8646 + 5.0416i) \cdot 10^{14} \end{bmatrix}^T.$$

The eigenvectors of $zA - E$ associated with the finite eigenvalue $\dfrac{1}{\lambda_4}$ are:

$$u_{4,1} = \begin{bmatrix} 0 \\ 0 \\ 0.0551 + 0.0619i \\ -0.9917 + 0.0083i \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_{4,1} = \begin{bmatrix} (-9.8646 + 5.0416i) \cdot 10^{14} \\ (2.1886 + 1.4414i) \cdot 10^{15} \\ (-3.9631 + 2.0168i) \cdot 10^{15} \\ (-2.5179 - 1.0083i) \cdot 10^{16} \\ (-5.8112 + 3.3219i) \cdot 10^{14} \\ (-2.5179 - 1.0083i) \cdot 10^{16} \\ (9.8646 - 5.0416i) \cdot 10^{14} \end{bmatrix}^T.$$

The eigenvectors of $zA-E$ associated with the finite eigenvalue $\dfrac{1}{\lambda_5} \to 0$ are:

$$u_{5,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0.11 \\ 0 \end{bmatrix}, \quad w_{5,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -0.02 \\ 0 \end{bmatrix}^T.$$

The eigenvectors of $zA-E$ associated with the finite eigenvalue $\frac{1}{\lambda_6} \rightarrow 0$ are:

$$u_{6,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.11 \\ -1 \\ 0 \end{bmatrix}, \quad w_{6,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.988 \\ 0 \end{bmatrix}^T.$$

The eigenvectors of $zA-E$ associated with the finite eigenvalue $\frac{1}{\lambda_7} \rightarrow 0$ are:

$$u_{7,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad w_{7,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T.$$

Considering $\Delta Y_\mu(0) = 1$, and $\Delta Y_i(0) = 0$, $i \neq \mu$, we calculate the participation factors of the dual system, which corresponds to the participation factors of the primal system associated with both finite and infinite eigenvalues. These are summarized in Table 2. We see that, the algebraic variables ΔT, ΔF, Δe do not participate in the finite dynamics, but are the ones that define the infinite eigenvalues of the system. Consider now the eigenvalues $\lambda_3 = -7.8832+9.0037i$, $\lambda_4 = -7.8832 - 9.0037i$. These eigenvalues represent an oscillatory mode of the dynamic system. The natural frequency of the oscillation is $f_n = \frac{9.0037}{2\pi} = 1.433$ Hz. The participation factors of Table 2 can be utilized to design a control scheme for such a mode. In particular, Table 2 suggests that effective control of the mode λ_3, λ_4 can be provided by utilizing the plate angle deviation $\Delta\theta$ or its angular speed deviation $\Delta\theta'$, which are in this case the mostly participating variables.

Table 2: Participation factors associated with finite and infinite modes.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
Δi	0.6648	0.3352	0	0	0	0	0
$\Delta\omega$	0.3352	0.6648	0	0	0	0	0
$\Delta\theta$	0	0	0.5	0.5	0	0	0
$\Delta\theta'$	0	0	0.5	0.5	0	0	0
ΔT	0	0	0	0	1	0	0
ΔF	0	0	0	0	0	1	0
Δe	0	0	0	0	0	0	1

4.3 Numerical example 3

Power system models for transient stability analysis are formulated as a set of differential algebraic equations [21]. Modal participation analysis, which is a widely utilized tool of the small-signal stability analysis of power systems, can be carried out by studying the linearized system around a valid operating point. The standard problem that commercial software tools solve to complete this task is the linear eigenvalue problem (LEP). However, a linearized power system can be also represented as a singular system of differential equations as follows:

$$E\Delta Y' = A\Delta Y \ , \tag{13}$$

where the matrices E, A formulate the generalized eigenvalue problem (GEP); and $\Delta Y = Y - Y(0)$.

Consider the simple example of a fourth order (two-axes) synchronous electrical generator connected through a transmission line to a bus, as shown in Fig. 1.

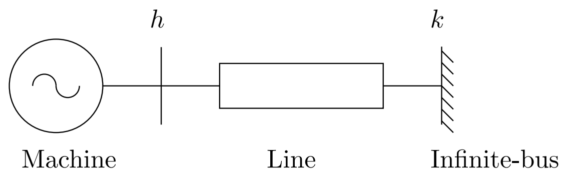

Figure 1: OMIB system.

This system, which in power systems is known as one-machine infinite-bus (OMIB) system, is widely employed for proof-of-concept in stability analysis studies. The term "infinite" refers to the fact that the voltage and the frequency at bus k are constant (infinite inertia). The non-linear equations and variables of the system are presented in Table 3. The system parameters are provided in Table 4.

The matrices that describe the linearized OMIB system are:

$$A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.41 & 0.23 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -0.70 & -0.73 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1.50 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.15 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5.19 & -5.19 & 0.71 & 0.20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5.14 & 5.14 & -0.71 & -0.19 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.20 & -0.20 & 4.89 & -5.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.72 & -0.72 & -5.09 & 5.24 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.73 & 0 & 0 & 0 & 0 & 0 & -0.73 & 0 & 0.69 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-0.70 & 0 & 0 & 0 & 0 & 0 & 0.70 & 0 & 0.72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.30 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.55 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.23 & 0.41 & 0.70 & 0.73 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.41 & 0.23 & 0.73 & -0.70 & 0 & -1
\end{bmatrix},$$

Table 3: OMIB system equations and variables

Devices	Equations	Variables
Generator	$\frac{1}{\Omega_b}\dot{\delta} = \omega - \omega_s$	δ: rotor angle
	$2H\dot{\omega} = \tau_m - \tau_e - D(\omega - \omega_s)$	ω: angular speed
	$T'_{d0}\dot{e}'_q = -e'_q - (x_d - x'_d)i_d + v_f$	τ_m: mechanical torque
	$T'_{q0}\dot{e}'_d = -e'_d + (x_q - x'_q)i_q$	τ_e: electrical torque
	$0 = -p_h + v_d i_d + v_q i_q$	e'_q: q-axis transient emf
	$0 = -q_h + v_q i_d - v_d i_q$	i_d: d-axis current
	$0 = v_h \sin(\delta - \theta_h) - v_d$	v_f: field voltage
	$0 = v_h \cos(\delta - \theta_h) - v_q$	e'_d: d-axis transient emf
	$0 = -\tau_e + \psi_d i_q - \psi_q i_d$	i_q: q-axis current
	$0 = \tau_{m0} - \tau_m$	v_d: d-axis voltage
	$0 = v_{f0} - v_f$	v_q: q-axis voltage
	$0 = r_\alpha i_d + \psi_q + v_d$	v_h: voltage at bus h
	$0 = r_\alpha i_q - \psi_d + v_q$	θ_h: voltage angle at bus h
	$0 = v_q + r_\alpha i_q - e'_q + x'_d i_d$	ψ_q: q-axis magnetic flux
	$0 = v_d + r_\alpha i_d - e'_d - x'_q i_q$	ψ_d: d-axis magnetic flux
Line	$0 = -p_h + v_h^2(g_L + g_{L,h}) - v_h v_k(g_L \cos\theta_{hk} + b_L \sin\theta_{hk})$	p_h: active power injection at bus h
	$0 = -q_h - v_h^2(b_L + b_{L,h}) - v_h v_k(g_L \sin\theta_{hk} - b_L \cos\theta_{hk})$	q_h: reactive power injection at bus h
	$0 = -p_k + v_k^2(g_L + g_{L,h}) - v_h v_k(g_L \cos\theta_{hk} - b_L \sin\theta_{hk})$	p_k: active power injection at bus k
	$0 = -q_k - v_k^2(b_L + b_{L,h}) - v_h v_k(g_L \sin\theta_{hk} + b_L \cos\theta_{hk})$, where $\theta_{hk} = \theta_h - \theta_k$.	q_k: reactive power injection at bus k
Infinite-bus	$0 = v_{G0,k} - v_k$	v_k: voltage at bus k
	$0 = \theta_{G0,k} - \theta_k$	θ_k: voltage angle at bus k

$$E = \begin{bmatrix}
0.003 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

The matrix pencil $sE-A$ has $\nu = 4$ finite eigenvalues $\lambda_1 = -0.6153+6.9086i$, $\lambda_2 = -0.6153 - 6.9086i$, $\lambda_3 = -5.4517$, $\lambda_4 = -0.1452$, of algebraic multiplicity $p_1 = p_2 = p_3 = p_4 = 1$, and infinite eigenvalues $\lambda_5, \lambda_6, \ldots, \lambda_{21}$. The geometric multiplicities are $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 1$.

The right and left eigenvectors of $sE-A$ associated with the finite eigenvalue

Table 4: OMIB system parameters

Device	Parameters
Generator	$\Omega_b = 314.16$ rad/s: base synchronous frequency, $\omega_s = 1$ pu [1] (rad/s): reference frequency, $H = 5$ MWs/MVA: inertia constant, $D = 0$ pu: damping coefficient, $T'_{d0} = 8$ s: d-axis transient time constant, $T'_{q0} = 0.4$ s: q-axis transient time constant, $x_d = 1.8$ pu (Ω): d-axis synchronous reactance, $x'_d = 0.3$ pu (Ω): d-axis transient reactance, $x_q = 1.7$ pu (Ω): q-axis synchronous reactance, $x'_q = 0.5$ pu (Ω): q-axis transient reactance, $\tau_{m0} = 0.46$ pu(MN·m): initial mechanical torque, $v_{f0} = 1.13$ pu (kV): initial field voltage, $r_\alpha = 0$ pu(Ω): armature resistance. $v_{G0,h} = 1.01$ pu (kV): initial voltage at bus h, $\theta_{G0,h} = 1.08^\circ$: initial voltage angle at bus h.
Line	$r_L = 0.01$ pu (Ω): series resistance, $g_{L,h} = 0.04$ pu (Ω^{-1}): shunt conductance of sending-end h, $x_L = 0.2$ pu (Ω): series reactance, $b_{L,h} = 0$ pu (Ω^{-1}): shunt susceptance of sending-end h, where $g_L + jb_L = (r_L + jx_L)^{-1}$.
Infinite-bus	$v_{G0,k} = 1.03$ pu (kV): initial voltage at bus k, $\theta_{G0,k} = 0^\circ$: initial voltage angle at bus k.

$\lambda_1 = -0.6153 + 6.9086i$ are:

$$u_{1,1} = \begin{bmatrix} 0.6053 - 0.0616i \\ 0.0002 + 0.0134i \\ 0.0031 + 0.0253i \\ 0.1005 - 0.1499i \\ -0.2790 + 0.0398i \\ -0.3350 + 0.0711i \\ 0.1748 + 0.0165i \\ 0 \\ 0.0308 - 0.0205i \\ 0 \\ 0 \\ -0.0309 + 0.1165i \\ -0.9212 - 0.0700i \\ 0 \\ 0 \\ 0.3350 - 0.0711i \\ -0.2790 + 0.0398i \\ 0.9404 - 0.0483i \\ 0.4262 + 0.1433i \\ 0.9291 + 0.0709i \\ 0.1861 - 0.0972i \end{bmatrix}, \quad w_{1,1} = \begin{bmatrix} 0.9867 - 0.0133i \\ -0.0011 + 0.0142i \\ 0.0004 \\ -0.0029 + 0.0020i \\ -0.0032 - 0.0128i \\ 0.0167i \\ -0.0004 - 0.0042i \\ 0 \\ 0.0005 - 0.0004i \\ 0 \\ 0 \\ -0.0024 + 0.0013i \\ -0.0022 - 0.0217i \\ -0.0011 + 0.0142i \\ 0.0004 + 0.0004i \\ -0.0029 - 0.0120i \\ 0.0001 + 0.0185i \\ -0.0004 + 0.0225i \\ 0.0035 + 0.0096i \\ 0.0004 + 0.0042i \\ -0.0005 + 0.0004i \end{bmatrix}^T.$$

The right and left eigenvectors of $sE - A$ associated with the finite eigenvalue $\lambda_2 = -0.6153 - 6.9086i$ are:

$$u_{2,1} = \begin{bmatrix} 0.6053 + 0.0616i \\ 0.0002 - 0.0134i \\ 0.0031 - 0.0253i \\ 0.1005 + 0.1499i \\ -0.2790 - 0.0398i \\ -0.3350 - 0.0711i \\ 0.1748 - 0.0165i \\ 0 \\ 0.0308 + 0.0205i \\ 0 \\ 0 \\ -0.0309 - 0.1165i \\ -0.9212 + 0.0700i \\ 0 \\ 0 \\ 0.3350 + 0.0711i \\ -0.2790 - 0.0398i \\ 0.9404 + 0.0483i \\ 0.4262 - 0.1433i \\ 0.9291 - 0.0709i \\ 0.1861 + 0.0972i \end{bmatrix}, \quad w_{2,1} = \begin{bmatrix} 0.9867 + 0.0133i \\ -0.0011 - 0.0142i \\ 0.0004 \\ -0.0029 - 0.0020i \\ -0.0032 + 0.0128i \\ -0.0167i \\ -0.0004 + 0.0042i \\ 0 \\ 0.0005 + 0.0004i \\ 0 \\ 0 \\ -0.0024 - 0.0013i \\ -0.0022 + 0.0217i \\ -0.0011 - 0.0142i \\ 0.0004 - 0.0004i \\ -0.0029 + 0.0120i \\ 0.0001 - 0.0185i \\ -0.0004 - 0.0225i \\ 0.0035 - 0.0096i \\ 0.0004 - 0.0042i \\ -0.0005 - 0.0004i \end{bmatrix}^T.$$

The right and left eigenvectors of $sE - A$ associated with the finite eigenvalue

$\lambda_3 = -5.4517$ are:

$$u_{3,1} = \begin{bmatrix} 0.3163 \\ -0.0055 \\ 0.0194 \\ 0.9614 \\ -0.1459 \\ -0.4185 \\ -0.0827 \\ 0 \\ 0.1846 \\ 0 \\ 0 \\ -1 \\ 0.2939 \\ 0 \\ 0 \\ 0.4185 \\ -0.1459 \\ 0.5511 \\ -0.9871 \\ -0.2992 \\ 0.8869 \end{bmatrix}, \quad w_{3,1} = \begin{bmatrix} -1 \\ 0.0183 \\ 0.0007 \\ 0.0467 \\ 0.0510 \\ 0.0222 \\ 0.0038 \\ 0 \\ -0.0104 \\ 0 \\ 0 \\ 0.0532 \\ 0.0174 \\ 0.0183 \\ 0.0007 \\ 0.0459 \\ 0.0230 \\ 0.0297 \\ -0.0551 \\ -0.0038 \\ 0.0104 \end{bmatrix}^T .$$

The right and left eigenvectors of $sE - A$ associated with the finite eigenvalue $\lambda_4 = -0.1452$ are:

$$u_{4,1} = \begin{bmatrix} 1 \\ -0.0005 \\ 0.8207 \\ 0.4199 \\ -0.7942 \\ -0.6091 \\ 0.0204 \\ 0 \\ -0.1509 \\ 0 \\ 0 \\ 0.7835 \\ 0.0028 \\ 0 \\ 0 \\ 0.6091 \\ -0.7942 \\ -0.0882 \\ 0.3440 \\ -0.0007 \\ -0.7346 \end{bmatrix}, \quad w_{4,1} = \begin{bmatrix} -1 \\ 0.6889 \\ -0.3825 \\ 0.2553 \\ -0.3958 \\ -0.3452 \\ -0.0040 \\ 0 \\ 0.1045 \\ 0 \\ 0 \\ -0.5273 \\ 0.0005 \\ 0.6889 \\ -0.3825 \\ -0.3519 \\ -0.3671 \\ -0.0617 \\ 0.2406 \\ 0.0040 \\ -0.1045 \end{bmatrix}^T .$$

Considering $Y_\mu(0) = 1$, and $Y_i(0) = 0$, $i \neq \mu$, we determine the participation factors associated with the finite modes and which in this example are of the form:

$$\pi_{\mu,h} = w_{h,1} E Y(0) u_{h,1}^{(\mu)} .$$

The results are summarized in Table 5. The oscillatory mode of this dynamic system is represented by the eigenvalues $\lambda_1 = -0.6153 + 6.9086i$, $\lambda_2 = -0.6153 - 6.9086i$. We see that the mostly participating variables in these eigenvalues are the rotor angle deviation $\Delta\delta$ and the rotor angular speed $\Delta\omega$. The differential equation of the generator rotor speed expresses the imbalance between its electrical and mechanical torque, see Table 3. In power engineering, such a mode is called electromechanical oscillatory mode. Stability analysis and control of electromechanical oscillations is crucial in power systems. In real world power systems, which consist of multiple generators, participation factors play an important role in identifying if an oscillatory mode is dominated by a single generator, or if different generators from (possibly) different areas are inherent to this mode.

Table 5: Participation factors associated to finite modes.

	λ_1	λ_2	λ_3	λ_4
$\Delta\delta$	0.4593	0.4593	0.0501	0.0012
$\Delta\omega$	0.4593	0.4593	0.0501	0.0012
$\Delta\tau_m$	0	0	0	0
$\Delta\tau_e$	0	0	0	0
$\Delta e'_q$	0.0199	0.0199	0.0054	0.9808
Δi_d	0	0	0	0
Δv_f	0	0	0	0
$\Delta e'_d$	0.0616	0.0616	0.8944	0.0167
Δi_q	0	0	0	0
Δv_d	0	0	0	0
Δv_q	0	0	0	0
Δv_h	0	0	0	0
$\Delta\theta_h$	0	0	0	0
$\Delta\psi_q$	0	0	0	0
$\Delta\psi_d$	0	0	0	0
Δp_h	0	0	0	0
Δq_h	0	0	0	0
Δp_k	0	0	0	0
Δq_k	0	0	0	0
Δv_k	0	0	0	0
$\Delta\theta_k$	0	0	0	0

Conclusions

In this article we provide a method to measure the participation of the h-th eigenvalue of the pencil of system (1) in $Y_\mu(t)$, the μ-th element of $Y(t)$. The method is necessarily a generalization of the conventional participation analysis problem for singular systems of differential equations with singular or regular pencils and eigenvalue multiplicities. All cases of finite and infinite eigenvalues are covered, by taking into account their algebraic and geometric multiplicity. A methodology to determine the participation factors associated with infinite eigenvalues is also provided. Numerical examples are given including a classical DC circuit and a 2-bus power system dynamic model. We will dedicate future work to study participation factors of systems of fractional discrete operators, see [7], [10], singular systems of fractional differential equations, see [9], [12], and fuzzy systems of differential equations, see [25], [27]. We also aim to further investigate applications of our approach for large-scale power systems.

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