Algorithmic Construction of Lyapunov Functions for Power System Stability Analysis

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***Abstract*—We present a methodology for the algorithmic construction of Lyapunov functions for the transient stability analysis of classical power system models. The proposed methodology uses recent advances in the theory of positive polynomials, semidefinite programming, and sum of squares decomposition, which have been powerful tools for the analysis of systems with polynomial vector fields. In order to apply these techniques to power grid systems described by trigonometric nonlinearities we use an algebraic reformulation technique to recast the system's dynamics into a set of polynomial differential algebraic equations. We demonstrate the application of these techniques to the transient stability analysis of power systems by estimating the region of attraction of the stable operating point. An algorithm to compute the local stability Lyapunov function is described together with an optimization algorithm designed to improve this estimate.**

***Index Terms*—Nonlinear systems, power system transient stability, sum of squares, Lyapunov methods, transient energy function.**

I. Introduction

A traditional approach to transient stability analysis of power systems involves the numerical integration of the nonlinear differential equations describing the complicated interactions between its components. This method provides an accurate description of transient phenomena but its computational cost prevents time-domain simulations from providing real-time transient stability assessments and significantly constraints the number of cases which can be analyzed [1].

Alternative approaches to transient stability analysis have been intensively explored [1]–[5]. Among the methods proposed, the so-called direct methods avoid the expensive time-domain integration of the postfault system dynamics. These methods rely on the estimation of the stability domain of the postfault equilibrium point. If the initial state of the postfault system lies inside this stability domain, then we can assert without numerically integrating the postfault trajectory that the system will eventually converge to its postfault equilibrium point. This inference is made by comparing the value of a carefully chosen scalar state function (energy and Lyapunov functions) at the clearing time to a critical value. In practice, finding analytical energy and Lyapunov functions for transient stability analysis has encountered significant difficulties. For example, the energy function approach to transient stability analysis relies on two strong requirements. First, we should be able to define an analytic energy function. This condition is generally violated in practice since energy functions for power systems with transfer conductances do not exist [5], [6]. Thus, for systems with losses, no analytical expressions are available for the estimated stability boundary of the operating point. Second, we should reliably compute the critical energy value. This task is also very difficult and can provide inaccurate stability assessments if it returns the wrong critical value [7]. The *closest* Unstable Equilibrium Point (UEP) method provides sufficient but not necessary conditions for stability and is conservative. This method requires the identification of *all* equilibrium points located on the boundary of the stability region. This requires a significant computational effort and it is impractical, but it offers mathematical guarantees. The *controlling* UEP provides less conservative estimates of the stability boundary than the closest UEP. It is generally very difficult to find the controlling UEP relative to the fault-on trajectory [7]. Nevertheless, a systematic method called the boundary of stability region based controlling UEP method (BCU method) has been developed to find this point [8], [9]. Extensive numerical simulations have found counter-examples [10] where the BCU method fails to give the correct answer, predicting stability for systems that in fact suffer from second-swing instability. Furthermore, it has been shown that the mathematical assumptions of the BCU method do not hold generically and that the theoretical guarantees for the BCU method are, at best, questionable [6], [11].

On the other hand the Lyapunov function approach to transient stability analysis has been traditionally considered very difficult due to the lack of a systematic methodology for constructing a Lyapunov function — see [12]–[14] for details and a systematic survey of Lyapunov functions in power system stability. The method of Zubov is an exception and, in principle, can find a Lyapunov function and determine the exact boundary of the Region Of Attraction (ROA). This method requires the solution of a Partial Differential Equation (PDE) which does not possess in general a closed form solution. Moreover, for power system models, the existence of transfer conductances has proven again to be a serious difficulty. This is the case, for example, when using the multivariable Popov stability criterion. This method can also construct a genuine Lyapunov function, but requires the satisfaction of sector conditions that break down in the presence of transfer conductances — see, for example, [15]–[17] and references

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therein.

More recent results in the literature, using a passivity-based control methodology (see [18] and references therein), show the existence of Lyapunov functions for small, but unspecified, transfer conductances and require the solution of a formidable system of PDEs (additionally, the angle differences in equilibrium are also required to be small). Another recent result [19], which is close methodologically to our approach, estimates the ROA for non-polynomial systems using truncated Taylor expansions and semidefinite programming optimizations — see also [20] for a comprehensive description of Sum Of Squares (SOS) programming techniques for the estimation and control of the domain of attraction of equilibrium points. Alternatively, the method in [21] shows that a local energy-like Lyapunov function exists, in general, for stable systems with transfer conductances. Since these results are local in character, they can only determine the stability of the equilibrium point and cannot be used to determine the domain of attraction. They cannot be used in transient stability assessments or in estimating the critical clearing time. The method in [22] proposes a procedure to construct Lyapunov functions for power systems with transfer conductances using dissipative systems theory for large scale interconnected systems. This approach is the only one that we found in the literature where the condition of small transfer conductances is not necessary. Nevertheless, it still contains some restrictive sector conditions on the nonlinearities which translates into conditions on the angle differences in equilibrium. It also contains many parameters that have to be finely tuned in order for the method to converge. The method in [23] uses an extension of LaSalle's Invariance Principle to find extended Lyapunov functions for power systems with transfer conductances. The derivative of the extended Lyapunov function is not required to be always negative semidefinite and can take positive values in some bounded regions. This is a very interesting and promising approach. Moreover, the authors propose a generic Lyapunov function for multimachine systems. The conditions in the Extended Invariance Principle require that the transfer conductances be small in order for the domain in which the derivative is positive to be included in the bounded domain defined by the Lyapunov function. Usually, these domain inclusions are very difficult to compute numerically and the assumption that the transfer conductances are small is necessary in order to guarantee these conditions.

The main contribution of this paper is twofold. First, we introduce an algorithm that constructs Lyapunov functions for classical power system models. Second, we embed this algorithm into an optimization loop which seeks to maximize the estimate of the region of attraction of the stable operating point. Our approach exploits recent system analysis methods that have opened the path toward the algorithmic analysis of nonlinear systems using Lyapunov methods [24]–[30]. We introduce three critical steps that are necessary in this formulation. For dynamical systems described by polynomial vector fields, the first step is to relax the non-negativity conditions in Lyapunov's theorem to appropriate Sum Of Squares (SOS) conditions which can be tested efficiently using semidefinite programming (SDP) [24]. The SOS technique cannot be applied directly to power grid systems since they are not defined by polynomial vector fields. Hence, the second step is to generalize the SOS formulation to non-polynomial systems using a procedure which recasts the original non-polynomial system into a set of polynomial differential algebraic equations [27]. Finally, since the recasted system evolves over algebraic equality constraints, we employ a fundamental theorem from real algebraic geometry [31] in order to provide a convex relaxation of the equality and inequality conditions required by Lyapunov's theorem in this case [25]. The proposed algorithm is used to find Lyapunov functions and estimates of the Region Of Attraction (ROA) for two power system models. We formulate an optimization algorithm that searches over the space of polynomial Lyapunov functions in order to improve these estimates. For the power system model without transfer conductances we compare the performance of the proposed algorithm to the energy function method. We apply the same analysis to the power system with transfer conductances for which an exact energy function does not exist but for which a Lyapunov function has been proposed in the literature. A critical discussion of the method is also presented. Extensions and a discussion of the steps required to generalize this analysis to large scale systems are also described. The SOS programming concepts introduced in this paper are not new but, to the best of our knowledge, they have never been applied to the transient stability analysis of power systems.

II. Classical Power System Model for Transient Stability Analysis

We will consider a power system consisting of n synchronous generators. Each generator is represented by a constant voltage behind a transient reactance, constant mechanical power, and its dynamics are modeled by the swing equation. The generator voltages are denoted by $E_1\angle\delta_1, \ldots, E_n\angle\delta_n$, where $\delta_1, \ldots, \delta_n$ are the generator phase angles with respect to the synchronously rotating frame. The magnitudes $E_1, \ldots, E_n$ are held constant during the transient in classical stability studies. Furthermore, the loads are represented as constant, passive impedances. Thus, the post fault mathematical model for this system is described by the following set of nonlinear differential equations [4]

$$\dot{\delta}_i = \omega_i \,, \tag{1a}$$

$$\dot{\omega}_i = -\lambda_i \omega_i + \frac{1}{M_i}(P_{mi} - P_{ei}(\delta)) \,, \tag{1b}$$

where M_i is the generator inertia constant, $\lambda_i = D_i/M_i$, where D_i is the generator damping coefficient, P_{mi} is mechanical power input, and P_{ei} is the electrical power output ,

$$\begin{aligned} P_{ei}(\delta) = E_i^2 G_{ii} + \sum_{j,j\neq i} E_i E_j [&B_{ij}\sin(\delta_i - \delta_j) \\ &+ G_{ij}\cos(\delta_i - \delta_j)] \,, \end{aligned} \tag{2}$$

where B_{ij} and G_{ij} are the line admittances and conductances.

We assume that the dynamical system has a post-fault Stable Equilibrium Point (SEP) given by $(\delta_s, \omega_s = 0)$ where δ_s is the solution of the following set of nonlinear equations,

$$P_{mi} - P_{ei}(\delta_s) = 0, \tag{3}$$

where $i = 1, \ldots, (n-1)$. Since the solution δ_s is invariant to a uniform translation of the angles ($\delta_s \to \delta_s + c$, where c is a constant), we work with the relative angles with respect to a reference node, for example, node n. Thus, the angle subspace has dimension $n-1$ and the one-dimensional equilibrium manifold collapses to a point in an $m = 2n-1$ phase space. Moreover, in the presence of uniform damping ($\lambda_i = \lambda, i = 1, \ldots, n$), including zero damping, we can further reduce the phase space by working with relative speeds. When this is the case the phase space dimension is $m = 2n-2$. The changes that we need to make to the equations of motion (1) in order to describe the dynamics of the relative angles and speeds are obvious [4] and are not explicitly presented here. Finally, we make the following change of variables $\delta \to \delta + \delta_s$ in (1) in order to transfer the stable equilibrium point to the origin in phase space.

A. Model A: Power System Without Transfer Conductances

The first model has no transfer conductances and it is a power system model which is discussed extensively in [5]. This example represents a three-machine system with machine number 3 as the reference machine:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\sin(x_1) - 0.5\sin(x_1 - x_3) - 0.4x_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -0.5\sin(x_3) - 0.5\sin(x_3 - x_1) - 0.5x_4 + 0.05
\end{aligned}
$$

where $x_1 = \delta_1$, $x_2 = \omega_1$, $x_3 = \delta_2$, and $x_4 = \omega_2$. Since there are no cosine terms in these equations, they model a lossless system for which $G_{ij} = 0$ in (2). The point $x_s = (0.02, 0, 0.06, 0)$ is a SEP for this system. Using a change of variables, $x \to x + x_s$, we shift the equilibrium point at the origin. The dynamic equations in shifted coordinates are:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 0.0200\cos(x_1)\cos(x_3) - 0.0200\cos(x_1) \\
&\quad - 0.9998\sin(x_1) - 0.4000x_2 \\
&\quad + 0.4996\cos(x_1)\sin(x_3) - 0.4996\cos(x_3)\sin(x_1) \\
&\quad + 0.0200\sin(x_1)\sin(x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= 0.4996\cos(x_3)\sin(x_1) - 0.0299\cos(x_3) \\
&\quad - 0.4991\sin(x_3) - 0.0200\cos(x_1)\cos(x_3) \\
&\quad - 0.4996\cos(x_1)\sin(x_3) - 0.5000x_4 \\
&\quad - 0.0200\sin(x_1)\sin(x_3) + 0.0500\,.
\end{aligned}
$$

This model has an energy function [5] whose expression in shifted coordinates is given by

$$
\begin{aligned}
V(x) &= x_2^2 + x_4^2 - 0.100x_3 - 1.9996\cos(x_1) \\
&\quad - 0.9982\cos(x_3) + 0.0400\sin(x_1) + 0.0600\sin(x_3) \\
&\quad - 0.9992\cos(x_1)\cos(x_3) + 0.0400\cos(x_1)\sin(x_3) \\
&\quad - 0.0400\cos(x_3)\sin(x_1) - 0.9992\sin(x_1)\sin(x_3) \\
&\quad - 0.0060\,.
\end{aligned}
$$

We will use both the closest UEP and the BCU method to estimate the region of attraction and to compare these results with the estimate obtained using SOS techniques.

B. Model B: Power System With Transfer Conductances

The second model has transfer conductances and represents a two-machine versus infinite bus system which has been discussed in [23]:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 33.5849 - 1.8868\cos(x_1 - x_3) - 5.2830\cos(x_1) \\
&\quad - 16.9811\sin(x_1 - x_3) - 59.6226\sin(x_1) \\
&\quad - 1.8868x_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= 11.3924\sin(x_1 - x_3) - 1.2658\cos(x_1 - x_3) \\
&\quad - 3.2278\cos(x_3) - 1.2658x_4 - 99.3671\sin(x_3) \\
&\quad + 48.4810
\end{aligned}
$$

where $x_1 = \delta_1$, $x_2 = \omega_1$, $x_3 = \delta_2$, and $x_4 = \omega_2$. This model has a stable equilibrium point at $x_s = (0.4680, 0, 0.4630, 0)$. The dynamic equations in shifted coordinates are:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 16.9715\cos(x_1)\sin(x_3) - 31.6131\cos(x_1) \\
&\quad - 50.8269\sin(x_1) - 1.9718\cos(x_1)\cos(x_3) \\
&\quad - 1.8868x_2 - 16.9715\cos(x_3)\sin(x_1) \\
&\quad - 1.9718\sin(x_1)\sin(x_3) + 33.5849 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= 11.3986\cos(x_3)\sin(x_1) - 47.2723\cos(x_3) \\
&\quad - 87.4618\sin(x_3) - 1.2088\cos(x_1)\cos(x_3) \\
&\quad - 11.3986\cos(x_1)\sin(x_3) - 1.2658x_4 \\
&\quad - 1.2088\sin(x_1)\sin(x_3) + 48.4810\,.
\end{aligned}
$$

In [23] an analytical Lyapunov function, $W(x)$, is proposed based on the extension of LaSalle's invariance principle — the expression for $W(x)$ is too long to reproduce here. The estimated ROA provided in [23] will be compared to the estimate obtained in this paper using SOS techniques.

III. Problem Formulation

We assume that our dynamical system is described by an autonomous set of nonlinear equations (1) which we write concisely as:

$$
\dot{x} = f(x)\,, \tag{4}
$$

where $x \in \mathbb{R}^m$ is the state vector and the vector field $f : \mathbb{R}^m \to \mathbb{R}^m$ satisfies the smoothness conditions for the existence and uniqueness of solutions. For the classical n generator model $m = 2(n-1)$ in the presence of uniform damping and $m = 2n-1$ otherwise. We assume without loss of generality that the origin is a SEP for this system, i.e. $x_s = 0$ and $f(x_s) = 0$.

We are now in a position to formulate the transient stability analysis problem. Assume that at the end of a disturbance controlled by fault-on dynamics, different from (4), the system reaches the state x_{cl} when the disturbance is finally cleared and its dynamics controlled by (4). The transient stability question is whether the trajectory $x(t)$ for (4) with initial conditions $x(0) = x_{cl}$ will converge to the stable equilibrium point of interest, i.e. $x_s = 0$, as time t goes to infinity. Mathematically,

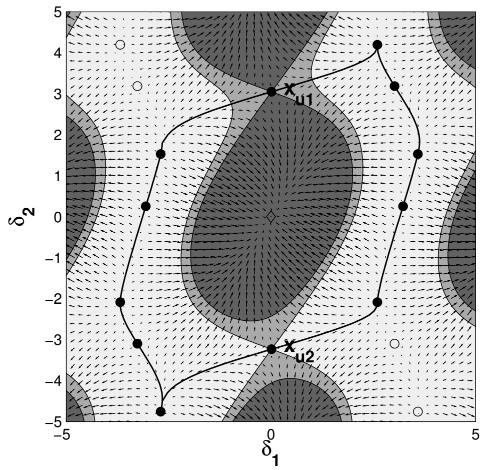

Fig. 1. Model A: The boundary of the region of attraction for the SEP x_s located at the origin (◇). This boundary contains 12 hyperbolic UEPs (•). The gray areas denote various estimates of the ROA based on energy function methods (see text for details).

we can answer this question by deciding if x_{cl} belongs to the ROA of x_s, defined as

$$A(x_s) = \{x \in \mathbb{R}^m \mid \lim_{t\to\infty} \phi(x,t) = x_s\}$$

where $\phi(x,t)$ is the system trajectory starting from x at time $t = 0$. The boundary of the stability region $A(x_s)$ is called the stability boundary of x_s and is denoted by $\partial A(x_s)$.

In order to estimate the stability region, or region of attraction (ROA), of the SEP x_s a mathematical characterization of its stability boundary $\partial A(x_s)$ is necessary. Under some generic mathematical conditions, it can be shown that for a fairly large class of nonlinear autonomous dynamical systems the stability boundary consists of the union of the stable manifolds of all unstable equilibrium points (and/or closed orbits) on the stability boundary [5], [32], [33].

For example, for model A, Fig. 1 shows the intersection of the stability boundary $\partial A(x_s)$ with the angle subspace $\{\delta_1, 0, \delta_2, 0\}$. There are 12 hyperbolic equilibrium points (•) lying on the stability boundary of x_s (◇) — the hyperbolicity of equilibrium points of the classical power system model is generic [5]. Four more UEPs are also shown (∘). The closest UEP x_{u1} defines a set $\{x \mid V(x) < V(x_{u1})\}$ which contains multiple connected components (dark gray areas). The connected component containing the SEP x_s estimates its stability region according to the closest UEP method. If the fault-on trajectory $x_f(t)$ intersects the stability boundary $\partial A(x_s)$ by crossing the stable manifold of x_{u2}, then this point is the controlling UEP relative to the fault-on trajectory. The set defined by $\{x \mid V(x) < V(x_{u2})\}$ (light gray areas) defines a *local* approximation to the stability boundary for all fault-on trajectories which intersect the stable manifold of x_{u2}.

While these mathematical results enable the exact computation of the stability region, the algorithmic implementation is numerically very expensive and often inaccurate. In particular, this approach requires the identification of all equilibrium points, which is extremely difficult for large-scale nonlinear dynamical systems. Moreover, the algorithm also needs to identify those equilibrium points whose unstable manifolds contain trajectories approaching the SEP and numerically expensive time-domain simulations are required to accomplish this task. For these reasons a number of methods have been proposed to approximate the ROA of stable equilbrium points. The so called direct methods use Lyapunov and energy functions to infer information about the system stability from the state of the system at the beginning of its post-fault phase.

IV. Lyapunov Function Theory

The use of Lyapunov functions for direct transient stability analysis relies on a stability theorem formulated by Lyapunov. This theorem defines the following sufficient conditions for the stability of the equilibrium point for the system (4) [34]:

Theorem 1 (Lyapunov): If there exists an open set $D \subset \mathbb{R}^m$ containing the equilibrium point $x = 0$ and a continuously differentiable function $V : D \to \mathbb{R}$ such that $V(0) = 0$ and

$$V(x) > 0\,, \quad \forall x \in D \setminus \{0\}\,, \tag{5a}$$

$$-\dot{V}(x) = -\left(\frac{\partial V}{\partial x}\right)^T \cdot f(x) \geq 0\,, \quad \forall x \in D\,, \tag{5b}$$

then $x = 0$ is a *stable* equilibrium point. Moreover, if $-\dot{V}(x)$ is *positive definite* in D then $x = 0$ is an *asymptotically stable* equilibrium of (4). In addition, any region $\Omega_c = \{x \in \mathbb{R}^m \mid V(x) \leq c\}$ such that $\Omega_c \subset D$ describes a positively invariant region contained in the ROA of the equilibrium point.

The continuously differentiable function V is called a *Lyapunov function* — the energy function is generally not a Lyapunov function, except in very specific cases. For a given Lyapunov function, the largest Ω_c region offers the best estimate of the region of attraction of the equilibrium point. Since the theorem leaves complete freedom in selecting both a Lyapunov function V and a domain D, an optimization algorithm that searches over V and D in order to maximize the estimate of the ROA will be formulated in Section VII.

The difficulties encountered in the application of Lyapunov theorem stem from the positivity conditions required in the theorem, which are notoriously difficult to test. Even in cases when both the vector field f and the Lyapunov function candidate V are polynomial, the Lyapunov conditions are essentially polynomial non-negativity conditions which are known to be $\mathcal{NP}$-hard to test [35]. Fortunately, as has been pointed out in [24], if we relax the polynomial non-negativity conditions to appropriate polynomial sum of squares (SOS) conditions, testing SOS conditions can then be done efficiently using semidefinite programming (SDP), as we discuss in Appendix A. To illustrate this point let us assume that $D = \mathbb{R}^m$ in Theorem 1. Then, the conditions of Theorem 1 become sufficient global stability conditions. They can be reformulated as SOS conditions as follows:

Proposition 1: Suppose that for the system (4) there exists a polynomial $V(x)$ of degree $2d$ such that $V(0) = 0$ and

$$V(x) - \phi_1(x) \in \Sigma_m\,, \tag{6a}$$

$$-\dot{V}(x) \in \Sigma_m\,, \tag{6b}$$

where Σ_m is the set of all SOS polynomials in m variables and $\phi_1(x) = \epsilon \sum_{i=1}^{m} \sum_{j=1}^{d} x_i^{2j}$, with $\epsilon > 0$, was introduced to guarantee the positive definiteness of V. Then, $x = 0$ is a globally stable equilibrium point. If we replace the second condition with $-\dot{V}(x) - \phi_2(x)$ is SOS, where $\phi_2(x) > 0$, for $x \neq 0$, then $x = 0$ is globally asymptotically stable.

The software SOSTOOLS [36], [28], in conjunction with a semidefinite programming solver such as SeDuMi [37], can be used to efficiently solve the LMIs that appear in the SOS conditions (6). For examples and extensions see [25], [28], [29], [36]. All the SOS programs formulated in this paper were solved using the freely-available MATLAB toolboxes SOSTOOLS, Version 2.0 [36], and SeDuMi, Version 1.1 [37].

V. Recasting the Power System Dynamics

SOS programming methods cannot be directly applied to study the stability of power system models because their dynamics contain trigonometric nonlinearities and are not polynomial. For this reason a systematic methodology to recast their dynamics into a polynomial form is necessary [25], [27]. It has been shown in [38] that any system with non-polynomial nonlinearities can be converted to a polynomial system with a larger state dimension. The recasting introduces a number of equality constraints restricting the states to a manifold having the original state dimension. For the classical power system model introduced in Section II recasting is trivially achieved by a non-linear change of variables

$$z_{3i-2} = \sin(x_{2i-1}) \tag{7a}$$
$$z_{3i-1} = 1 - \cos(x_{2i-1}) \tag{7b}$$
$$z_{3i} = x_{2i}\,, \tag{7c}$$

for $i = 1, \ldots, n-1$. Here we assume a model with uniform damping so that $x_{2i-1} = \delta_i - \delta_n$ and $x_{2i} = \omega_i - \omega_n$ represent the relative angles and speeds of the generators. Recasting produces a dynamical system with a larger state dimension, $z \in \mathbb{R}^M$, where $M = 3(n-1)$ for a model with uniform damping. When the damping is not uniform $M = 2(n-1)+n$, $x_{2i} = \omega_i$, and the recasted variables include $z_{3n-2} = \omega_n$ in addition to (7). Recasting also introduces $(n-1)$ equality constraints,

$$G_i(z) = z_{3i-2}^2 + z_{3i-1}^2 - 2z_{3i-1} = 0\,, \tag{8}$$

where $i = 1, \ldots, n-1$, which restrict the dynamics of the new system to a nonlinear manifold of dimension m in $\mathbb{R}^M$.

Note that we have chosen the recasted variables in such a way that the stable equilibrium point of the original system, $x_s = 0$, is mapped to $z_s = 0$ in the recasted system space.

A. Recasting the dynamics of Model A

Let us consider first the differential equations describing the dynamics of model A. We define the new state variables $z_1 = \sin(x_1)$, $z_2 = 1 - \cos(x_1)$, $z_3 = x_2$, and $z_4 = \sin(x_3)$, $z_5 = 1 - \cos(x_3)$, $z_6 = x_4$. The dynamics for these new state variables can be derived from the model equations by using the chain rule of differentiation and by replacing everywhere in the derived equations $\sin(x_1), \cos(x_1), x_2$ with z_1, z_2, z_3, and $\sin(x_3), \cos(x_3), x_4$ with z_4, z_5, z_6. Thus, we obtain the following dynamical system

$$\dot{z}_1 = z_3 - z_2 z_3 \tag{9a}$$
$$\dot{z}_2 = z_1 z_3 \tag{9b}$$
$$\begin{aligned}\dot{z}_3 &= 0.5z_4 - 0.4z_3 - 1.5z_1 - 0.02z_5 + 0.02z_1z_4 \\ &\quad + 0.5z_1z_5 - 0.5z_2z_4 + 0.02z_2z_5\end{aligned} \tag{9c}$$
$$\dot{z}_4 = z_6 - z_5 z_6 \tag{9d}$$
$$\dot{z}_5 = z_4 z_6 \tag{9e}$$
$$\begin{aligned}\dot{z}_6 &= 0.5z_1 + 0.02z_2 - 1.00z_4 + 0.05z_5 - 0.5z_6 \\ &\quad - 0.02z_1z_4 - 0.5z_1z_5 + 0.5z_2z_4 - 0.02z_2z_5\end{aligned} \tag{9f}$$

The dynamics are constrained by the following equations,

$$G_1(z) = z_1^2 + z_2^2 - 2.0z_2 = 0 \tag{10a}$$
$$G_2(z) = z_4^2 + z_5^2 - 2.0z_5 = 0\,, \tag{10b}$$

which restrict the evolution of the new system in its 6-dimensional state space to a 4-dimensional manifold.

B. Recasting the dynamics of Model B

The reacasted dynamics of model B is given by

$$\dot{z}_1 = z_3 - z_2 z_3 \tag{11a}$$
$$\dot{z}_2 = z_1 z_3 \tag{11b}$$
$$\begin{aligned}\dot{z}_3 &= 33.6z_2 - 67.8z_1 - 1.89z_3 \\ &\quad + 16.9715z_4 + 1.9718z_5 - 1.9718z_1z_4 \\ &\quad + 16.9715z_1z_5 - 16.9715z_2z_4 - 1.9718z_2z_5\end{aligned} \tag{11c}$$
$$\dot{z}_4 = z_6 - z_5 z_6 \tag{11d}$$
$$\dot{z}_5 = z_4 z_6 \tag{11e}$$
$$\begin{aligned}\dot{z}_6 &= 11.3986z_1 + 1.2088z_2 - 98.8604z_4 \\ &\quad + 48.4810z_5 - 1.2658z_6 - 1.2088z_1z_4 \\ &\quad - 11.3986z_1z_5 + 11.3986z_2z_4 - 1.2088z_2z_5\,,\end{aligned} \tag{11f}$$

while its constraints are defined by the following equalities:

$$G_1(z) = z_1^2 + z_2^2 - 2.0z_2 = 0 \tag{12}$$
$$G_2(z) = z_4^2 + z_5^2 - 2.0z_5 = 0\,. \tag{13}$$

For both models recasting produces a system whose dynamics are described by polynomial Differential Algebraic Equations (DAE).

VI. Analysis of Recasted Models

We have just shown that for a classical power system consisting of n generators recasting is trivially achieved by a non-linear change of variables (7), which we write as

$$z = H(x)\,, \tag{14}$$

with $H : \mathbb{R}^m \to \mathbb{R}^M$. Recasting produces a dynamical system whose dynamics are modeled by polynomial DAE

$$\dot{z} = F(z) \tag{15a}$$
$$0 = G(z)\,, \tag{15b}$$

where $z \in \mathbb{R}^M$, and $F : \mathbb{R}^M \to \mathbb{R}^M$, and $G : \mathbb{R}^{2(n-1)} \to \mathbb{R}^{n-1}$ are vectors of polynomial functions.

In the new state space we assume a semi-algebraic domain $\tilde{D}$ defined by the following inequality and equality constraints,

$$\tilde{D} = \{z \in \mathbb{R}^M \mid \beta - p(z) \geq 0, G(z) = 0\}\,, \tag{16}$$

with $p(z)$ a positive definite polynomial and $\beta > 0$ to ensure that $\tilde{D}$ is connected and contains the origin. For the recasted system (15) the following extension of Theorem 1 provides sufficient conditions that guarantee the existence of a Lyapunov function for the original non-polynomial system [27]:

Theorem 2: If there exists an open set $\tilde{D} \subset \mathbb{R}^M$ containing the equilibrium point $z = 0$ and a continuously differentiable function $\tilde{V} : \tilde{D} \to \mathbb{R}$ such that $\tilde{V}(0) = 0$, and

$$\tilde{V}(z) > 0\,, \forall z \in \{\beta - p(z) \geq 0, G(z) = 0\} \diagdown \{0\}\,, \tag{17}$$

$$-\dot{\tilde{V}}(z) > 0\,, \forall z \in \{\beta - p(x) \geq 0, G(z) = 0\} \diagdown \{0\}\,, \tag{18}$$

then $z = 0$ is an asymptotically stable equilibrium of (15). Moreover, any region $\Omega_c = \{z \in \mathbb{R}^M \mid \tilde{V}(z) \leq c\}$ such that $\Omega_c \in \tilde{D}$ describes a positively invariant region contained in the ROA of the equilibrium point.

This theorem expresses the fact that $\tilde{V}(z)$ only needs to be positive on the domain $\tilde{D}$ defined by (16). Finally, $V(x) = \tilde{V}(H(x))$ is a Lyapunov function for the original non-polynomial system.

A. Local Stability Analysis

The conditions of Theorem 2 for asymptotic stability can be formulated as set inclusion conditions:

$$\begin{aligned} \{z \in \mathbb{R}^M \,|\beta - p(z) \geq 0, G(z) = 0\} \diagdown \{0\} \subseteq \\ \{z \in \mathbb{R}^M \mid \tilde{V}(z) > 0\} \end{aligned} \tag{19a}$$

$$\begin{aligned} \{z \in \mathbb{R}^M \,|\beta - p(z) \geq 0, G(z) = 0\} \diagdown \{0\} \subseteq \\ \{z \in \mathbb{R}^M \mid \dot{\tilde{V}}(z) < 0\}\,. \end{aligned} \tag{19b}$$

If we can find a constant $\beta > 0$ and a $\tilde{V}(z)$ to satisfy these conditions then system (15) is asymptotically stable about the fixed point $z = 0$. We assume that the positive polynomial $p(z)$ defining the level sets of the domain $\tilde{D}$ is fixed.

We further replace the non-polynomial constraint $z \neq 0$ with $l_1(z) \neq 0$ and $l_2(z) \neq 0$, where $l_1, l_2 \in \Sigma_M$, and formulate the conditions (19) as the following set emptiness conditions:

$$\begin{aligned} \{z \in \mathbb{R}^M \,|\beta - p(z) \geq 0, G(z) = 0, \\ l_1(z) \neq 0, \tilde{V}(z) \leq 0\} = \emptyset \end{aligned} \tag{20a}$$

$$\begin{aligned} \{z \in \mathbb{R}^M \,|\beta - p(z) \geq 0, G(z) = 0, \\ l_2(z) \neq 0, \dot{\tilde{V}}(z) \geq 0\} = \emptyset\,. \end{aligned} \tag{20b}$$

According to the Positivstellensatz (P-satz) theorem discussed in Appendix B, these conditions hold if and only if we can find $\tilde{V}(z)$ and $\tilde{f}_1 \in \mathcal{C}(\beta - p(z), -\tilde{V}(z))$, $\tilde{f}_2 \in \mathcal{C}(\beta - p(z), \dot{\tilde{V}}(z))$, $\tilde{g}_1 \in \mathcal{M}(l_1(z))$, $\tilde{g}_2 \in \mathcal{M}(l_2(z))$, and $\tilde{h}_{1,2} \in \mathcal{I}(G(z))$ such that

$$\tilde{f}_1 + \tilde{g}_1^2 + \tilde{h}_1 = 0 \tag{21}$$

$$\tilde{f}_2 + \tilde{g}_2^2 + \tilde{h}_2 = 0\,. \tag{22}$$

Using the definitions of the cone $\mathcal{C}$, monoid $\mathcal{M}$, and ideal $\mathcal{I}$, we can rewrite these set emptiness constraints as a search for $\tilde{V}(z)$, $s_1, \ldots, s_8 \in \Sigma_M$, $\lambda_{1,2} \in \mathcal{R}_M^{n-1}$, and $k_{1,2} \in \mathbb{Z}_+$ such that

$$\begin{aligned} s_1 + s_2(\beta - p) - s_3\tilde{V} - s_4(\beta - p)\tilde{V} \\ + \lambda_1^T G + l_1^{2k_1} = 0 \end{aligned} \tag{23a}$$

$$\begin{aligned} s_5 + s_6(\beta - p) + s_7\dot{\tilde{V}} + s_8(\beta - p)\dot{\tilde{V}} \\ + \lambda_2^T G + l_2^{2k_2} = 0 \end{aligned} \tag{23b}$$

Note that $\lambda_{1,2}$ are $(n-1)$-dimensional vectors of polynomials in $\mathcal{R}_M$.

To limit the degree of the polynomials, and implicitly the size of the SOS program, we select $k_1 = k_2 = 1$. To further reduce the size of the SOS program we replace $s_1, \ldots, s_4$ with $s_1 l_1, \ldots, s_4 l_1$ and $s_5, \ldots, s_8$ with $s_5 l_2, \ldots, s_8 l_2$, since the product of two SOS polynomials is SOS. Similarly, we replace λ_1 and λ_2 with $\lambda_1 l_1$ and $\lambda_2 l_2$. We can now factor out the $l_{1,2}$ terms to get the following convex relaxation of Theorem 2:

Proposition 2: If there exists a constant $\beta > 0$ and polynomial functions $\tilde{V}, \lambda_{1,2} \in \mathcal{R}_M^{n-1}$, and $s_1, \ldots, s_8 \in \Sigma_M$ such that $\tilde{V}(0) = 0$ and

$$-s_2(\beta - p) + s_3\tilde{V} + s_4(\beta - p)\tilde{V} - \lambda_1^T G - l_1 \in \Sigma_M \tag{24a}$$

$$-s_6(\beta - p) - s_7\dot{\tilde{V}} - s_8(\beta - p)\dot{\tilde{V}} - \lambda_2^T G - l_2 \in \Sigma_M \tag{24b}$$

then $z = 0$ is a stable equilibrium point of (15) and $V(x) = \tilde{V}(z(x))$ is a Lyapunov function for the original non-polynomial system.

Note that by choosing $s_4 = s_8 = 0$ and $s_3 = s_7 = 1$ we recover Proposition 4 in [25]. This choice also removes the bilinear constraints in $\tilde{V}$ and s.

1) Lyapunov Function for Model A: We define $p(z) = z_1^2 + z_2^2 + 2.0z_3^2 + z_4^2 + z_5^2 + 2.0z_6^2$ and search for β and for a Lyapunov function $\tilde{V}$ of maximum degree $d_{\tilde{V}} = 2$ and without any constant term (degree zero monomial) since we have to enforce the constraint $\tilde{V}(0) = 0$. We choose $l_i(z) = \sum_{j=1}^{6} \epsilon_{ij} z_j^2$, $i = 1, 2$, where $\epsilon_{ij} \geq 0$ and $\sum_{j=1}^{6} \epsilon_{ij} \geq 0.01$, $i = 1, 2$. We select $s_4 = s_8 = 0$ and $s_3 = s_7 = 1$ and the maximum degrees d_{s_2}, d_{s_6} of the SOS multipliers and $d_{\lambda_1}, d_{\lambda_2}$ of the λ polynomials. These are two component vectors since the constraints G are two component vectors of polynomials. These degrees have to be chosen so that the following relations hold:

$$\begin{aligned} \max\{\deg(ps_2), \deg(\tilde{V})\} \geq \\ \max\{\deg(\lambda_{11}G_1), \deg(\lambda_{12}G_2), d_{l_1}\} \end{aligned}$$

$$\begin{aligned} \max\{\deg(ps_6), \deg(\dot{\tilde{V}})\} \geq \\ \max\{\deg(\lambda_{21}G_1), \deg(\lambda_{22}G_2), d_{l_2}\}. \end{aligned}$$

We now search for a feasible solution of the following problem with SOS constraints

$$-s_2(\beta - p) + \tilde{V} - \lambda_{11}G_1 - \lambda_{12}G_2 - l_1 \in \Sigma_M \tag{25a}$$

$$-s_6(\beta - p) - \dot{\tilde{V}} - \lambda_{21}G_1 - \lambda_{22}G_2 - l_2 \in \Sigma_M \tag{25b}$$

where we choose $d_{\lambda_1(1)} = 0$, $d_{\lambda_1(2)} = 0$, $d_{\lambda_2(1)} = 1$, $d_{\lambda_2(2)} = 1$, $d_{s_2} = 0$,$d_{s_6} = 1$. We find that for $\beta = 0.2$ the SOS problem is feasible and it has the following solution in the original phase space coordinates:

$$\begin{aligned}V(x) &= 0.0932\sin(x_1) - 0.2920x_4 - 25.3499\cos(x_1)\\ &- 21.0067\cos(x_3) - 0.0408x_2 - 0.3359\sin(x_3)\\ &- 2.6408\cos(x_1)\cos(x_3) + 0.0165\cos(x_1)\sin(x_1)\\ &+ 0.1450\cos(x_1)\sin(x_3) - 0.1098\cos(x_3)\sin(x_1)\\ &+ 0.1909\cos(x_3)\sin(x_3) - 5.0017\sin(x_1)\sin(x_3)\\ &- 1.6016\cos(x_1)^2 - 1.1354\cos(x_3)^2 + 4.6283x_2x_4\\ &- 0.02086x_2\cos(x_1) + 0.0616x_2\cos(x_3)\\ &+ 0.0199x_4\cos(x_1) + 0.2721x_4\cos(x_3)\\ &+ 3.5181x_2\sin(x_1) + 1.52425x_2\sin(x_3)\\ &+ 0.6551x_4\sin(x_1) + 5.2582x_4\sin(x_3) + 11.0457x_2^2\\ &+ 12.8486x_4^2 + 51.7345\,.\end{aligned}$$

According to Theorem 2 the operating point at the origin is asymptotically stable.

2) Lyapunov Function for Model B: For this model we chose $p(z) = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2$. We have made the same choices for the degree of the Lyapunov function $\tilde{V}$ and for the degrees of the various polynomials involved in the SOS problem (25). We found that for $\beta = 0.1$ the SOS problem is feasible and it has the following solution in the original phase space coordinates:

$$\begin{aligned}V(x) &= 1.2468\cos(x_1)\sin(x_1) - 0.3646x_4 - 18.7585\cos(x_1)\\ &- 27.6219\cos(x_3) - 6.9358\sin(x_1) - 4.1573\sin(x_3)\\ &- 7.2379\cos(x_1)\cos(x_3) - 0.3399x_2\\ &+ 2.5142\cos(x_1)\sin(x_3) + 5.6889\cos(x_3)\sin(x_1)\\ &+ 1.6431\cos(x_3)\sin(x_3) - 2.5392\sin(x_1)\sin(x_3)\\ &- 11.3052\cos(x_1)^2 - 13.3274\cos(x_3)^2 + 0.0841x_2x_4\\ &+ 0.0939x_2\cos(x_1) + 0.2461x_2\cos(x_3)\\ &+ 0.2212x_4\cos(x_1) + 0.1434x_4\cos(x_3)\\ &+ 0.7038x_2\sin(x_1) - 0.1629x_2\sin(x_3)\\ &+ 0.2459x_4\sin(x_1) + 0.4671x_4\sin(x_3)\\ &+ 0.3647x_2^2 + 0.3158x_4^2 + 78.2509\,.\end{aligned}$$

As for model A, this shows that the operating point at the origin is asymptotically stable.

B. Estimating the Region of Attraction

These Lyapunov functions enable us to estimate the domain of attraction of the stable operating point for these two models. Indeed, assume that for a given scalar c the level set $\Omega_c = \{z \in \mathbb{R}^M \mid \tilde{V}(z) \le c, G(z) = 0\}$, is included in the domain $\tilde{D}$, i.e. $\Omega_c \subseteq \tilde{D}$. Then Ω_c describes a positively invariant region contained in the domain of attraction of the equilibrium point. For a given domain $\tilde{D}$ and Lyapunov function $\tilde{V}(z)$, the best estimate of the region of attraction of the stable fixed point at the origin is given by the largest c such that $\Omega_c \subseteq \tilde{D}$. To find c we have to solve the following optimization problem

$$\begin{gathered}\max \quad c\\ \text{s.t.}\\ \{z \in \mathbb{R}^M \mid c - \tilde{V}(z) \ge 0, G(z) = 0\} \subseteq\\ \{z \in \mathbb{R}^M \mid \beta - p(z) \ge 0, G(z) = 0\}\end{gathered}$$

where $\tilde{V}, p, G,$ and β are fixed. This can be formulated as an SOS programming problem by constructing the following empty set constraint version

$$\begin{gathered}\max \quad c\\ \text{s.t.}\\ \{z \in \mathbb{R}^M \mid c - \tilde{V}(z) \ge 0, G(z) = 0,\\ p(z) - \beta \ge 0, p(z) - \beta \ne 0\} = \emptyset\end{gathered}$$

According to the the P-satz theorem this condition holds if and only if we can find $c > 0$, $\tilde{f} \in \mathcal{C}(p(z) - \beta, c - \tilde{V}(z))$, $\tilde{g} \in \mathcal{M}(p(z) - \beta)$, and $\tilde{h} \in \mathcal{I}(G(z))$ such that

$$\tilde{f} + \tilde{g}^2 + \tilde{h} = 0 \tag{26}$$

By picking $k = 1$ in the definition of the monoid the set emptiness condition is cast into a search for $c > 0$, $s_0, s_1, s_2, s_4 \in \Sigma_M$, and $\lambda \in \mathcal{R}_M^{n-1}$ such that

$$s_0 + s_1(c - \tilde{V}) + s_2(p - \beta) + s_3(c - \tilde{V})(p - \beta) + \lambda^T G + (p - \beta)^2 = 0 \tag{27}$$

Thus, the best estimation of the ROA can be defined as the following SOS programming problem

$$\max_{s_1, s_2, s_3 \in \Sigma_M, \lambda \in \mathcal{R}_M} \quad c \tag{28a}$$

s.t.

$$-s_1(c - \tilde{V}) - s_2(p - \beta) - s_3(c - \tilde{V})(p - \beta) \tag{28b}$$

$$-\lambda^T G - (p - \beta)^2 \in \Sigma_M \tag{28c}$$

which is solved using a bisection search on c.

1) ROA Estimation for Model A: In Fig. 2 the dark gray area represents the largest invariant set $\Omega_c = \{z \in \mathbb{R}^6 \mid \tilde{V}(z) \le c, G(z) = 0\}$ which was obtained for $c = 0.922$. This represents a poor estimate of the exact ROA (the thin line connecting the UEPs (•) on the boundary of the stable fixed point). Compare this estimate to the constant energy surface passing through the closest UEP x_{u1} and, locally, to the energy surface passing through the UEP x_{u2} (thick black lines). The light gray area defines the domain $\tilde{D} = \{z \in \mathbb{R}^6 \mid \beta - p(z) \ge 0, G(z) = 0\}$, projected in the angle space, for $\beta = 0.2$. An algorithm to maximize the size of the invariant subset is needed in order to improve the estimated ROA.

2) ROA Estimation for Model B: In Fig. 3 the dark gray area represents the largest invariant set Ω_c obtained for $c = 0.868$. It represents a poor estimate of the exact ROA (the outermost thin black line). This estimate should be compared to the level set $\Omega_L = \{x \in \mathbb{R}^4 \mid W(x) \le L\}$, for $L = 3.2$, where $W(x)$ is the Lyapunov function computed for this model in [23] (the intermediate thick black line). The light gray area defines the domain $\tilde{D}$, projected in the angle space, for $\beta = 0.1$. For model B it is also necessary to devise an algorithm to improve the estimated ROA.

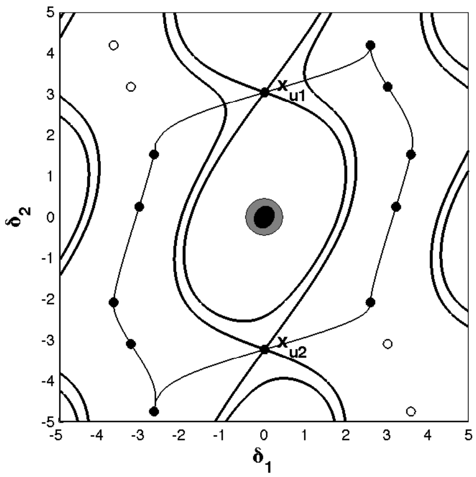

Fig. 2. Model A: The region of attraction for the SEP located at the origin, projected in the angle space ($\omega_1 = \omega_2 = 0$), is shown in thin black line connecting the UEPs (•) on its boundary. The light gray area defines the domain $\tilde{D}$ for $\beta = 0.2$. The dark gray area inside $\tilde{D}$ represents Ω_c, for $c = 0.922$, and is an (under)estimate of the ROA.

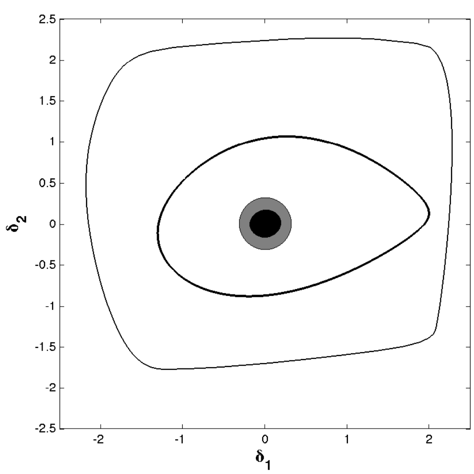

Fig. 3. Model B: The region of attraction for the SEP located at the origin, projected in the angle space ($\omega_1 = \omega_2 = 0$), is the outermost thin black line. The light gray area defines the domain $\tilde{D}$. The dark gray area inside $\tilde{D}$ represents Ω_c, for $c = 0.868$, which is an (under)estimate of the ROA. The thick black line defines the estimated ROA provided in [23].

VII. Optimizing the Region of Attraction

An obvious choice to improve the estimate of the fixed point's region of attraction is to expand the domain $\tilde{D}$ by maximizing β. A bisection search over β can be used to search for the maximum β value for which a feasible solution $\tilde{V}(z)$ for the problem (19) can be found. Then, by solving (28) and finding the largest level set of $\tilde{V}$ included in $\tilde{D}$, an improved estimate of the fixed point's region of attraction can be found. This is the essence of the *expanding* $\tilde{D}$ algorithm first proposed in [39]. Its extension to the analysis of non-polynomial systems can be easily obtained by replacing the relevant steps in the algorithm with their non-polynomial extensions described in sections VI-A and VI-B. However, expanding $\tilde{D}$ does not guarantee the expansion of Ω_c, the largest invariant set contained in $\tilde{D}$. For this reason, as Figs. 2 and 3 already suggest, the algorithm often finds a large $\tilde{D}$ that contains a much smaller invariant set Ω_c. We do not provide more details here because this algorithm does not perform as well as the *expanding interior algorithm* which we describe next.

A. Expanding Interior Algorithm

The idea of the algorithm is to expand a domain that is contained in a level set of the Lyapunov function $\tilde{V}$. This improves the estimate of the ROA since the domain expansion always guarantees the expansion of the invariant region defined by the level set of $\tilde{V}$. This algorithm was also introduced in [39]. We modify this algorithm in two ways. First, we extend the algorithm to analyze non-polynomial systems. Then, we introduce an iteration loop designed to improve the estimate of the region of attraction.

The basic idea of the algorithm is to select a positive definite polynomial $p \in \Sigma_M$ and to define a variable sized domain

$$P_\beta = \{z \in \mathbb{R}^n \mid p(z) \le \beta\}, \tag{29}$$

subject to the constraint that all points in P_β converge to the origin under the flow defined by the system's dynamics. In order to satisfy this constraint we define a second domain

$$\tilde{D} = \{z \in \mathbb{R}^M \mid \tilde{V}(z) \le c\}, \quad c > 0, \tag{30}$$

for a yet unspecified candidate Lyapunov function $\tilde{V}$ and impose the constraint that P_β is contained in $\tilde{D}$. Then by maximizing β over the set of Lyapunov functions $\tilde{V}$, while keeping the constraint $P_\beta \subset \tilde{D}$, we guarantee the expansion of the domain $\tilde{D}$ which provides an estimate of the fixed point's ROA.

Theorem 2 imposes additional constraints which can be formulated as set inclusion conditions. The first constraint,

$$\{z \in \mathbb{R}^M \,|\tilde{V}(z) \le c, G(z) = 0\} \setminus \{0\} \subseteq \{z \in \mathbb{R}^M \mid \dot{\tilde{V}}(z) < 0\}, \tag{31}$$

requires the derivative of the Lyapunov function $\tilde{V}$ to be negative over the manifold defined by $G = 0$ inside the domain $\tilde{D}$. The second constraint requires the Lyapunov function $\tilde{V}$ to be positive on the manifold defined by $G = 0$ inside the domain $\tilde{D} \setminus \{0\}$. Since $\tilde{V}$ and thus $\tilde{D}$ are unknown, the only effective way to ensure this constraint is to require that $\tilde{V}$ is positive everywhere on the manifold defined by $G = 0$:

$$\tilde{V}(z) > 0, \quad \forall z \in \{G(z) = 0\} \setminus \{0\}. \tag{32}$$

Thus, the problem of finding the best estimate of the region of attraction can be written as an optimization problem with set emptiness constraints

$$\max_{\tilde{V} \in \mathcal{R}_M, \tilde{V}(0)=0} \beta$$
$$\text{s.t.}$$
$$\{z \in \mathbb{R}^M \mid \tilde{V}(z) \le 0, G(z) = 0, z \ne 0\} = \emptyset$$
$$\{z \in \mathbb{R}^M \mid p(z) \le \beta, G(z) = 0, \tilde{V}(z) \ge c, \tilde{V}(z) \ne c\} = \emptyset$$
$$\{z \in \mathbb{R}^M \mid \tilde{V}(z) \le c, G(z) = 0, \dot{\tilde{V}}(z) \ge 0, z \ne 0\} = \emptyset$$

If we replace the two $z \neq 0$ non-polynomial constraints with $l_1(z) \neq 0$ and $l_2(z) \neq 0$ for $l_1, l_2 \in \Sigma_M$, positive definite, the formulation becomes

$$\max_{\tilde{V} \in \mathcal{R}_M, \tilde{V}(0)=0} \beta$$
$$\text{s.t.}$$
$$\{z \in \mathbb{R}^M \mid \tilde{V}(z) \leq 0, G(z) = 0, l_1(z) \neq 0\} = \emptyset$$
$$\{z \in \mathbb{R}^M \mid p(z) \leq \beta, G(z) = 0, \tilde{V}(z) \geq c, \tilde{V}(z) \neq c\} = \emptyset$$
$$\{z \in \mathbb{R}^M \mid \tilde{V}(z) \leq c, G(z) = 0, \dot{\tilde{V}}(z) \geq 0, l_2(z) \neq 0\} = \emptyset$$

By selecting $c = 1$ we recover the formulation in [39], extended to handle equality constraints introduced by the recasting procedure.

By applying the P-satz theorem, this optimization problem can be now formulated as the SOS programming problem

$$\max_{\substack{\tilde{V} \in \mathcal{R}_M, \tilde{V}(0)=0, k_1, k_2, k_3 \in \mathbb{Z}_+ \\ s_1, \ldots, s_{10} \in \Sigma_M, \lambda_1, \lambda_2, \lambda_3 \in \mathcal{R}_M^{n-1}}} \beta$$
$$\text{s.t.}$$
$$s_1 - s_2 \tilde{V} + \lambda_1^T G + l_1^{2k_1} = 0$$
$$s_3 + s_4(\beta - p) + s_5(\tilde{V} - c) + s_6(\beta - p)(\tilde{V} - c) + \lambda_2^T G + (\tilde{V} - c)^{2k_2} = 0$$
$$s_7 + s_8(c - \tilde{V}) + s_9 \dot{\tilde{V}} + s_{10}(c - \tilde{V})\dot{\tilde{V}} + \lambda_3^T G + l_2^{2k_3} = 0$$

Again, in order to limit the size of the SOS problem, we make a number of simplifications. First, we select $k_1 = k_2 = k_3 = 1$. Then, we simplify the first constraint by selecting $s_2 = l_1$ and factoring out l_1 from s_1 and the polynomials λ_1. Since the second constraint contains quadratic terms in the coefficents of $\tilde{V}$, we select $s_3 = s_4 = 0$, replace λ_2 with $\lambda_2(\tilde{V} - c)$, and factor out $(\tilde{V} - c)$ from all the terms. Finally, we select $s_{10} = 0$ in the third constraint in order to eliminate the quadratic terms in $\tilde{V}$ and factor out l_2. Thus, we reduce the SOS problem to the following formulation

$$\max_{\substack{\tilde{V} \in \mathcal{R}_M, \tilde{V}(0)=0, \\ s_6, s_8, s_9 \in \Sigma_M, \lambda_1, \lambda_2, \lambda_3 \in \mathcal{R}_M^{n-1}}} \beta$$
$$\text{s.t.}$$
$$\tilde{V} - \lambda_1^T G - l_1 \in \Sigma_M \quad (36a)$$
$$-s_6(\beta - p) - \lambda_2^T G - (\tilde{V} - c) \in \Sigma_M \quad (36b)$$
$$-s_8(c - \tilde{V}) - s_9 \dot{\tilde{V}} - \lambda_3^T G - l_2 \in \Sigma_M \quad (36c)$$

The algorithm performs an iterative search to expand the domain $\tilde{D}$ starting from some initial Lyapunov function $\tilde{V}$. At each iteration step, due to the presence of bilinear terms in the decision variables, the algorithm alternates between two SOS optimization problems. When no improvement in β is possible, the algorithm stops and $\tilde{D}$ offers the best estimate of the region of attraction. The quality of the estimate critically depends on the choice of the polynomial $p(z)$. By improving this choice we can find better estimates and the following observation suggests how this can be done. Notice that the Lyapunov function changes as the iteration progresses and that by expanding the domain P_β the algorithm forces the level sets of the Lyapunov function to better approximate the shape of the region of attraction. This observation suggests that the algorithm can be improved by introducing another iteration loop over $p(z)$: when the algorithm defined above converges and no improvements in β can be found, we use the Lyapunov function $\tilde{V}$ to define the new $p(z)$. Since we required $\tilde{V}$ to be positive definite everywhere, this substitution is always possible. This substitution guarantees that the next β optimization loop starts from the point $\beta = c$ where $P_\beta = \tilde{D}$. Due to the constraint $P_\beta = \{z \mid p(z) \leq \beta\} \subset \tilde{D} = \{z \mid \tilde{V}(z) \leq c\}$ the algorithm stops when it reaches a fixed point where $p(z) = \tilde{V}(z)$, and $\beta = c$. Finally, we noticed that we cannot always guarantee that a domain $\tilde{D}$ can be found while keeping the constant c fixed. For this reason we have included a search over c at each iteration step. The detailed description of the algorithm is as follows — see [39] for a comparison to its original formulation.

B. SOS Formulation

The algorithm contains two iteration loops to expand the region P_β and, implicitly, the domain $\tilde{D}$ that provides an estimate of the region of attraction of the stable fixed point $z = 0$. The outer iteration loop is over the polynomial $p(z)$ defining $P_\beta = \{p(z) < \beta\}$. The iteration index for this loop is j. The inner iteration loop is over the parameter β and i defines its iteration index. The outer iteration starts from a candidate polynomial $p^{(j=0)}(z) > 0$ for $\forall z \in \mathbb{R}^M$. The inner iteration starts from a candidate Lyapunov function $\tilde{V}^{(i=0)}$ which can be found by solving the SOS program described in Theorem 2. For the two power grid systems we select the quadratic polynomial $p(z)$ and the Lyapunov function $\tilde{V}(z)$ found in Section VI-A.

Select the maximum degrees of the Lyapunov function, the SOS multipliers, the polynomials λ, and the l polynomials as $d_{\tilde{V}}$, $d_{s_6}, d_{s_8}, d_{s_9}$, $d_{\lambda_1}, d_{\lambda_2}, d_{\lambda_3}$ and d_{l_1}, d_{l_2} respectively. Fix $l_k = \epsilon \sum_{k=1}^M z_k^{d_{l_k}}$ for $k = 1, 2$ and some small $\epsilon > 0$. Finally, select $\beta^{(i=0)} = 0$.

(1a) Set $\tilde{V} = \tilde{V}^{(i-1)}$, $\beta = \beta^{(i-1)}$. We expect the SOS problem to be infeasible until c reaches the level at which $\{x \mid p^{(j-1)}(x) < \beta^{(i-1)}\} \in \{x \mid \tilde{V}^{(i-1)}(x) < c\}$ The problem remains feasible until we reach a c value at which $d\tilde{V}/dt$ is no longer negative inside $\tilde{V}^{(i-1)}(x) < c$ level set. Therefore, for given $\tilde{V} = \tilde{V}^{(i-1)}$, $\beta = \beta^{(i-1)}$ we will find that the SOS problem is feasible for $c \in [c_{\min}, c_{\max}]$. Therefore, we search on c in order to solve the following SOS optimization problem

$$\max_{s_6, s_8, s_9 \in \Sigma_M, \lambda_1, \lambda_2, \lambda_3 \in \mathcal{R}_M^{n-1}} c$$
$$\text{s.t.}$$
$$-s_6(\beta - p^{(j-1)}) - \lambda_2^T G - (\tilde{V} - c) \in \Sigma_M \quad (37)$$
$$-s_8(c - \tilde{V}) - s_9 \dot{\tilde{V}} - \lambda_3^T G - l_2 \in \Sigma_M \quad (38)$$

where the decision variables are: $s_6 \in \Sigma_{M,d_{s_6}}$, $s_8 \in \Sigma_{M,d_{s_8}}$, $s_9 \in \Sigma_{M,d_{s_9}}$ and $\lambda_1 \in \mathcal{R}_{M,d_{\lambda_1}}^{n-1}$ and $\lambda_2 \in \mathcal{R}_{M,d_{\lambda_2}}^{n-1}$. Set $s_8^{(i)} = s_8$, $s_9^{(i)} = s_9$, $\lambda_1^{(i)} = \lambda_1$, and $\lambda_2^{(i)} = \lambda_2$. Set $c^{(i)} = c$.

(1b) Set $\tilde{V} = \tilde{V}^{(i-1)}$ and $c = c^{(i)}$ and perform a line search on β in order to find the largest domain $p^{(j-1)}(x) < \beta$

included in $\tilde{V}^{(i-1)}(x) < c^{(i)}$. To solve this problem we formulate the following SOS optimization problem

$$\max_{s_6,s_8,s_9\in\Sigma_M,\lambda_1,\lambda_2,\lambda_3\in\mathcal{R}_M^{n-1}} \beta$$
$$\text{s.t.}$$
$$-s_6(\beta - p^{(j-1)}) - \lambda_2^T G - (\tilde{V} - c) \in \Sigma_M \tag{39}$$
$$-s_8(c - \tilde{V}) - s_9\dot{\tilde{V}} - \lambda_3^T G - l_2 \in \Sigma_M \tag{40}$$

where the decision variables are: $s_6 \in \Sigma_{M,d_{s_6}}$, $s_8 \in \Sigma_{M,d_{s_8}}$, $s_9 \in \Sigma_{M,d_{s_9}}$ and $\lambda_1 \in \mathcal{R}^{n-1}_{M,d_{\lambda_1}}$ and $\lambda_2 \in \mathcal{R}^{n-1}_{M,d_{\lambda_2}}$. Set $s_8^{(i)} = s_8$, $s_9^{(i)} = s_9$, $\lambda_1^{(i)} = \lambda_1$, and $\lambda_2^{(i)} = \lambda_2$. Set $\beta^{(i)} = \beta$.

(2a) Set $\beta = \beta^{(i)}$ fixed and $s_8 = s_8^{(i)}$, and $s_9 = s_9^{(i)}$. We want to find a c and a $\tilde{V} > 0$ on the manifold $G = 0$ so that $p^{(j-1)}(x) < \beta$ is included in $\tilde{V}(z) < c$. Thus, we solve

$$\min_{\tilde{V},\tilde{V}(0)=0,s_6,\lambda_1,\lambda_2,\lambda_3} c$$
$$\text{s.t.}$$
$$\tilde{V}(z) - \lambda_1^T G(z) - l_1 \in \Sigma_M \tag{41}$$
$$-s_6(\beta - p) - \lambda_2^T G - (\tilde{V} - c) \in \Sigma_M \tag{42}$$
$$-s_8(c - \tilde{V}) - s_9\dot{\tilde{V}} - \lambda_3^T G - l_2 \in \Sigma_M \tag{43}$$

and set $c^{(i)} = c$.

(2b) Fix $c = c^{(i)}$ and set $s_8 = s_8^{(i)}$, and $s_9 = s_9^{(i)}$. We search over $\tilde{V}$ and s_6 so that we can maximize β:

$$\max_{\tilde{V},\tilde{V}(0)=0,s_6,\lambda_1,\lambda_2,\lambda_3} \beta$$
$$\text{s.t.}$$
$$\tilde{V}(z) - \lambda_1^T G(z) - l_1 \in \Sigma_M \tag{44}$$
$$-s_6(\beta - p) - \lambda_2^T G - (\tilde{V} - c) \in \Sigma_M \tag{45}$$
$$-s_8(c - \tilde{V}) - s_9\dot{\tilde{V}} - \lambda_3^T G - l_2 \in \Sigma_M \tag{46}$$

Set $\beta^{(i)} = \beta$ and $V^{(i)} = \tilde{V}$. If $\beta^{(i)} - \beta^{(i-1)}$ is smaller than a given tolerance go to step (3). Otherwise, increment i and go to step (1a).

(3) If $j = 0$ set $p^{(1)} = \tilde{V}^{(i)}$ and go to step (1a). If $j \geq 1$ and the largest (in absolute value) coefficient of the polynomial $p^j(z) - p^{(j-1)}(z)$ is smaller than a given tolerance, the outer iteration loop ends. Otherwise, advance j, set $p^{(j)} = \tilde{V}^{(i)}$ and go to step (1a).

(4) When the outer iteration loop stops the set $\tilde{D}^{(i)} = \{z \in \mathbb{R}^M \mid \tilde{V}^{(i)}(z) \leq c^i\}$ contains the domain $P^{(j)}_{\beta^{(i)}} = \{z \in \mathbb{R}^M \mid p^{(j)}(z) \leq \beta^i\}$ and is the largest estimate of the fixed point's region of attraction. In practice, we noticed that when the outer iteration loop stops, the algorithm reaches a fixed point where the domain $\tilde{D}^{(i)}$ becomes essentially indistinguishable from the domain $P^{(j)}_{\beta^{(i)}}$.

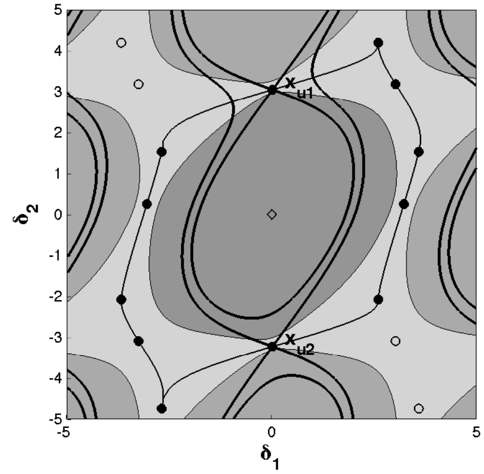

Fig. 4. The region of attraction for the SEP located at the origin (◇), projected in the angle space ($\omega_1 = \omega_2 = 0$), is shown in thin black line connecting the UEPs (•) on its boundary. The thick black lines show the constant energy surface passing through the closest UEP x_{u1} and the one passing through the UEP x_{u2}. The dark gray area shows the best estimate of the ROA according to the expanding interior algorithm.

C. Analysis of Model A

For this model the optimization algorithm described in the previous section returns the following Lyapunov function

$$\begin{aligned}
V(x) = {} & 0.0030\sin(x_1) - 0.00008x_4 - 0.2683\cos(x_1) \\
& - 0.2649\cos(x_3) - 0.0030x_2 + 0.0044\sin(x_3) \\
& - 0.2377\cos(x_1)\cos(x_3) + 0.0008\cos(x_1)\sin(x_1) \\
& + 0.0047\cos(x_1)\sin(x_3) - 0.0037\cos(x_3)\sin(x_1) \\
& - 0.0092\cos(x_3)\sin(x_3) - 0.1588\sin(x_1)\sin(x_3) \\
& - 0.0109\cos(x_1)^2 + 0.0203\cos(x_3)^2 - 0.0004x_2x_4 \\
& - 0.0016x_2\cos(x_1) + 0.0047x_2\cos(x_3) \\
& + 0.0011x_4\cos(x_1) - 0.0010x_4\cos(x_3) \\
& + 0.0579x_2\sin(x_1) + 0.0219x_2\sin(x_3) \\
& + 0.0195x_4\sin(x_1) + 0.0972x_4\sin(x_3) \\
& + 0.1461x_2^2 + 0.1703x_4^2 + 0.7614\,.
\end{aligned}$$

The Lyapunov function has been rescaled so that the best estimate of the ROA is provided by the level set $\{x \in \mathbb{R}^4 \mid V(x) \leq c\}$ with $c = 1.0$. This estimate is shown by the dark gray area in Fig. 4. We notice that this estimate significantly improves the one provided by the closest UEP method. We also notice that the algorithm provides a good *global* estimate of the ROA which compares well with the *local* estimates returned by the controlling UEP method. For example, compare locally the approximation returned by our algorithm with the one provided by the controlling UEP x_{u2}: our estimate is better except very close to x_{u2}. This property holds for many other possible controlling UEPs on the boundary of the ROA. Finally, our algorithm avoids the computationally difficult task of estimating the controlling UEP.

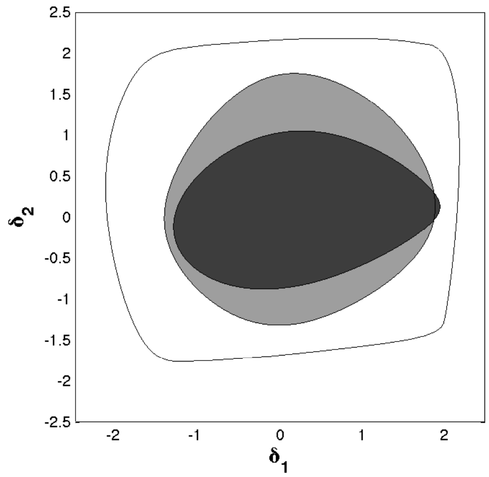

Fig. 5. The region of attraction for the SEP located at the origin, projected in the angle space ($\omega_1 = \omega_2 = 0.80$), is the outermost thin black line. The expanding interior algorithm produces an estimate of the ROA shown in light gray. The dark gray area represents the estimated ROA provided in [23].

D. Analysis of Model B

For this model the expanding interior algorithm returns the following Lyapunov function (projected back in the original phase space coordinates)

$$\begin{aligned} V(x) = {} & 0.0036x_2 - 0.0026x_4 - 0.7007\cos(x_1) \\ & - 0.7866\cos(x_3) - 0.2762\sin(x_1) - 0.2702\sin(x_3) \\ & - 0.1905\cos(x_1)\cos(x_3) + 0.2072\cos(x_1)\sin(x_1) \\ & + 0.0467\cos(x_1)\sin(x_3) + 0.0690\cos(x_3)\sin(x_1) \\ & + 0.2235\cos(x_3)\sin(x_3) - 0.0559\sin(x_1)\sin(x_3) \\ & - 0.0744\cos(x_1)^2 - 0.1044\cos(x_3)^2 + 0.0015x_2x_4 \\ & - 0.0076x_2\cos(x_1) + 0.0040x_2\cos(x_3) \\ & + 0.0042x_4\cos(x_1) - 0.0016x_4\cos(x_3) \\ & + 0.0138x_2\sin(x_1) - 0.0018x_2\sin(x_3) \\ & + 0.0056x_4\sin(x_1) + 0.0091x_4\sin(x_3) \\ & + 0.0075x_2^2 + 0.0059x_4^2 + 1.8567\,. \end{aligned}$$

This Lyapunov function has also been rescaled so that the best estimate of the ROA is provided by the level set $\{z \in \mathbb{R}^4 \mid V(x) \leq c\}$ with $c = 1.0$. Our estimate should be compared to the dark gray area which is the estimated ROA provided by $\Omega_L = \{x \in \mathbb{R}^4 \mid W(x) \leq L\}$ for $L = 3.2$, where $W(x)$ is the Lyapunov function computed in [23] for this model. Except for a very small region of the phase space (for this particular $\omega_1 = \omega_2 = 0.80$ projection) our estimate is better. In fact, the analysis of multiple two-dimensional projections in phase space shows that our estimate outperforms the estimate provided by Ω_L. Perhaps this comparison is not fair since the elegant method proposed in [23] contains multiple parameters that can be optimized in order to improve the estimated ROA. More importantly, the domain inclusions and the boundedness of the set Ω_L which are required by the Extended Invariance Principle in [23] are very difficult to check numerically. For this reason the assumption that the transfer conductances are small is necessary in order to guarantee some of these constraints. Many of these difficulties could be overcome by applying the algebraic methods proposed in this paper and a synthesis of these two approaches might provide improved ROA estimates.

VIII. Discussion and Future Work

We have introduced an algorithm for the construction of Lyapunov functions for classical power system models. The algorithm we propose provides mathematical guarantees and avoids the major computational difficulties engendered by the computation of the controlling UEP in the energy function method. Moreover, we have also shown that systems with transfer conductances can be analyzed as well, without any conceptual difficulties. This is a significant result because analytical energy functions do not exist for these systems and the proposed SOS analysis provides a constructive approach for computing analytical Lyapunov functions for these systems. The approaches proposed in [18], [19], [23] for constructing Lyapunov functions for power systems with transfer conductances have to assume that the transfer conductances are small. Our approach is free of these parametric constraints. Moreover, these approaches impose structural constraints on the class of Lyapunov functions. The approach we propose is structure-free and for this reason the function space in which we search for Lyapunov functions includes all these structured Lyapunov subspaces. If well designed, our proposed algorithm should outperform these alternative approaches. The generalization of this approach to network preserving models, which also include more realistic load and generator models [40]–[44], can in principle be achieved. Moreover, further improvements in estimating the ROA might be achieved by increasing the dimension of the Lyapunov function.

Another possible generalization is the inclusion of parametric uncertainties. For power systems these uncertainties can reflect changes in line impedances or uncertainties in some of the system parameters (for example the inertia and damping coefficient of generators). When this is the case, the location of the equilibrium usually changes when the parameters are varied. In the presence of parametric uncertainties the use of equality and inequality constraints is natural: the region of the parameter space that is of interest can be described by inequality constraints, and if the equilibrium moves as the parameters change, one can impose an equality constraint on the corresponding variables. As we have already shown in this paper, the stability of systems with constraints can be elegantly handled using SOS techniques as demonstrated in [29].

This fact can be used to handle the following difficulty.[1] The Lyapunov function derived in this paper is valid for a particular operating point and any change in parameters or operating point will require the solution of another optimization problem to obtain a new Lyapunov function for the new configuration. Apparently, new Lyapunov functions have to be computed, solving a high-dimensional optimization problem, every time a change in the system occurs. Nevertheless, by expressing the dependence of the equilibrium point on the uncertain

[1] We thank one of our reviewers for pointing out this difficulty to us.

parameters using equality constraints, parameterized Lyapunov function can be constructed as has been discussed in [29]. Conceptually this approach can produce Lyapunov functions which depend explicitly on some of the system parameters.

Nevertheless, there are serious difficulties before these algebraic methods, and the generalizations discussed above, can be applied to large power systems. The difficulties are not conceptual but numerical because one of the major limitations of the SOS framework is the complexity of the system description that can currently be analyzed. Indeed, the size of the SDP that needs to be solved in order to compute the SOS decomposition grows with the number of variables and the degree of the polynomial. This is a serious limitation, which renders the proposed algorithm impractical in its current formulation, as many systems of interest are of significantly higher dimension.

However, some of these numerical problems can be partially overcome by using decomposition techniques. In this regard, the approach in [22] is very significant for a couple of reasons. First, it provides the only alternative that we found in the literature for computing Lyapunov functions for systems with transfer conductances that do not suffer from the difficulties mentioned before. Second, it contains conditions on the interconnection of a large scale system such that a weighted sum of the subsystems energy functions give a Lyapunov function for the overall system. Similar conditions can be employed by our method in order to analyze larger power systems.

Alternatively, decomposition techniques that have been proposed for the analysis of large-scale systems — see for example [45] and the references therein — can be used in order to address this problem. The underlying assumption is that stability certificates can be constructed for the individual subsystems and patched together to form a composite Lyapunov function [30]. Finally, one can employ clustering and aggregation techniques [46] to generate a low-dimensional system of equivalent generators and to apply the proposed analysis techniques to this reduced model.

Appendix A
The Sum of Squares Decomposition

In this appendix we give a brief introduction to sum of squares (SOS) polynomials and describe how the existence of a SOS decomposition can be verified using semidefinite programming [47]. The notation used is as follows. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Z}_+$ denote the set of nonnegative integers. The set of $n \times m$ matrices is represented by $\mathbb{R}^{n\times m}$. A matrix $P \in \mathbb{R}^{n\times n}$ is positive definite if $x^T P x > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$ and positive semidefinite if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$; we denote these by $P \succ 0$ and $P \succeq 0$ respectively. A monomial m_α in n independent real variables $x \in \mathbb{R}^n$ is a function of the form $m_\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha_i \in \mathbb{Z}_+$, and the degree of the monomial is $\deg m_\alpha := \alpha_1 + \ldots + \alpha_n$. Given $c \in \mathbb{R}^k$ and $\alpha \in \mathbb{Z}_+^k$ a polynomial is defined as $p(x) = \sum_{j=1}^k c_j m_{\alpha_j}$. The degree of p is defined by $\deg p := \max_j(\deg m_{\alpha_j})$. We will denote the set of polynomials in n variables with real coefficients as $\mathcal{R}_n$ and the subset of polynomials in n variables that have maximum degree d as $\mathcal{R}_{n,d}$.

Definition 1: For $x \in \mathbb{R}^n$, a multivariate polynomial $p(x) \overset{\text{def}}{=} p(x_1, \ldots, x_n)$ is a sum of squares (SOS) if there exist some polynomial functions $h_i(x), i = 1 \ldots r$ such that

$$p(x) = \sum_{i=1}^{r} h_i^2(x) \tag{47}$$

Note that $p(x)$ being a SOS implies that $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. However, the converse is not always true except in special cases [48]. The set of all SOS polynomials in n variables will be denoted as Σ_n and we define $\Sigma_{n,d} = \Sigma_n \bigcap \mathcal{R}_{n,d}$.

An equivalent characterization of SOS polynomials is given in the following proposition [24]:

Proposition 3: A polynomial $p(x) \in \mathcal{R}_n$ of degree $2d$ is a SOS if and only if there exists a positive semidefinite matrix Q and a vector of monomials $Z_{n,d}(x)$ in n variables of degree less than or equal to d such that $p = Z_{n,d}(x)^T Q Z_{n,d}(x)$.

In general, since the monomials in $Z_{n,d}(x)$ are not algebraically independent, the matrix Q in the quadratic representation of the polynomial $p(x)$ is not unique and the set of matrices that make the quadratic equality in Proposition 3 hold are an affine subspace of the symmetric matrices [49]:

$$\mathcal{Q}_p = \left\{Q \mid Z_{n,d}(x)^T Q Z_{n,d}(x) = p(x)\right\} = \left\{Q_0 + \sum_{i=1}^{p} \lambda_i Q_i\right\} \tag{48}$$

where Q_0 is any symmetric matrix such that $p(x) = Z_{n,d}(x)^T Q_0 Z_{n,d}(x)$ and $\{Q_i\}_{i=1}^p$ is the set of symmetric matrices such that $Z_{n,d}(x)^T Q_i Z_{n,d}(x) = 0$. Since $p(x)$ being SOS is equivalent to $Q \succeq 0$, the problem of finding a Q which proves that $p(x)$ is an SOS is equivalent to checking if there exist λ_i such that $Q_0 + \sum_{i=1}^p \lambda_i Q_i \succeq 0$. This Linear Matrix Inequality is a convex feasibility problem, as was first noticed in [24], and can be solved efficiently using semidefinite programming techniques which have worst-case polynomial time complexity. Note that, as the degree of $p(x)$ or its number of variables is increased, the computational complexity for testing whether $p(x)$ is an SOS increases. Nonetheless, the complexity overload is still a polynomial function of these parameters.

An important extension, widely used in this paper, was introduced in [50] and refers to the case when $p(x)$ is a linear combination of polynomials with unknown coefficients, and we want to search for feasible values of those coefficients such that $p(x)$ is nonnegative.

Theorem 3: Given a finite set of polynomials $\{p_i\}_{i=0}^r \in \mathcal{R}_n$, the existence of $\{a_i\}_{i=1}^r \in \mathbb{R}$ such that

$$p = p_0 + \sum_{i=1}^{r} a_i p_i \quad \text{is an SOS} \tag{49}$$

is an LMI feasibility problem.

When supplemented by the following optimization objective

$$\max \sum_{i=1}^{r} a_i w_i \,, \tag{50}$$

where the the a_j are scalar, real decision variables and the w_j are some given real numbers, (49) and (50) define a SOS program. This SOS program can be converted to a convex semidefinite program (SDP) which can be solved numerically with great efficiency. The software SOSTOOLS [36], [28] automatically performs this conversion for general SOS programs. It also calls a SDP solver, such as SeDuMi [37], and converts the SDP solution back to the solution of the original SOS program. We have used SOSTOOLS, Version 2.0, in conjunction with SeDuMi, Version 1.1, to solve all SOS programs formulated in this paper.

APPENDIX B
BASIC ALGEBRAIC GEOMETRY

In this section we introduce the basic algebraic definitions that are necessary in order to present one of the most important theorems in real algebraic geometry.

Definition 2: Given $\{g_1, \dots, g_t\} \in \mathcal{R}_n$, the *Multiplicative Monoid* generated by g_j's is

$$\mathcal{M}(g_1, \dots, g_t) = \{{g_1}^{k_1} {g_2}^{k_2} \dots {g_t}^{k_t} | k_1, \dots, k_t \in \mathbb{Z}_+\} \quad (51)$$

which is the set of all finite products of g_j's including the empty product, defined to be 1. It is denoted as $\mathcal{M}(g_1, \dots, g_t)$.

Definition 3: Given $\{f_1, \dots, f_s\} \in \mathcal{R}_n$, the *Cone* generated by f_j's is

$$\mathcal{C}(f_1, \dots, f_s) := \left\{ s_0 + \sum s_i b_i | s_i \in \Sigma_n, b_i \in \mathcal{M}(f_1, \dots, f_s) \right\} \quad (52)$$

Definition 4: Given $\{h_1, \dots, h_u\} \in \mathcal{R}_n$, the *Ideal* generated by h_k's is

$$\mathcal{I}(h_1, \dots, h_u) := \left\{ \sum h_k p_k | p_k \in \mathcal{R}_n \right\} \quad (53)$$

With these definitions we can now state the following fundamental theorem.

Theorem 4 (Positivstellensatz): Given polynomials $\{f_1, \dots, f_s\}$, $\{g_1, \dots, g_t\}$, and $\{h_1, \dots, h_u\}$ in $\mathcal{R}_n$, the following are equivalent:

1) The set

$$\left\{ x \in \mathbb{R}^n \left| \begin{array}{l} f_1(x) \geq 0, \dots, f_s(x) \geq 0 \\ g_1(x) \neq 0, \dots, g_t(x) \neq 0 \\ h_1(x) = 0, \dots, h_u(x) = 0 \end{array} \right. \right\} \quad (54)$$

is empty.

2) There exist polynomials $f \in \mathcal{C}(f_1, \dots, f_s)$, $g \in \mathcal{M}(g_1, \dots, g_t)$, and $h \in \mathcal{I}(h_1, \dots, h_u)$ such that

$$f + g^2 + h = 0. \quad (55)$$

The LMI based tests for SOS polynomials can be used to prove that the set emptiness condition from Positivstellensatz (P-satz) holds, by finding specific f, g and h such that $f + g^2 + h = 0$. These f, g and h are known as P-satz certificates since they certify that the equality holds.

It is important to notice that P-satz offers no guidance on how to select the degrees of the polynomials involved in the definition of the monoid $\mathcal{M}$, cone $\mathcal{C}$, and ideal $\mathcal{I}$. By putting an upper bound on these degrees and checking whether (55) holds, one can create a series of tests for the emptiness of (54). Each of these tests requires the construction of some sum of squares and polynomial multipliers, resulting in a sum of squares program that can be solved using SOSTOOLS.

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