Analytical Framework for Power System Strength

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***Abstract*—This paper proposes a general framework to evaluate power system strength. The formulation features twelve indicators, grouped in three dynamical orders, that quantify the resistance of bus voltage phasors and their first and second order rates of change to sudden current injection changes. To quantify such changes the paper introduces a novel finite differentiation technique, that we named *Delta operator*, able to properly capture "jumps" of algebraic variables and utilizes the recently developed concept of complex frequency. The paper also shows how the proposed framework can be systematically applied to any system device, and provides a variety of examples based on synchronous machines, converters and loads models are given. Numerical results in a benchmark system validate the exactness of the formulation.**

***Index Terms*—System strength, low-inertia systems, short-circuit ratio, nodal inertia, complex frequency.**

I. Introduction

A. Motivation

The term 'strength' refers to the resistance of a system to disturbances. Intuitively, a 'stronger' system is less sensitive to perturbations, whereas a 'weaker' system experiences higher deviations when subjected to the same disturbance [1]. It is typically the voltage the representative variable over which strength is evaluated. For instance, the Australian Energy Market Commission (AEMC) defines system strength as *the power system's ability to resist the changes in the magnitude, phase angle, and waveform of the voltage at any given location under different operating conditions* [2]. How to quantify this ability to resist changes, however, is currently an open question. In this paper, we propose a general analytical framework to evaluate 'strength' based on a novel finite-difference operator specifically developed for this purpose.

B. Literature review

Analytical developments for quantifying voltage strength are separated into specific indicators to assess voltage magnitude strength, and separate metrics aimed at evaluating frequency strength. This separation establishes a natural relationship with conventional types of stability analysis, namely rotor angle, frequency, and voltage stability [3].

The foundation of voltage magnitude strength is the short-circuit level (SCL) at a given location [4]. The SCL has been historically used as a measure of strength, and has become a critical aspect for ensuring a stable operation of modern systems with presence of inverter-based resources (IBRs). In particular, the short-circuit ratio (SCR), defined as as the ratio of the SCL to the rated power of the IBR, has emerged as a basic metric to assess the ability of the IBR to withstand low strength conditions [5]. The SCR ignores the effect of neighboring converters and can lead to optimistic results in networks with multiple IBRs. This issue has motivated the proposal the weighted SCR [6], the composite SCR [7], and a wide list of more elaborated indicators addressing the shortcomings of the SCR [8]–[13]. A comparison between these metrics can be found in [14].

With regard to frequency, strength metrics are focused on the system dynamic performance. Conventional AC systems are dominated by synchronous machines (SMs), which operate by nature at a unique frequency, rendering it a rather global variable of the grid. Consequently, the principles of frequency dynamics have been studied based on simplified single-node models, or using global averaged quantities, such as the frequency of the center of inertia. This scenario has also led to the use of a global metric for quantifying frequency strength, namely the *system inertia*, defined as the sum of the inertia of the SMs. It is a common assumption that low inertia levels combined with a high penetration of IBRs endanger power systems operation. This is a very common scenario in power systems nowadays, and has given rise to several challenges for frequency control, and created a paradigm shift in the study of frequency dynamics [15].

Recent investigations have shown that the spatial distribution of inertia significantly impacts the post-disturbance local frequency response [16]–[18]. These observations have motivated research on novel indicators aimed at reflecting local frequency strength, quantifying the inertia concentration at different locations of the network [19]–[24]. Despite notable methodological differences, most works are based on the so-called 'nodal inertia', whose foundation assumes the existence of an equivalent swing equation at buses [21]. While some of these proposal's ultimate goal is to calculate this nodal-level metric [21]–[23], others use it as an intermediate step to get regional frequency indicators, recognizing frequency coherency in areas of the grid [24].

Current research efforts on strength metrics still rely on strong approximations on the system representation, and often involve equations proposed rather empirically than derived analytically. In addition, indicators for quantifying the voltage magnitude strength and voltage frequency strength are derived independently with approaches fundamentally different, even though both variables are, ultimately, two components of a unique entity: a three phase AC voltage. Recent developments have enabled the search for a more general and unifying formulation. In particular, the complex frequency (CF) is a concept recently proposed in [25] that has given rise to a variety of improvements in power system modeling and control [26]–[30]. This article exploits the properties of the CF to develop the proposed analytical framework to evaluate system strength.

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C. Contribution

The contributions of the paper are threefold.

- A novel general analytical framework to evaluate power system strength in steady-state and dynamic conditions.
- A systematic methodology to study the effect of diverse device models on system strength. Specific expressions for relevant devices, such as synchronous machines, converters, and loads, are provided.
- Definition of a novel mathematical operator, called *Delta operator*, along with some of its properties and identities.

A relevant consequence of the proposed analytical framework is the unification of the voltage magnitude and frequency strength assessment. It also gives further information on buses strength when subjected to remote perturbations. The proposed framework is purely analytical, minimizes assumptions on the system representation, and avoids approximations, rendering it 'exact', provided the system model is exact.

D. Paper organization

The remainder of this document is organized as follows. Section II introduces key mathematical foundations required for the analytical developments of the paper. Section III introduces the proposed formulation of system strength and its derivation. Section IV presents specific expressions for basic power system devices that are relevant to system strength. Section V offers study cases implementing the proposal in benchmark systems. Finally, Section VI draws main conclusions and outlines future work.

II. Mathematical background

This section presents the mathematical foundations required to build the derivation of the system strength indicators presented in this work.

A. Complex frequency

Consider the voltage represented as a dynamic Clarke vector $\bar{v} \in \mathbb{C} \,|\, \bar{v} = v\cos\theta + \jmath\, v \sin\theta$. The complex frequency (CF) of the voltage is a complex quantity denoted as $\bar{\eta}$ and defined in [25] as follows:

$$\bar{\eta} = \frac{\dot{v}}{v} + \jmath\,\dot{\theta} = \rho + \jmath\,\omega, \quad v \neq 0\,. \tag{1}$$

The CF is related to the total time derivative of the voltage as in the equation below:

$$\bar{p}\{\bar{v}\} = \bar{v}\,\bar{\eta}\,, \tag{2}$$

where $\bar{p}\{\cdot\} = \frac{d}{dt}\{\cdot\} + \{\cdot\}\,\jmath\,\dot{\theta}_{\rm dq}$ is the total derivative operator of a dynamic Clarke vector as defined in [31], and $\theta_{\rm dq}$ the reference angle of the $\rm dq$ coordinates with respect to a fixed reference angle. Therefore, (2) gives the 'absolute' time derivative of $\bar{v}$, i.e., the rate of change of the vector relative to the fixed reference frame, independently of the $\rm dq$ coordinates used to represent $\bar{v}$. We are interested in calculating the time derivative of the vector relative to a rotating reference frame. To do so, (2) becomes:

$$\dot{\bar{v}} = \bar{v}\,\left(\bar{\eta} - \jmath\,\omega_0\right)\,, \tag{3}$$

where ω_0 is the fundamental frequency of the system.

Equation (3) is a very useful property of the CF that allows using it as a time derivative operator. In this work, we are also interested in calculating the second-order time derivative of the voltage, $\ddot{\bar{v}}$, for which a quantity with a property similar to (3) would highly facilitate the formulation. This motivates defining a second-order CF, $\bar{\eta}''$, as the complex quantity satisfying the following equation:

$$\ddot{\bar{v}} = \bar{v}\,\bar{\eta}''\,, \tag{4}$$

An expression for $\bar{\eta}''$ is found by taking (3) and applying the time derivative at both sides:

$$\ddot{\bar{v}} = \dot{\bar{v}}\,(\bar{\eta} - \jmath\,\omega_0) + \bar{v}\,\dot{\bar{\eta}} \tag{5}$$

$$\Leftrightarrow \ddot{\bar{v}} = \bar{v}\,\left((\bar{\eta} - \jmath\,\omega_0)^2 + \dot{\bar{\eta}}\right) \tag{6}$$

$$\Rightarrow \bar{\eta}'' = (\bar{\eta} - \jmath\,\omega_0)^2 + \dot{\bar{\eta}}\,. \tag{7}$$

The real and imaginary parts of $\bar{\eta}''$ are hereafter denoted as σ and γ, respectively, i.e., $\bar{\eta}'' = \sigma + \jmath\,\gamma$, where:

$$\sigma = \rho^2 - (\omega - \omega_0)^2 + \dot{\rho}\,; \quad \gamma = 2\rho\,(\omega - \omega_0) + \dot{\omega}\,. \tag{8}$$

In the remainder of this work, the (original) first-order relative CF (i.e., $\bar{\eta} - \jmath\,\omega_0$) is denoted as $\bar{\eta}'$ and the second-order CF is denoted as $\bar{\eta}''$. For this paper, it is enough to recognize that $\bar{\eta}''$ contains the information of the second-order dynamics of the voltage vector, similarly to how the CF packs the first-order dynamics of the vector. Providing a complete physical interpretation of this quantity is out of the scope of this work.

B. Delta operator

Consider a standard dynamic model of power systems in the form of a set of differential algebraic equations (DAEs). Let $f(t)$ be a scalar function of time $f(t) : \mathbb{R}^+ \to \mathbb{R}$ representing an arbitrary variable of the set of DAEs, which can be an input, state, or algebraic variable. Note f might have discontinuities in the latter case as algebraic variables can *jump* at specific times, e.g., due to the ocurrence of faults.

Definition 1: Delta (Δ) operator applied to $f(t)$:

$$\Delta f(t) := \lim_{\tau \to t^+} f(\tau) - \lim_{\tau \to t^-} f(\tau)\,. \tag{9}$$

In simple words, $\Delta f(t)$ gives the difference between the value of f evaluated at a time infinitesimally after t (t^+) and infinitesimally before t (t^-). Note $\Delta f(t)$ is always null unless f is discontinuous at t. Hereafter, the notation used for the limits of the function when τ approaches t^+ and t^- is simplified as f^+ and f^-, respectively. Thus:

$$\Delta f(t) = f^+ - f^-\,. \tag{10}$$

Definition 2: The *instantaneous arithmetic mean* of f:

$$\widetilde{f}(t) := \frac{f^+ + f^-}{2}\,. \tag{11}$$

Definition 3: The *instantaneous geometric mean* of f:

$$\hat{f}(t) := \sqrt{f^+ f^-}\,. \tag{12}$$

Let $f(t)$, $g(t)$ be variables of the set of DAEs, and α, β constants. The Δ operator satisfies the properties presented below, whose proofs can be found in the addendum provided with the paper.

Property 1: Δ of a constant with time is null.

$$\Delta\alpha = 0\,. \tag{13}$$

Property 2: Linearity.

$$\Delta\{\alpha f(t) + \beta g(t)\} = \alpha\Delta f(t) + \beta\Delta g(t)\,. \tag{14}$$

Property 3: Multiplication rule.

$$\Delta\{f(t)g(t)\} = \Delta f(t)\widetilde{g}(t) + \widetilde{f}(t)\Delta g(t)\,. \tag{15}$$

Property 4: Division rule.

$$\Delta\left\{\frac{f(t)}{g(t)}\right\} = \frac{\Delta f(t)\widetilde{g}(t) - \widetilde{f}(t)\Delta g(t)}{\hat{g}(t)^2}\,. \tag{16}$$

Property 5: Chain rule of the complex exponential function:

$$\Delta e^{\jmath f(t)} = \widetilde{e^{\jmath f(t)}}\,\jmath\,\frac{\tan(\Delta f(t)/2)}{1/2}\,. \tag{17}$$

In case we calculate the limit when $f^+ \to f^-$ of the definitions and properties stated above, the Δ operator becomes equivalent to the absolute derivative operator (d), i.e., $\lim_{f^+\to f^-} \Delta f(t) = df(t)$. The proof for properties 1 and 2 is trivial. It also comes straightforwardly for properties 3 and 4 after noting that, in this situation, $\lim_{f^+\to f^-} \widetilde{f}(t) = \lim_{f^+\to f^-} \hat{f}(t) = f(t)$. Finally, the proof of property 5 needs recalling that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. This highlights the consistency of the definition and properties found for Δ.

Based on the properties above, some identities regarding the use of the Δ operator with Clarke vectors are found. Hereafter, the dependency on time (t) is omitted for simplicity.

Identity 1: Δ of a Clarke vector $\bar{v} = v\,\mathrm{e}^{\jmath\,\theta}$:

$$\Delta\bar{v} = \widetilde{v}\widetilde{\mathrm{e}^{\jmath\,\theta}}\left(\frac{\Delta v}{\widetilde{v}} + \jmath\,\frac{\tan(\Delta\theta/2)}{1/2}\right). \tag{18}$$

Identity 2: Consider a Clarke vector $\bar{\imath}$ and its equivalent after applying the Park transform with an angle θ_{dq}, $\bar{\imath}_{\mathrm{dq}} = \bar{\imath}e^{-\jmath\,\theta_{\mathrm{dq}}}$. The following property holds after aplying Δ to the original vector:

$$\Delta\bar{\imath} = \widetilde{e^{\jmath\,\theta_{\mathrm{dq}}}}\left(\Delta\bar{\imath}_{\mathrm{dq}} + \widetilde{\bar{\imath}}_{\mathrm{dq}}\,\jmath\,\frac{\tan(\Delta\theta_{\mathrm{dq}}/2)}{1/2}\right). \tag{19}$$

The definitions, properties and identities given in this section are, to the best of our knowledge, a novelty of this paper.

III. Generalization of System Strength

A. Preliminaries

We aim to provide a more general set of indicators to quantify system strength. As mentioned in the introduction, given the broad nature of the concept, we start by introducing the following considerations to treat system strength:

- It is conceived as a property of each bus of the network.
- The voltage at the bus is taken as the representative variable over which strength is to be quantified.
- The voltage strength is evaluated with respect to changes in the current injected by a fictitious independent current source at the bus.
- The concept of strength is not restricted to a sensitivity in a small-signal sense, namely, variations of the voltage due to infinitesimal current changes (in which case we would look at a derivative-like expression such as $\frac{dV}{d\imath}$). In turn, strength is a measure applicable to large-signal or discrete events (e.g., $\frac{\Delta V}{\Delta\imath}$).

B. Proposed formulation

The usual approach to define system strength is to establish a relationship between the sensitivity of the voltage vector to a sudden change in the current injection at the bus. Such a formulation can only take into account how the voltage vector changes instantaneously due to the current jump. However, recalling the essence of the concept of strength as the voltage's resistance to perturbations, such an approach falls short in representing strength, because it cannot evaluate how much the voltage will continue to deviate right after the disturbance, something we consider an important aspect of the evaluation of strength.

Thus we aim at a more general formulation to evaluate voltage strength that is able to reveal: (i) how much voltage is expected to jump, and (ii) how fast it will continue to deviate right after the current change. Specifically, we propose a formulation composed of three categories of indicators, depending on the order of the time derivative involved: zero, first, and second-order strength.

1) Zero-order strength: Sensitivity of the voltage vector $\Delta\bar{V}$ to changes in the current of the independent source $\Delta\bar{\imath}$. As $\bar{V}$ and $\bar{\imath}$ are 2-dimensional quantities, the sensitivity is 4-dimensional, i.e., there are four sensitivity factors corresponding to the possible combinations between the two components used to represent each vector. These components have to be chosen and can be different for the voltage and current. Without lack of generality, the coordinates chosen for the independent current source are its active and reactive components ($\imath_{\mathrm{p}}$, $\imath_{\mathrm{q}}$), namely, the component that is in phase and in quadrature with the voltage vector, respectively. In turn, we use polar coordinates for the voltage (v, θ). Therefore, ideally, we would evaluate the sensitivity of v and θ to changes in $\imath_{\mathrm{p}}$ and $\imath_{\mathrm{q}}$. However, note the voltage components have different dimensions: $volt$ and rad/s, which would make the strength indicators of different dimensions as well, complicating their interpretation and use. For this reason, we formulate the strength indicators over functions of v and θ that avoid this problem. We formalize this formulation using a matrix representation as shown below:

$$\begin{bmatrix} \Delta v/\widetilde{v} \\ 2\tan(\Delta\theta/2) \end{bmatrix} = \begin{bmatrix} S_{v\imath_{\mathrm{p}}} & S_{v\imath_{\mathrm{q}}} \\ S_{\theta\imath_{\mathrm{p}}} & S_{\theta\imath_{\mathrm{q}}} \end{bmatrix} \begin{bmatrix} \Delta\imath_{\mathrm{p}} \\ \Delta\imath_{\mathrm{q}} \end{bmatrix}, \tag{20}$$

$$\Delta\underline{v}_{\mathrm{v}\theta} = \underline{\underline{S}}\,\Delta\underline{\imath}_{\mathrm{pq}}\,. \tag{21}$$

The magnitude of the voltage is normalized by its instantaneous arithmetic mean. In the case of the angle, note the tangent is a bijective function for $\Delta\theta \in (-\pi, \pi)$, and for small angles in a vicinty of zero $2\tan(\Delta\theta/2) \approx \Delta\theta$. Hereafter, subscript v$\theta$ denote the voltage vector parametrized using these functions. The convenience of the form chosen will be evident in the derivation presented later in Section III-D.

2) First-order strength: Sensitivity of the first-order dynamic of the voltage vector, represented through the first-order CF $\Delta\bar{\eta}'$, in real and imaginary parts (ρ, ω), to changes in the current of the independent source $\Delta\bar{\imath}$, in active and reactive components ($\imath_{\rm p}$, $\imath_{\rm q}$). Formally:

$$\left[\begin{array}{c}\Delta\rho\\ \Delta\omega\end{array}\right]=\left[\begin{array}{cc}S_{\rho\imath_{\rm p}} & S_{\rho\imath_{\rm q}}\\ S_{\omega\imath_{\rm p}} & S_{\omega\imath_{\rm q}}\end{array}\right]\left[\begin{array}{c}\Delta\imath_{\rm p}\\ \Delta\imath_{\rm q}\end{array}\right], \tag{22}$$

$$\Delta\underline{\eta}'=\underline{\underline{S}}'\,\Delta\underline{\imath}_{\rm pq}\,. \tag{23}$$

3) Second-order strength: Sensitivity of the second-order dynamic of the voltage vector, represented through the second-order CF $\Delta\bar{\eta}''$, in real and imaginary parts (σ, γ), to changes in the current of the independent source $\Delta\bar{\imath}$, in active and reactive components ($\imath_{\rm p}$, $\imath_{\rm q}$). Formally:

$$\left[\begin{array}{c}\Delta\sigma\\ \Delta\gamma\end{array}\right]=\left[\begin{array}{cc}S_{\sigma\imath_{\rm p}} & S_{\sigma\imath_{\rm q}}\\ S_{\gamma\imath_{\rm p}} & S_{\gamma\imath_{\rm q}}\end{array}\right]\left[\begin{array}{c}\Delta\imath_{\rm p}\\ \Delta\imath_{\rm q}\end{array}\right], \tag{24}$$

$$\Delta\underline{\eta}''=\underline{\underline{S}}''\,\Delta\underline{\imath}_{\rm pq}\,. \tag{25}$$

According to the proposed general formulation, voltage strength at a single bus is quantified through twelve individual indicators, four per strength order. In practice, some symmetries in actual power system models cause some indicators to be equal. Nevertheless, in the most general case each bus has a set of twelve individual strength indicators. For instance, $S_{\upsilon\imath_{\rm q}}$ is the zero-order strength metric for the voltage's magnitude with respect to changes in reactive current injections. $S_{\gamma\imath_{\rm p}}$ is the second-order strength metric for the imaginary part of the second-order CF with respect to changes in the active current injections, and so forth.

C. Assumptions for the Evaluation of Strength Indicators

To estimate the the set of indicators presented in (20)-(25), we propose an analytical approach based on a dynamic model of the system. The resulting indicators are therefore a function of the variables and parameters of the model. As any mathematical model representing a physical system, it is based on a set of working assumptions. The most important assumptions are as follows.

- Three-phase systems in balanced conditions. This allows representing voltages and currents using dynamic Clarke vectors, and the objects introduced in the Addendum provided with this paper. In particular, the reference used for voltages and currents is an angle rotating at the fundamental frequency ω_0.
- The time scale of interest for evaluating system strength is considered the one corresponding to electromechanical phenomena. Fast electromagnetic dynamics are treated as algebraic constraints. The most important consequence of this assumption on the derivation is the use of a constant admittance matrix for the transmission network.

This model is the reference of exactness in this work, and is the starting point of the derivation presented below. Note that the proposed indicators can be also obtained using data-driven approaches, but a discussion on these approaches is beyond the scope of this paper.

D. Derivation

The starting point is the dynamic model in the standard form of a set of Differential-Algebraic Equations (DAEs). The goal is to find an analytical expression for the strength indicators introduced above, in terms of the parameters $\boldsymbol{p}$ and variables ($\boldsymbol{x}(t)$ $\boldsymbol{y}(t)$) of the system of DAEs. Formally:

$$\Delta\underline{\boldsymbol{v}}_{{\rm v}\theta}=\underline{\underline{\boldsymbol{S}}}(\boldsymbol{p},\boldsymbol{x}(t),\boldsymbol{y}(t))\Delta\underline{\boldsymbol{\imath}}_{\rm pq}\,, \tag{26}$$

$$\Delta\underline{\boldsymbol{\eta}}'=\underline{\underline{\boldsymbol{S}}}'(\boldsymbol{p},\boldsymbol{x}(t),\boldsymbol{y}(t))\Delta\underline{\boldsymbol{\imath}}_{\rm pq}\,, \tag{27}$$

$$\Delta\underline{\boldsymbol{\eta}}''=\underline{\underline{\boldsymbol{S}}}''(\boldsymbol{p},\boldsymbol{x}(t),\boldsymbol{y}(t))\Delta\underline{\boldsymbol{\imath}}_{\rm pq}\,, \tag{28}$$

where $\Delta\underline{\boldsymbol{\imath}}_{\rm pq}$ is a column vector containing the current injection of an independent current source at each bus (in pq coordinates relative to each bus), whereas $\Delta\underline{\boldsymbol{v}}_{{\rm v}\theta}$, $\Delta\underline{\boldsymbol{\eta}}'$, $\Delta\underline{\boldsymbol{\eta}}''$ are column vectors containing the change due to $\Delta\underline{\boldsymbol{\imath}}_{\rm pq}$ in the voltage, the first-order CF, and the second-order CF, at each bus, respectively. Therefore, $\underline{\underline{\boldsymbol{S}}}$, $\underline{\underline{\boldsymbol{S}}}'$ and $\underline{\underline{\boldsymbol{S}}}''$ are square matrices whose elements are smaller square matrices (2x2) containing the strength metrics as defined in (20)-(25). We are particularly interested in the diagonal elements of the matrices, which are the bus sensitivities with respect to a change in the current injection at the same bus, i.e., the sought strength indicators. The off-diagonal elements are in turn a measure of the sensitivity of a bus with respect to changes in the current injection at a different bus, and is a byproduct of the formulation with relevant information that can be used in future work.

The dependence on time of $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ is at two specific times: right before ($-$) and right after ($+$) the current injection change introduced by $\Delta\underline{\boldsymbol{\imath}}_{\rm pq}$. Thus, (26)-(28) can be expressed more precisely as follows:

$$\Delta\underline{\boldsymbol{v}}_{{\rm v}\theta}=\underline{\underline{\boldsymbol{S}}}(\boldsymbol{p},\boldsymbol{x}^-,\boldsymbol{y}^-,\boldsymbol{x}^+,\boldsymbol{y}^+)\Delta\underline{\boldsymbol{\imath}}_{\rm pq}\,, \tag{29}$$

$$\Delta\underline{\boldsymbol{\eta}}'=\underline{\underline{\boldsymbol{S}}}'(\boldsymbol{p},\boldsymbol{x}^-,\boldsymbol{y}^-,\boldsymbol{x}^+,\boldsymbol{y}^+)\Delta\underline{\boldsymbol{\imath}}_{\rm pq}\,, \tag{30}$$

$$\Delta\underline{\boldsymbol{\eta}}''=\underline{\underline{\boldsymbol{S}}}''(\boldsymbol{p},\boldsymbol{x}^-,\boldsymbol{y}^-,\boldsymbol{x}^+,\boldsymbol{y}^+)\Delta\underline{\boldsymbol{\imath}}_{\rm pq}\,, \tag{31}$$

The dependence on pre-disturbance values $\boldsymbol{x}^-$ and $\boldsymbol{y}^-$ highlights the strength indicators are specific to a given operating condition. This also implies calculating their numerical values requires initializing the set of DAEs beforehand. The strength metrics are also a function of post-disturbance values $\boldsymbol{x}^+,\boldsymbol{y}^+$, i.e., the resistance of the voltage to a perturbation depends not only on the system parameters and current operating condition, but also on the perturbation itself. This is a very important aspect of the formulation, which retains the nonlinearity of the system equations, and is a fundamental difference with respect to calculating small signal sensitivities. Obtaining $\boldsymbol{x}^+$ is straightforward as states do not jump on discrete events, i.e. $\boldsymbol{x}^+=\boldsymbol{x}^-$.

The derivation starts by considering the set of DAEs of the system. Without lack of generality, we assume it is in a current injection form, i.e., each shunt-connected device interfaces with the transmission network through its current injection at the connection bus [32]. The algebraic equations of the network voltages and current injections are:

$$\bar{\boldsymbol{v}}=\bar{\boldsymbol{Z}}\,\bar{\boldsymbol{\imath}}_{\rm dev}\,, \tag{32}$$

where $\bar{\boldsymbol{v}}$ and $\bar{\boldsymbol{\imath}}_{\rm dev}$ are column vectors containing the voltage and net current injected by shunt-connected devices at each bus, respectively. $\bar{\boldsymbol{Z}}$ is the impedance matrix, i.e., the inverse of the admittance matrix of the network.

We add the current injected by the fictitious independent source at each bus $\bar{\boldsymbol{\imath}}$ to (32):

$$\bar{\boldsymbol{v}} = \bar{\boldsymbol{Z}}\,\bar{\boldsymbol{\imath}}_{\rm dev} + \bar{\boldsymbol{Z}}\,\bar{\boldsymbol{\imath}}\,. \tag{33}$$

At this point, we organize the remainder of the derivation into five steps:

1) Use (33) to find expressions for $\bar{\boldsymbol{v}}$, $\dot{\bar{\boldsymbol{v}}}$ and $\ddot{\bar{\boldsymbol{v}}}$ in terms of $\bar{\boldsymbol{\imath}}_{\rm dev}$, $\dot{\bar{\boldsymbol{\imath}}}_{\rm dev}$, $\ddot{\bar{\boldsymbol{\imath}}}_{\rm dev}$, and $\bar{\boldsymbol{\imath}}$.
2) Apply the Δ operator to the equations found in the previous step to get expressions for $\Delta\bar{\boldsymbol{v}}$, $\Delta\dot{\bar{\boldsymbol{v}}}$ and $\Delta\ddot{\bar{\boldsymbol{v}}}$ in terms of $\Delta\bar{\boldsymbol{\imath}}_{\rm dev}$, $\Delta\dot{\bar{\boldsymbol{\imath}}}_{\rm dev}$, $\Delta\ddot{\bar{\boldsymbol{\imath}}}_{\rm dev}$, and $\Delta\bar{\boldsymbol{\imath}}$. Note that at this point, continuing the derivation requires replacing $\Delta\bar{\boldsymbol{\imath}}_{\rm dev}$, $\Delta\dot{\bar{\boldsymbol{\imath}}}_{\rm dev}$, and $\Delta\ddot{\bar{\boldsymbol{\imath}}}_{\rm dev}$ depending on device models.
3) For a single generic device, write expressions for $\Delta\bar{\imath}_{\rm dev}$, $\Delta\dot{\bar{\imath}}_{\rm dev}$, and $\Delta\ddot{\bar{\imath}}_{\rm dev}$, as a function of $\Delta\bar{v}$, $\Delta\dot{\bar{v}}$, and $\Delta\ddot{\bar{v}}$.
4) Combine the equations of steps 2 and 3 to solve for $\Delta\bar{\boldsymbol{v}}$, $\Delta\dot{\bar{\boldsymbol{v}}}$ and $\Delta\ddot{\bar{\boldsymbol{v}}}$. The solutions depends on $\bar{\boldsymbol{\imath}}$ and the parameter and variables of the system.
5) Apply the transformations required to express $\Delta\bar{\boldsymbol{v}}$, $\Delta\dot{\bar{\boldsymbol{v}}}$ and $\Delta\ddot{\bar{\boldsymbol{v}}}$ in the desired form, i.e., as $\Delta\underline{\boldsymbol{v}}_{{\rm v}\theta}$, $\Delta\underline{\boldsymbol{\eta}}'$ and $\Delta\underline{\boldsymbol{\eta}}''$, and also $\Delta\bar{\boldsymbol{\imath}}$ as $\Delta\underline{\boldsymbol{\imath}}_{\rm pq}$.
6) The derivation concludes by combining the results of steps 4 and 5 to get the sought expressions for the three orders of strength metrics, i.e., $\underline{\underline{\boldsymbol{S}}}$, $\underline{\underline{\boldsymbol{S}}}'$, and $\underline{\underline{\boldsymbol{S}}}''$.

1) Step 1: Equation (33) is already in the desired form at this stage:

$$\bar{\boldsymbol{v}} = \bar{\boldsymbol{Z}}\,\bar{\boldsymbol{\imath}}_{\rm dev} + \bar{\boldsymbol{Z}}\,\bar{\boldsymbol{\imath}}\,. \tag{34}$$

By applying the time derivative to (34), we find the expressions for $\dot{\bar{\boldsymbol{v}}}$ and $\ddot{\bar{\boldsymbol{v}}}$:

$$\dot{\bar{\boldsymbol{v}}} = \bar{\boldsymbol{Z}}\,\dot{\bar{\boldsymbol{\imath}}}_{\rm dev}\,, \tag{35}$$
$$\ddot{\bar{\boldsymbol{v}}} = \bar{\boldsymbol{Z}}\,\ddot{\bar{\boldsymbol{\imath}}}_{\rm dev}\,, \tag{36}$$

where $\dot{\bar{\boldsymbol{\imath}}} = \ddot{\bar{\boldsymbol{\imath}}} = 0$ as $\bar{\boldsymbol{\imath}}$ is a fictitious independent source, acting as a step-like input.

2) Step 2: We apply the Δ operator to (34)-(36):

$$\Delta\bar{\boldsymbol{v}} = \bar{\boldsymbol{Z}}\,\Delta\bar{\boldsymbol{\imath}}_{\rm dev} + \bar{\boldsymbol{Z}}\,\Delta\bar{\boldsymbol{\imath}}\,, \tag{37}$$
$$\Delta\dot{\bar{\boldsymbol{v}}} = \bar{\boldsymbol{Z}}\,\Delta\dot{\bar{\boldsymbol{\imath}}}_{\rm dev}\,, \tag{38}$$
$$\Delta\ddot{\bar{\boldsymbol{v}}} = \bar{\boldsymbol{Z}}\,\Delta\ddot{\bar{\boldsymbol{\imath}}}_{\rm dev}\,. \tag{39}$$

3) Step 3: Consider a single device shunt-connected at a generic bus as illustrated in Fig. 1.

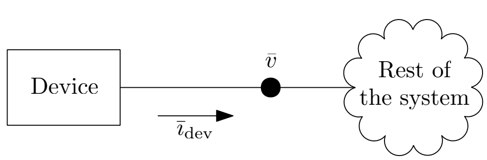

Fig. 1: Generic device interface with the rest of the system.

In a current injection form, the device can be seen as an input-output block, where the input is the terminal voltage, $\bar{v}$, and the output is the current injected at the bus, $\bar{\imath}_{\rm dev}$. This allows us to express the change in the current as a linear function of the change of the voltage:

$$\Delta\underline{\imath}_{\rm dev} = \underline{\underline{a}}\,\Delta\underline{v}\,, \tag{40}$$

where $\underline{\underline{a}}$ depends on variables and parameters of the specific model of the device. Note that we have replaced the use of complex numbers with an equivalent representation with vectors and matrices (see the Addendum provided with this paper for a detailed explanation on the mathematical objects used and their notation). Consequently, $\underline{\underline{a}}$ has four degrees of freedom instead of two, making (40) more general.

Similarly, we can assume the time derivatives of the current are, ultimately, a function of the voltage and its time derivatives. This allows us to express $\Delta\dot{\underline{\imath}}_{\rm dev}$ and $\Delta\ddot{\underline{\imath}}_{\rm dev}$ as follows:

$$\Delta\dot{\underline{\imath}}_{\rm dev} = \underline{\underline{a}}'\,\Delta\underline{v} + \underline{\underline{b}}'\,\Delta\dot{\underline{v}}\,, \tag{41}$$
$$\Delta\ddot{\underline{\imath}}_{\rm dev} = \underline{\underline{a}}''\,\Delta\underline{v} + \underline{\underline{b}}''\,\Delta\dot{\underline{v}} + \underline{\underline{c}}''\,\Delta\ddot{\underline{v}}\,. \tag{42}$$

Expressions for $\underline{\underline{a}}$, $\underline{\underline{a}}'$, $\underline{\underline{a}}''$, $\underline{\underline{b}}'$, $\underline{\underline{b}}''$ and $\underline{\underline{c}}''$ specific to relevant power system device models are presented in Section IV.

4) Step 4: The results of the previous step can be written for all buses as follows:

$$\Delta\underline{\boldsymbol{\imath}}_{\bf dev} = \underline{\underline{\boldsymbol{A}}}\,\Delta\underline{\boldsymbol{v}}\,, \tag{43}$$
$$\Delta\dot{\underline{\boldsymbol{\imath}}}_{\bf dev} = \underline{\underline{\boldsymbol{A}}}'\,\Delta\underline{\boldsymbol{v}} + \underline{\underline{\boldsymbol{B}}}'\,\Delta\dot{\underline{\boldsymbol{v}}}\,, \tag{44}$$
$$\Delta\ddot{\underline{\boldsymbol{\imath}}}_{\bf dev} = \underline{\underline{\boldsymbol{A}}}''\,\Delta\underline{\boldsymbol{v}} + \underline{\underline{\boldsymbol{B}}}''\,\Delta\dot{\underline{\boldsymbol{v}}} + \underline{\underline{\boldsymbol{C}}}''\,\Delta\ddot{\underline{\boldsymbol{v}}}\,. \tag{45}$$

Replacing (43) in (37):

$$\Delta\underline{\boldsymbol{v}} = \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}\,\Delta\underline{\boldsymbol{v}} + \underline{\underline{\boldsymbol{Z}}}\,\Delta\underline{\boldsymbol{\imath}}\,, \tag{46}$$
$$\Leftrightarrow \Delta\underline{\boldsymbol{v}} = \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}\right)^{-1}\,\underline{\underline{\boldsymbol{Z}}}\,\Delta\underline{\boldsymbol{\imath}}\,, \tag{47}$$

where $\underline{\underline{\mathbf{I}}}$ is the identity matrix. Denote:

$$\underline{\underline{\boldsymbol{Z}_{\rm eq}}} := \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}\right)^{-1}\,\underline{\underline{\boldsymbol{Z}}}\,, \tag{48}$$

then:

$$\Delta\underline{\boldsymbol{v}} = \underline{\underline{\boldsymbol{Z}_{\rm eq}}}\,\Delta\underline{\boldsymbol{\imath}}\,. \tag{49}$$

Equation (49) is the sought expression at this step for $\Delta\underline{\boldsymbol{v}}$. We continue replacing (44) in (38):

$$\Delta\dot{\underline{\boldsymbol{v}}} = \underline{\underline{\boldsymbol{Z}}}\,\left(\underline{\underline{\boldsymbol{A}}}'\,\Delta\underline{\boldsymbol{v}} + \underline{\underline{\boldsymbol{B}}}'\,\Delta\dot{\underline{\boldsymbol{v}}}\right)\,, \tag{50}$$
$$\Leftrightarrow \Delta\dot{\underline{\boldsymbol{v}}} = \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{B}}}'\right)^{-1}\,\underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}'\,\Delta\underline{\boldsymbol{v}}\,. \tag{51}$$

Using the previous result in (49):

$$\Delta\dot{\underline{\boldsymbol{v}}} = \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{B}}}'\right)^{-1}\,\underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}'\,\underline{\underline{\boldsymbol{Z}_{\rm eq}}}\,\Delta\underline{\boldsymbol{\imath}}\,. \tag{52}$$

Denote:

$$\underline{\underline{\boldsymbol{Z}'_{\rm eq}}} := \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{B}}}'\right)^{-1}\,\underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}'\,\underline{\underline{\boldsymbol{Z}_{\rm eq}}}\,, \tag{53}$$

then:

$$\Delta\dot{\underline{\boldsymbol{v}}} = \underline{\underline{\boldsymbol{Z}'_{\rm eq}}}\,\Delta\underline{\boldsymbol{\imath}}\,. \tag{54}$$

Equation (54) is the sought expression at this step for $\Delta\dot{\underline{\boldsymbol{v}}}$. Repeating the procedure for $\Delta\ddot{\underline{\boldsymbol{v}}}$:

$$\Delta\ddot{\underline{\boldsymbol{v}}} = \underline{\underline{\boldsymbol{Z}}}\,\left(\underline{\underline{\boldsymbol{A}}}''\,\Delta\underline{\boldsymbol{v}} + \underline{\underline{\boldsymbol{B}}}''\,\Delta\dot{\underline{\boldsymbol{v}}} + \underline{\underline{\boldsymbol{C}}}''\,\Delta\ddot{\underline{\boldsymbol{v}}}\right)\,, \tag{55}$$
$$\Leftrightarrow \Delta\ddot{\underline{\boldsymbol{v}}} = \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{C}}}''\right)^{-1}\,\underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}''\,\Delta\underline{\boldsymbol{v}} + \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{B}}}''\,\Delta\dot{\underline{\boldsymbol{v}}}\,. \tag{56}$$

Using the previous results in (49) and (54):

$$\Delta \underline{\ddot{\boldsymbol{v}}} = \left(\left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{C}}}'' \right)^{-1} \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}''\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq} + \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{B}}}''\,\underline{\underline{\boldsymbol{Z}}}'_{\rm eq} \right) \Delta \underline{\boldsymbol{\imath}} \,. \tag{57}$$

Denote:

$$\underline{\underline{\boldsymbol{Z}}}''_{\rm eq} := \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{C}}}'' \right)^{-1} \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}''\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq} + \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{B}}}''\,\underline{\underline{\boldsymbol{Z}}}'_{\rm eq} \,, \tag{58}$$

then:

$$\Delta \underline{\ddot{\boldsymbol{v}}} = \underline{\underline{\boldsymbol{Z}}}''_{\rm eq}\,\Delta \underline{\boldsymbol{\imath}} \,. \tag{59}$$

Equation (59) is the sought expression for $\Delta \underline{\ddot{\boldsymbol{v}}}$.

5) Step 5: We take the resulting expressions of the previous step and transform them into the desired form proposed in III-B. We start with the current, which we require to transform from $\Delta \underline{\boldsymbol{\imath}}$ to $\Delta \underline{\boldsymbol{\imath}}_{\rm pq}$. The former is in global $\rm dq$ coordinates with respect to a common rotating reference frame at ω_0, and the latter is with respect to the angle of the voltage corresponding to each bus. This transformation makes the d-axis and q-axis current to be the active and reactive components, which motivates the subscript $\rm pq$.

We start by applying the inverse Park transform to $\underline{\boldsymbol{\imath}}$:

$$\Delta \underline{\boldsymbol{\imath}} = \Delta \left\{ \underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta}\,\underline{\boldsymbol{\imath}}_{\rm pq} \right\} \,, \tag{60}$$

where $\underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta}$ is a diagonal matrix whose k-th component is:

$$\underline{\underline{e}}_k^{\jmath\,\theta} = \begin{bmatrix} \cos(\theta_k) & -\sin(\theta_k) \\ \sin(\theta_k) & \cos(\theta_k) \end{bmatrix} \,, \tag{61}$$

and θ_k is the angle of the voltage vector at bus k with respect to the global $\rm dq$ reference frame rotating at ω_0.

Next, we apply the definition of the Δ operator (equation (10). Hence, (60) becomes:

$$\Delta \underline{\boldsymbol{\imath}} = \underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta^+}\,\underline{\boldsymbol{\imath}}^+_{\rm pq} - \underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta^-}\,\underline{\boldsymbol{\imath}}^-_{\rm pq} \,. \tag{62}$$

Note that $\underline{\boldsymbol{\imath}}^-_{\rm pq} = 0$ by definition, and thus $\underline{\boldsymbol{\imath}}^+_{\rm pq} = \Delta \underline{\boldsymbol{\imath}}$. Therefore:

$$\Delta \underline{\boldsymbol{\imath}} = \underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta^+}\,\Delta \underline{\boldsymbol{\imath}}_{\rm pq} \,. \tag{63}$$

Equation (63) is the sought transformation for the current. We continue by deriving the expression required to transform $\Delta \underline{\boldsymbol{v}}$ into $\Delta \underline{\boldsymbol{v}}_{{\rm v}\theta}$. Recalling the alternative form of identity 1 (equation (18) and using a matrix representation:

$$\Delta \underline{\boldsymbol{v}} = \widetilde{\boldsymbol{v}}\,\widetilde{\underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta}}\,\Delta \underline{\boldsymbol{v}}_{{\rm v}\theta} \,. \tag{64}$$

Equation (64) is the sought transformation for the voltage. Next, we find the transformation from $\Delta \underline{\dot{\boldsymbol{v}}}$ to $\Delta \underline{\boldsymbol{\eta}}'$. To do so, we use the complex frequency property of acting as time derivative operator as presented in equation (3):

$$\Delta \underline{\dot{\boldsymbol{v}}} = \Delta \left\{ \underline{\underline{\boldsymbol{v}}}\,\underline{\boldsymbol{\eta}}' \right\} \,, \tag{65}$$

$$\Delta \underline{\dot{\boldsymbol{v}}} = \widetilde{\underline{\underline{\boldsymbol{\eta}}}}'\,\Delta \underline{\boldsymbol{v}} + \widetilde{\underline{\underline{\boldsymbol{v}}}}\,\Delta \underline{\boldsymbol{\eta}}' \,. \tag{66}$$

Therefore:

$$\Delta \underline{\boldsymbol{\eta}}' = \widetilde{\underline{\underline{\boldsymbol{v}}}}^{-1}\,\Delta \underline{\dot{\boldsymbol{v}}} - \widetilde{\underline{\underline{\boldsymbol{v}}}}^{-1}\,\widetilde{\underline{\underline{\boldsymbol{\eta}}}}'\,\Delta \underline{\boldsymbol{v}} \,. \tag{67}$$

Equation (67) is the sought transformation for the first-order complex frequency. Finally, the transformation for the second-order complex frequency is found following a similar procedure:

$$\Delta \underline{\boldsymbol{\eta}}'' = \widetilde{\underline{\underline{\boldsymbol{v}}}}^{-1}\,\Delta \underline{\ddot{\boldsymbol{v}}} - \widetilde{\underline{\underline{\boldsymbol{v}}}}^{-1}\,\widetilde{\underline{\underline{\boldsymbol{\eta}}}''}\,\Delta \underline{\boldsymbol{v}} \,. \tag{68}$$

6) Step 6: We combine the results of steps 4 and step 5 to get the final expressions for $\underline{\underline{\boldsymbol{S}}}$, $\underline{\underline{\boldsymbol{S}}}'$ and $\underline{\underline{\boldsymbol{S}}}''$. First, we replace (63) and (64) into (49):

$$\Delta \underline{\boldsymbol{v}}_{{\rm v}\theta} = \widetilde{\boldsymbol{v}}^{-1}\,(\widetilde{\underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta}})^{-1}\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq}\,\underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta^+}\,\Delta \underline{\boldsymbol{\imath}}_{\rm pq} \,. \tag{69}$$

Therefore, the sought expression for the zero-order is:

$$\boxed{\underline{\underline{\boldsymbol{S}}} = \widetilde{\boldsymbol{v}}^{-1}\,(\widetilde{\underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta}})^{-1}\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq}\,\underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta^+}} \tag{70}$$

Next, we replace (49), (54) and (63) into (67):

$$\Delta \underline{\boldsymbol{\eta}}' = \widetilde{\underline{\underline{\boldsymbol{v}}}}^{-1} \left(\underline{\underline{\boldsymbol{Z}}}'_{\rm eq} - \widetilde{\underline{\underline{\boldsymbol{\eta}}}}'\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq} \right) \underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta^+}\,\Delta \underline{\boldsymbol{\imath}}_{\rm pq} \,. \tag{71}$$

Therefore, the sought expression for the first-order is:

$$\boxed{\underline{\underline{\boldsymbol{S}}}' = \widetilde{\underline{\underline{\boldsymbol{v}}}}^{-1} \left(\underline{\underline{\boldsymbol{Z}}}'_{\rm eq} - \widetilde{\underline{\underline{\boldsymbol{\eta}}}}'\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq} \right) \underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta^+}} \tag{72}$$

We continue by replacing (49), (59) and (63) into (68):

$$\Delta \underline{\boldsymbol{\eta}}'' = \widetilde{\underline{\underline{\boldsymbol{v}}}}^{-1} \left(\underline{\underline{\boldsymbol{Z}}}''_{\rm eq} - \widetilde{\underline{\underline{\boldsymbol{\eta}}}''}\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq} \right) \underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta^+}\,\Delta \underline{\boldsymbol{\imath}}_{\rm pq} \,. \tag{73}$$

Hence, the sought expression for the second-order is:

$$\boxed{\underline{\underline{\boldsymbol{S}}}'' = \widetilde{\underline{\underline{\boldsymbol{v}}}}^{-1} \left(\underline{\underline{\boldsymbol{Z}}}''_{\rm eq} - \widetilde{\underline{\underline{\boldsymbol{\eta}}}''}\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq} \right) \underline{\underline{\boldsymbol{e}}}^{\jmath\,\theta^+}} \tag{74}$$

where:

$$\underline{\underline{\boldsymbol{Z}}}_{\rm eq} = \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}} \right)^{-1} \underline{\underline{\boldsymbol{Z}}} \,, \tag{75}$$

$$\underline{\underline{\boldsymbol{Z}}}'_{\rm eq} = \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{B}}}' \right)^{-1} \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}'\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq} \,, \tag{76}$$

$$\underline{\underline{\boldsymbol{Z}}}''_{\rm eq} = \left(\underline{\underline{\mathbf{I}}} - \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{C}}}'' \right)^{-1} \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{A}}}''\,\underline{\underline{\boldsymbol{Z}}}_{\rm eq} + \underline{\underline{\boldsymbol{Z}}}\,\underline{\underline{\boldsymbol{B}}}''\,\underline{\underline{\boldsymbol{Z}}}'_{\rm eq} \,. \tag{77}$$

The resulting strength metrics depend on the device models through $\underline{\underline{\boldsymbol{A}}}$, $\underline{\underline{\boldsymbol{A}}}'$, $\underline{\underline{\boldsymbol{A}}}''$, $\underline{\underline{\boldsymbol{B}}}'$, $\underline{\underline{\boldsymbol{B}}}''$ and $\underline{\underline{\boldsymbol{C}}}''$ matrices. Consequently, evaluating strength as a function of the parameters and variables of a specific system requires the knowledge of the devices composing the grid. The problem thus reduces to find expressions for $\underline{\underline{a}}$, $\underline{\underline{a}}'$, $\underline{\underline{a}}''$, $\underline{\underline{b}}'$, $\underline{\underline{b}}''$ and $\underline{\underline{c}}''$ for specific devices. We address this task in the next section.

IV. Device Models

In this section, we work with a variety of basic power system device models that are relevant to system strength, e.g., synchronous machines, inverter-based resources, and basic load models. For each case, the goal is to find analytical expressions for $\underline{\underline{a}}$, $\underline{\underline{a}}'$, $\underline{\underline{a}}''$, $\underline{\underline{b}}'$, $\underline{\underline{b}}''$ and $\underline{\underline{c}}''$, which can be replaced into the results of the previous section to evaluate strength.

To do so, we follow a systematic methodology divided into two steps. First, starting from the set of DAEs composing the dynamic model of the device, find expressions for its current injection at terminals and its time derivatives, i.e., $\underline{\imath}_{\rm dev}$, $\underline{\dot{\imath}}_{\rm dev}$ and $\underline{\ddot{\imath}}_{\rm dev}$. These expressions must be a function only of model inputs, states, the terminal voltage and/or its time derivatives. Second, apply the Δ operator to the equations found. Finally, the sought expressions for $\underline{\underline{a}}$, $\underline{\underline{a}}'$, $\underline{\underline{a}}''$, $\underline{\underline{b}}'$, $\underline{\underline{b}}''$ and $\underline{\underline{c}}''$ can be identified from the results of the second step.

A. Synchronous machines

Consider the classical model of synchronous machines [32], where the symbols have the usual meanings:

$$\dot{\delta} = \Omega_{\rm b}(\omega_{\rm r} - 1)\,, \tag{78}$$

$$M\dot{\omega}_{\rm r} = p_{\rm m} - p_{\rm e} - D(\omega_{\rm r} - 1)\,, \tag{79}$$

along with the algebraic equations:

$$0 = (r_{\rm a} + \jmath\, x_{\rm 1d})\,\bar{\imath}_{\rm dev} - \bar{E} + \bar{v}\,, \tag{80}$$

$$0 = \bar{E}\exp(-\jmath\,(\delta - \pi/2)) - \jmath\, e_{\rm 1q}\,, \tag{81}$$

$$0 = \Re\{\bar{E}\,\bar{\imath}^{*}_{\rm dev}\} - p_{\rm e}\,, \tag{82}$$

where $p_{\rm m}$ and $e_{\rm 1q}$ are the inputs of the model, normally states of the machine controllers.

From the set of DAEs of the machine's model, the following expressions are found:

$$\bar{\imath}_{\rm dev} = (r_{\rm a} + \jmath\, x_{\rm 1d})^{-1}\left(\bar{E} - \bar{v}\right)\,, \tag{83}$$

$$\dot{\bar{\imath}}_{\rm dev} = (r_{\rm a} + \jmath\, x_{\rm 1d})^{-1}\left(\dot{\bar{E}} - \dot{\bar{v}}\right)\,, \tag{84}$$

$$\ddot{\bar{\imath}}_{\rm dev} = (r_{\rm a} + \jmath\, x_{\rm 1d})^{-1}\left(\ddot{\bar{E}} - \ddot{\bar{v}}\right)\,, \tag{85}$$

where:

$$\bar{E} = e_{\rm 1q}\exp(\jmath\,\delta)\,, \tag{86}$$

$$\dot{\bar{E}} = \bar{E}\left(\frac{\dot{e}_{\rm 1q}}{e_{\rm 1q}} + \jmath\,\dot{\delta}\right)\,, \tag{87}$$

$$\ddot{\bar{E}} = \bar{E}\left(\left(\frac{\dot{e}_{\rm 1q}}{e_{\rm 1q}} + \jmath\,\dot{\delta}\right)^{2} + \frac{\ddot{e}_{\rm 1q}e_{\rm 1q} - \dot{e}^{2}_{\rm 1q}}{{e_{\rm 1q}}^{2}} + \jmath\,\ddot{\delta}\right)\,. \tag{88}$$

Considering a constant $e_{\rm 1q}$ and the differential equations of the model, equations (86)-(88) can be rewritten as:

$$\bar{E} = e_{\rm 1q}\exp(\jmath\,\delta) \tag{89}$$

$$\dot{\bar{E}} = \jmath\,\Omega_{\rm b}\bar{E}(\omega_{\rm r} - 1) \tag{90}$$

$$\ddot{\bar{E}} = \bar{E}(-\Omega^{2}_{\rm b}(\omega_{\rm r} - 1)^{2} + \jmath\,\Omega_{\rm b}\dot{\omega}_{\rm r}) \tag{91}$$

$$= \bar{E}(-\Omega^{2}_{\rm b}(\omega_{\rm r} - 1)^{2} + \jmath\,\frac{\Omega_{\rm b}}{M}\left(p_{\rm m} - p_{\rm e} - D(\omega_{\rm r} - 1)\right))\,. \tag{92}$$

The Δ operator is applied to (83):

$$\Delta\bar{\imath}_{\rm dev} = (r_{\rm a} + \jmath\, x_{\rm 1d})^{-1}\left(\Delta\bar{E} - \Delta\bar{v}\right)\,, \tag{93}$$

where, using identity 1 (equation (18):

$$\Delta\bar{E} = \widetilde{e}_{\rm 1q}\,\widetilde{{\rm e}^{\jmath\,\delta}}\left(\frac{\Delta e_{\rm 1q}}{\widetilde{e}_{\rm 1q}} + \jmath\,\frac{\tan(\Delta\delta/2)}{1/2}\right)\,. \tag{94}$$

As δ is a state and $e_{\rm 1q}$ a constant input, $\Delta\delta = \Delta e_{\rm 1q} = 0$. Therefore:

$$\Delta\bar{E} = 0\,. \tag{95}$$

Finally, we obtain:

$$\Delta\bar{\imath}_{\rm dev} = -(r_{\rm a} + \jmath\, x_{\rm 1d})^{-1}\Delta\bar{v}\,, \tag{96}$$

from where $\underline{\underline{a}}$ is identified:

$$\boxed{\underline{\underline{a}} = -\begin{bmatrix} r_{\rm a} & -x_{\rm 1d} \\ x_{\rm 1d} & r_{\rm a} \end{bmatrix}^{-1}} \tag{97}$$

Next, the Δ operator is applied to (84):

$$\Delta\dot{\bar{\imath}}_{\rm dev} = (r_{\rm a} + \jmath\, x_{\rm 1d})^{-1}\left(\Delta\dot{\bar{E}} - \Delta\dot{\bar{v}}\right)\,, \tag{98}$$

where:

$$\Delta\dot{\bar{E}} = \jmath\,\Omega_{\rm b}\left(\Delta\bar{E}(\widetilde{\omega}_{\rm r} - 1) + \bar{E}\Delta\omega_{\rm r}\right)\,. \tag{99}$$

Again, as $\omega_{\rm r}$ is a state and using (95):

$$\Delta\dot{\bar{E}} = 0\,. \tag{100}$$

Hence:

$$\Delta\dot{\bar{\imath}}_{\rm dev} = -(r_{\rm a} + \jmath\, x_{\rm 1d})^{-1}\Delta\dot{\bar{v}}\,, \tag{101}$$

from where $\underline{\underline{a}}'$ and $\underline{\underline{b}}'$ are identified:

$$\boxed{\underline{\underline{a}}' = 0\,; \qquad \underline{\underline{b}}' = \underline{\underline{a}}\,.} \tag{102}$$

Finally, we apply the Δ operator to (85):

$$\Delta\ddot{\bar{\imath}}_{\rm dev} = (r_{\rm a} + \jmath\, x_{\rm 1d})^{-1}\left(\Delta\ddot{\bar{E}} - \Delta\ddot{\bar{v}}\right)\,, \tag{103}$$

where:

$$\Delta\ddot{\bar{E}} = \jmath\,\bar{E}\frac{\Omega_{\rm b}}{M}\left(\Delta p_{\rm m} - \Delta p_{\rm e}\right)\,. \tag{104}$$

As $p_{\rm m}$ is considered a constant input:

$$\Delta\ddot{\bar{E}} = -\jmath\,\bar{E}\frac{\Omega_{\rm b}}{M}\Delta p_{\rm e}\,. \tag{105}$$

Note $p_{\rm e}$ is an algebraic variable of the machine model. Finding the desired form formulated in (45) needs expressing $p_{\rm e}$ as a function of the current and terminal voltage. To do so, note (82) can be equivalently rewritten as (see Addendum):

$$\begin{bmatrix} p_{\rm e} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\underline{\underline{E}}^{*}\,\underline{\imath}_{\rm dev}\,. \tag{106}$$

Replacing in (105) and using the previous result in (83) for $\Delta\underline{\imath}_{\rm dev}$:

$$\Delta\underline{\ddot{E}} = -\jmath\,\underline{\underline{E}}\,\frac{\Omega_{\rm b}}{M}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\underline{\underline{E}}^{*}\,\underline{\underline{a}}\,\Delta\underline{v}\,. \tag{107}$$

The resulting equation for $\Delta\underline{\ddot{E}}$ is replaced in (103):

$$\Delta\ddot{\underline{\imath}}_{\rm dev} = \jmath\,\underline{\underline{a}}\,\underline{\underline{E}}\,\frac{\Omega_{\rm b}}{M}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\underline{\underline{E}}^{*}\,\underline{\underline{a}}\,\Delta\underline{v} + \underline{\underline{a}}\,\Delta\ddot{\underline{v}}\,, \tag{108}$$

from where $\underline{\underline{a}}''$, $\underline{\underline{b}}''$ and $\underline{\underline{c}}''$ are identified:

$$\boxed{\begin{aligned} &\underline{\underline{a}}'' = \jmath\,\underline{\underline{a}}\,\underline{\underline{E}}\,\frac{\Omega_{\rm b}}{M}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\underline{\underline{E}}^{*}\,\underline{\underline{a}}\,, \\ &\underline{\underline{b}}'' = 0\,; \qquad \underline{\underline{c}}'' = \underline{\underline{a}}\,. \end{aligned}} \tag{109}$$

B. Grid-following converters

Consider a GFL converter model with active and reactive power control, ideal synchronization, and a droop frequency control. The set of DAEs describing such model are shown below, where the symbols have the usual meanings:

$$T\dot{\imath}_{\rm d} = \imath_{\rm dref} - \imath_{\rm d}\,, \tag{110}$$

$$T\dot{\imath}_{\rm q} = \imath_{\rm qref} - \imath_{\rm q}\,, \tag{111}$$

$$T_{\rm f}\dot{x}_{\rm p} = \frac{1}{R}(\omega - \omega_{\rm ref}) - x_{\rm p}\,, \tag{112}$$

along with the algebraic equations:

$$
\begin{aligned}
0 &= \bar{s}_{\mathrm{ref0}} + x_{\mathrm{p}} - \bar{s}_{\mathrm{ref}} \,, &(113)\\
0 &= \bar{v}_{\mathrm{dq}}\, \bar{\imath}^{*}_{\mathrm{ref}} - \bar{s}_{\mathrm{ref}} \,, &(114)\\
0 &= \bar{v}_{\mathrm{dq}} - v \,, &(115)\\
0 &= \bar{\imath}\, \mathrm{e}^{\jmath\theta} - \bar{\imath}_{\mathrm{dev}} \,. &(116)
\end{aligned}
$$

From the set of DAEs of the GFL model, the following expressions are found:

$$
\bar{\imath}_{\mathrm{dev}} = \bar{\imath}\, \mathrm{e}^{\jmath\theta} \,; \quad \dot{\bar{\imath}}_{\mathrm{dev}} = \bar{\imath}_{\mathrm{dev}} \left(\frac{\dot{\bar{\imath}}}{\bar{\imath}} + \jmath\,(\omega - \omega_0) \right) , \tag{117}
$$

$$
\ddot{\bar{\imath}}_{\mathrm{dev}} = \bar{\imath}_{\mathrm{dev}} \left(\left(\frac{\dot{\bar{\imath}}}{\bar{\imath}} + \jmath\,(\omega - \omega_0) \right)^2 + \jmath\,\dot{\omega} + \frac{\dot{\bar{\imath}}_{\mathrm{ref}}\, \bar{\imath} - \bar{\imath}_{\mathrm{ref}}\, \dot{\bar{\imath}}}{T\, \bar{\imath}^2} \right) ,
$$

where:

$$
\dot{\bar{\imath}} = \frac{\bar{\imath}_{\mathrm{ref}} - \bar{\imath}}{T} \,, \tag{118}
$$

$$
\dot{\bar{\imath}}_{\mathrm{ref}} = -K_{\mathrm{p}} \left(\frac{\dot{\omega}}{R} + \bar{\imath}^{*}_{\mathrm{dev}}\, \dot{\bar{v}} + \dot{\bar{\imath}}^{*}_{\mathrm{dev}}\, \bar{v} \right) + K_{\mathrm{i}}\, (\bar{s}_{\mathrm{ref}} - \bar{s}) \,. \tag{119}
$$

Next, we apply the Δ operator to (117) to find the expressions for $\Delta\bar{\imath}_{\mathrm{dev}}$, $\Delta\dot{\bar{\imath}}_{\mathrm{dev}}$ and $\Delta\ddot{\bar{\imath}}_{\mathrm{dev}}$:

$$
\Delta\bar{\imath}_{\mathrm{dev}} = \bar{\imath}\, \Delta\mathrm{e}^{\jmath\theta} . \tag{120}
$$

Recalling property 5 and identity 1 (see (17) and (18)):

$$
\Delta\bar{\imath}_{\mathrm{dev}} = \bar{\imath}\, \widetilde{\mathrm{e}^{\jmath\theta}}\, \jmath\, \frac{\tan(\Delta\theta/2)}{1/2} \,, \tag{121}
$$

which, in matrix form:

$$
\Delta\underline{\imath}_{\mathrm{dev}} = \underline{\underline{\imath}}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tilde{v}^{-1}\, (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1}\, \Delta\underline{v} \,. \tag{122}
$$

Therefore, $\underline{\underline{a}}$ is identified:

$$
\boxed{\underline{\underline{a}} = \underline{\underline{\imath}}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tilde{v}^{-1}\, (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1} .} \tag{123}
$$

In the case of $\Delta\dot{\bar{\imath}}_{\mathrm{dev}}$:

$$
\begin{aligned}
\Delta\dot{\bar{\imath}}_{\mathrm{dev}} = \Delta\bar{\imath}_{\mathrm{dev}} &\left(\frac{\tilde{\bar{\imath}}_{\mathrm{ref}} - \bar{\imath}}{T\, \bar{\imath}} + \jmath\,(\tilde{\omega} - \omega_0) \right) + \\
&\tilde{\bar{\imath}}_{\mathrm{dev}} \left(\frac{\Delta\bar{\imath}_{\mathrm{ref}}}{T\, \bar{\imath}} + \jmath\, \Delta\omega \right) ,
\end{aligned} \tag{124}
$$

where:

$$
\Delta\omega = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{\underline{\tilde{v}}}^{-1} \left(\Delta\dot{\underline{v}} - \underline{\underline{\tilde{\eta}}}'\, \Delta\underline{v} \right) , \tag{125}
$$

$$
\Delta\underline{\imath_{\mathrm{ref}}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \underline{\underline{\tilde{v}}}^{-1}\, \underline{\underline{\widetilde{\imath_{\mathrm{ref}}}}}^{*} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1}\, \Delta\underline{v} \,. \tag{126}
$$

Note the first term of (124) and $\underline{\underline{\tilde{\eta}}}'$ are negligible compared to the other terms. Adopting this simplification and combining (124) to (126) leads to the sought expression for $\Delta\dot{\underline{\imath}}_{\mathrm{dev}}$:

$$
\begin{aligned}
&\Delta\dot{\underline{\imath}}_{\mathrm{dev}} = \underline{\underline{\imath}}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{\underline{\tilde{v}}}^{-1}\, \Delta\dot{\underline{v}} + \\
&\frac{1}{T}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \hat{v}^{-2}\, (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1}\, \underline{\underline{s_{\mathrm{ref}}}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1}\, \Delta\underline{v} \,.
\end{aligned} \tag{127}
$$

Therefore, $\underline{\underline{a}}'$ and $\underline{\underline{b}}'$ are identified:

$$
\boxed{
\begin{aligned}
\underline{\underline{b}}' &= \underline{\underline{\imath}}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{\underline{\tilde{v}}}^{-1} , \\
\underline{\underline{a}}' &= \frac{1}{T}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \hat{v}^{-2}\, (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1}\, \underline{\underline{s_{\mathrm{ref}}}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1} .
\end{aligned}
}
$$

Finally, a similar procedure is followed to find an expression for $\Delta\ddot{\bar{\imath}}_{\mathrm{dev}}$:

$$
\begin{aligned}
&\Delta\ddot{\underline{\imath}}_{\mathrm{dev}} = \underline{\underline{\imath}}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{\underline{\tilde{v}}}^{-1}\, \Delta\ddot{\underline{v}} + \\
&\frac{1}{T}\, \hat{v}^{-2}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}} \left(\frac{1}{T_{\mathrm{f}} R} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \underline{\underline{s_{\mathrm{ref}}}}^{*} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1}\, \Delta\dot{\underline{v}} + \\
&\frac{1}{T^2}\, \tilde{v}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}}\, \hat{v}^{-4}\, \underline{\underline{\imath}}^{-1}\, \underline{\underline{s_{\mathrm{ref}}}}^{*2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1}\, \Delta\underline{v} \,.
\end{aligned} \tag{128}
$$

Therefore, $\underline{\underline{a}}''$, $\underline{\underline{b}}''$ and $\underline{\underline{c}}''$ are found:

$$
\boxed{
\begin{aligned}
\underline{\underline{c}}'' &= \underline{\underline{b}}' , \\
\underline{\underline{a}}'' &= \frac{1}{T^2}\, \tilde{v}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}}\, \hat{v}^{-4}\, \underline{\underline{\imath}}^{-1}\, \underline{\underline{s_{\mathrm{ref}}}}^{*2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1} , \\
\underline{\underline{b}}'' &= \frac{1}{T}\, \hat{v}^{-2}\, \widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}} \left(\frac{1}{T_{\mathrm{f}} R} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \underline{\underline{s_{\mathrm{ref}}}}^{*} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) (\widetilde{\underline{\underline{\mathrm{e}}}^{\jmath\theta}})^{-1} .
\end{aligned}
}
$$

C. Constant-impedance loads

Consider a standard constant shunt impedance load model:

$$
0 = \bar{v} + \bar{z}\, \bar{\imath}_{\mathrm{dev}} \,. \tag{129}
$$

From (129), the following expressions are found:

$$
\begin{aligned}
\bar{\imath}_{\mathrm{dev}} &= -\bar{z}^{-1}\, \bar{v} \;\Rightarrow\; \Delta\bar{\imath}_{\mathrm{dev}} = -\bar{z}^{-1}\, \Delta\bar{v} \,, &(130)\\
\dot{\bar{\imath}}_{\mathrm{dev}} &= -\bar{z}^{-1}\, \dot{\bar{v}} \;\Rightarrow\; \Delta\dot{\bar{\imath}}_{\mathrm{dev}} = -\bar{z}^{-1}\, \Delta\dot{\bar{v}} \,, &(131)\\
\ddot{\bar{\imath}}_{\mathrm{dev}} &= -\bar{z}^{-1}\, \ddot{\bar{v}} \;\Rightarrow\; \Delta\ddot{\bar{\imath}}_{\mathrm{dev}} = -\bar{z}^{-1}\, \Delta\ddot{\bar{v}} \,. &(132)
\end{aligned}
$$

Therefore:

$$
\boxed{
\begin{aligned}
&\underline{\underline{a}} = \underline{\underline{b}}' = \underline{\underline{c}}'' = -\underline{\underline{z}}^{-1} , \\
&\underline{\underline{a}}' = \underline{\underline{a}}'' = \underline{\underline{b}}'' = 0 \,.
\end{aligned}
} \tag{133}
$$

V. Study cases

The proposed framework is discussed based on the well-known IEEE 39-bus system. Synchronous machines are represented using the classical model described in Section IV-A. Loads are considered constant-impedances according to the model presented in Section IV-C. Standard system parameters and operating conditions are used and can be found in [33].

The complete set of strength indicators $\underline{\underline{\boldsymbol{S}}}$, $\underline{\underline{\boldsymbol{S}}}'$ and $\underline{\underline{\boldsymbol{S}}}''$ is calculated at every bus using (70), (72) and (74). Without lack of generality, the test perturbation used for this example is an active current sudden change of $\Delta\bar{\imath}_{\mathrm{pq}} = 1 + \jmath\,0$ pu. The results for two representative strength indicators are presented in Fig. 2. The values are normalized to the maximum of all the indicators of the same order, and the absolute value is taken for an easier comparison among different buses using a grayscale. Fig. 2a shows the results for $S_{v\imath_{\mathrm{q}}}$, namely, the sensitivity of the magnitude of the voltage with respect to reactive current changes. As expected, the results of this zero-order metric are driven by the topology of the network plus the internal impedance of SMs. Nodes with a higher Thevenin equivalent are more sensitive, such as nodes 29, 28 and 38. In turn, nodes in more meshed areas of the grid or closer to bigger SMs are stronger, such as nodes 02, 03 and 39. The results for the opposite zero-order metric $S_{\theta \imath_{\mathrm{p}}}$, namely, the sensitivity of the angle of the voltage with respect to active current changes, is equal in magnitude to $S_{v \imath_{\mathrm{q}}}$. This is a consequence of the device models used, particularly, the classical model of SMs, which has a unique internal impedance for both coordinates.

The other two zero-order metrics $S_{v \imath_{\mathrm{p}}}$ and $S_{\theta \imath_{\mathrm{q}}}$ are much lower than those discussed previously. Fig. 2b shows the results for $S_{\gamma \imath_{\mathrm{p}}}$, namely, the sensitivity of the RoCoF (γ as defined in (8) with respect to active current changes. In this case, buses closer to SMs terminals tend to be weaker than those further from them. The rationale behind this result is that buses in 'central' locations of the grid leverage the contribution of the inertia of all the SMs. In turn, a SM terminal bus is mostly affected by the (relatively lower) inertia of the local machine. In particular, bus 36 is the weakest bus, and coincides with the terminal bus of the machine of the lowest inertia in the system. The rest of second-order metrics $S_{\gamma \imath_{\mathrm{q}}}$, $S_{\sigma \imath_{\mathrm{p}}}$ and $S_{\sigma \imath_{\mathrm{q}}}$ are negligible compared to $S_{\gamma \imath_{\mathrm{p}}}$.

Finally, the four first-order metrics $S_{\rho \imath_{\mathrm{p}}}$, $S_{\rho \imath_{\mathrm{q}}}$, $S_{\omega \imath_{\mathrm{p}}}$, $S_{\omega \imath_{\mathrm{q}}}$ are null for this system. This means that the first-order complex-frequency $\bar{\eta}'$ is infinitely strong, i.e., it does not jump after a sudden current change. This is a consequence of the device models present in the system, particularly because of the classical model of SMs. A system entirely composed of SMs makes the CF of the voltage at every node continuous.

(a) Indicator $S_{v \imath_{\mathrm{q}}}$. (b) Indicator $S_{\gamma \imath_{\mathrm{p}}}$.

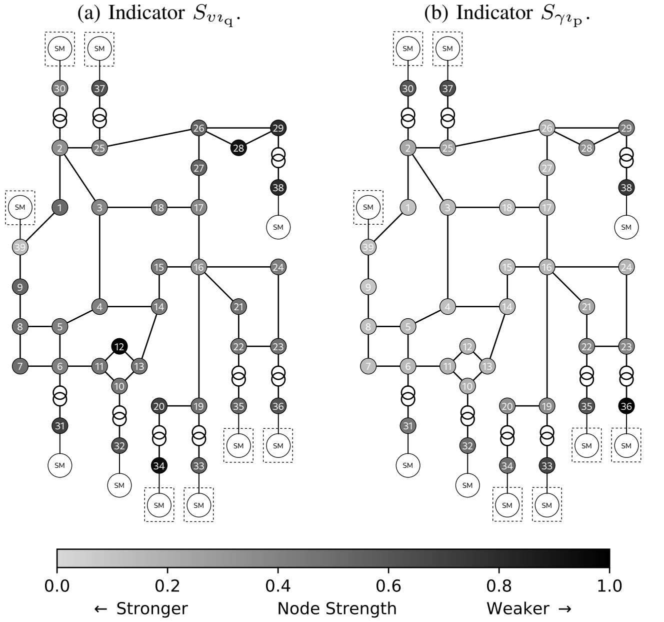

Fig. 2: Normalized results of the system strength metrics considering a conventional power system. (Note: generators marked with dashed rectangles are replaced by GLF converters in the second case study.)

Having calculated the strength metrics of the system, a dynamic validation for a sample perturbation at bus 15 at simulation time $t = 1$ s is done. The initial deviations of $\Delta \bar{v}_{15}$, $\Delta \bar{\eta}'_{15}$ and $\Delta \bar{\eta}''_{15}$ predicted by the strength indicators for the given contingency are calculated using (21), (23) and (25), and compared with the actual trajectories of these variables observed after a time-domain simulation (TDS). While $v(t)$ and $\theta(t)$ are directly available as they are part of the set of system DAEs, the exact trajectories of $\rho(t)$ and $\omega(t)$ are calculated using the Complex Frequency Divider Formula [28]. The results for these four variables are presented in Table I, which verifies the exactness of the formulation. Regarding $\sigma(t)$ and $\gamma(t)$, their trajectories are unfortunately not available directly from the TDS. However, as they are approximately equal to $\dot{\rho}$ and $\dot{\omega}$ (see (8)), their prediction can be compared to the slope of ρ and ω immediately after the perturbation. Fig. 3 shows ρ_{15}, ω_{15} and their predicted initial rate of change (PRoC) based on $\Delta\sigma_{15}$ and $\Delta\gamma_{15}$, respectively. The exactness of the second-order strength metrics is also verified.

TABLE I: Dynamic validation results.

Variable	Predicted using $\underline{\underline{S}}'_{15}$	Read from TDS	Error
Δv_{15} (pu)	-0.00461526	-0.00461524	-1.273e-08
$\Delta \theta_{15}$ (rad)	-0.01730211	-0.01730211	6.545e-09
$\Delta \rho_{15}$ (pu/s)	0.0	2.984e-06	-2.984e-06
$\Delta \omega_{15}$ (pu/s)	0.0	-4.067e-05	4.067e-05

(a) ρ_{15} and its PRoC. (b) ω_{15} and its PRoC.

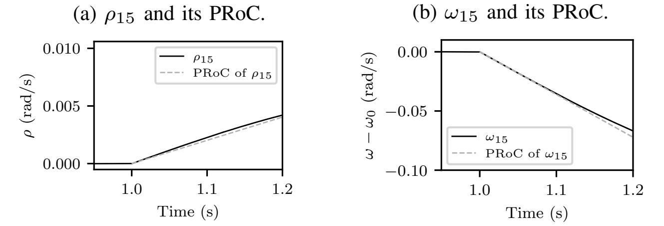

Fig. 3: Trajectories of $\bar{\eta}'_{15}$ and their PRoC.

The system is modified by replacing the SMs marked with dashed rectangles in Fig. 2 by GFL converters using the model presented in Section IV-B. For this modified system and the same perturbation simulated before, Fig. 4 shows the trajectories of v_{15}, θ_{15}, ρ_{15} and ω_{15} and their respective PRoC calculated using our strength formulation. A relevant difference with respect to the original system is that $\underline{\underline{\boldsymbol{A}}}'$ is no longer null due to the $\underline{\underline{a}}'$ component of GFLs (see (127)). This implies that $\bar{\eta}'$ is not continuous anymore, i.e., the CF at buses can now experience 'jumps', as demonstrated in Fig. 4c and Fig. 4d. The results again verify the accuracy in the prediction of $\Delta \bar{v}_{15}$, $\Delta \bar{\eta}'_{15}$ and $\Delta \bar{\eta}''_{15}$.

VI. Conclusions

This paper establishes the theoretical foundations for a general and unifying framework for power system strength. The formulation features a set of twelve indicators organized in three different dynamical orders that capture the voltage strength when subjected to current injections changes.

The paper also presents a systematic way to study the impact of different devices on strength, and shown examples featuring basic models. For instance, the key parameters of SMs are the internal reactance, mostly affecting the zero-order

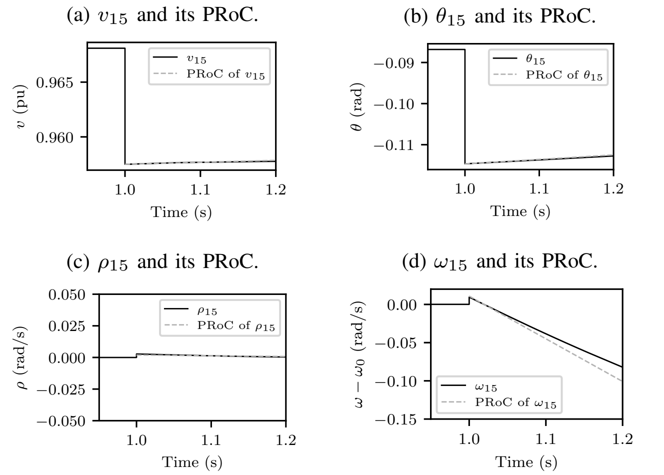

Fig. 4: Trajectories of $\bar{v}_{15}$, $\bar{\eta}'_{15}$ and their PRoC in the modified system.

strength, and the inertia, dominating the second-order strength. Furthermore, a network composed exclusively of SMs forces the first-order CF to be continuous, a characteristic that is lost under presence of GFLs due to their first-order component.

Future work will focus on applications and practical aspects of the calculation of the proposed metrics.

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Addendum to the Manuscript: Analytical Framework for Power System Strength

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This addendum provides supplementary material to support the content presented in the above-mentioned manuscript. It includes three sections:

1) Notation: A detailed explanation of the notation used throughout the paper.
2) Mathematical proofs: Formal proofs of the properties of the *Delta operator* introduced in the paper.
3) Matrix form of complex operators: Equivalent matrix form of complex operators used in the paper.

These additions are intended to enhance the technical completeness of the manuscript and facilitate reproducibility for interested readers.

I. Notation

A. Mathematical objects

The paper requires using a variety of mathematical objects. First, a bar over a symbol indicates a complex quantity:

$$\bar{v} \in \mathbb{C}, \quad \bar{v} = v_{\mathrm{d}} + \jmath\, v_{\mathrm{q}}\,, \tag{1}$$

where $v_{\mathrm{d}}, v_{\mathrm{q}} \in \mathbb{R}$ and $\jmath$ is the imaginary number. Complex numbers are widely used for representing electrical variables in three-phase balanced systems. Unfortunately, some sources of asymmetry that arise when generalizing system strength raise the need for a higher dimensional representation of them. A double bar below a symbol indicates a two by two square matrix:

$$\underline{\underline{v}} \in \mathbb{R}^{2\times 2}, \quad \underline{\underline{v}} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \tag{2}$$

where $v_{11}, v_{12}, v_{21}, v_{22} \in \mathbb{R}$. This object will typically represent a complex quantity, in which case the diagonal elements are equal to the real part, and the off-diagonal elements are the imaginary part and its additive inverse:

$$\underline{\underline{v}} \in \mathbb{R}^{2\times 2}, \, \underline{\underline{v}} = \begin{bmatrix} v_{\mathrm{d}} & -v_{\mathrm{q}} \\ v_{\mathrm{q}} & v_{\mathrm{d}} \end{bmatrix}. \tag{3}$$

Nevertheless, these objects are not restricted to a matrix representation of complex quantities, as, in general, they have four degrees of freedom instead of two. Finally, it is also convenient to define another object for representing complex quantities as a two-element vector. A single bar below a symbol denotes this object:

$$\underline{v} \in \mathbb{R}^{2\times 1} = \begin{bmatrix} v_{\mathrm{d}} \\ v_{\mathrm{q}} \end{bmatrix} = [v_{\mathrm{d}},\, v_{\mathrm{q}}]^{\mathsf{T}}, \tag{4}$$

where $v_{\mathrm{d}}, v_{\mathrm{q}} \in \mathbb{R}$ and T is the transpose operator.

We also require notation to distinguish vectors and matrices representing a set of electrical quantities for the network. Bold lowercase letters (e.g., $\boldsymbol{v}$) denote column vectors containing variables for every bus of the network. Bold uppercase letters (e.g., $\boldsymbol{Y}$) denote square matrices containing variables that relate every pair of buses of the network. Whenever bars are used above or below bold letters, it indicates that each of their elements are of the corresponding type. Some examples are given below for an n-bus system:

- $\bar{\boldsymbol{v}} \in \mathbb{C}^{n\times 1}$ contains the voltage at every bus, each one represented as a complex quantity:

$$\bar{\boldsymbol{v}} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \cdot \\ \cdot \\ \bar{v}_n \end{bmatrix} = \begin{bmatrix} v_{\mathrm{d}1} + \jmath\, v_{\mathrm{q}1} \\ v_{\mathrm{d}2} + \jmath\, v_{\mathrm{q}2} \\ \cdot \\ \cdot \\ v_{\mathrm{d}n} + \jmath\, v_{\mathrm{q}n} \end{bmatrix}, \tag{5}$$

  where $v_{\mathrm{d}k}$, $v_{\mathrm{q}k}$ are the d-axis and q-axis voltages at bus k, respectively.
- $\underline{\boldsymbol{s}} \in \mathbb{R}^{2\times 1^{n\times 1}}$ contains the complex power injected at every bus, each one represented as a two element vector:

$$\underline{\boldsymbol{s}} = \begin{bmatrix} \underline{s}_1 \\ \underline{s}_2 \\ \cdot \\ \cdot \\ \underline{s}_n \end{bmatrix} = \begin{bmatrix} [p_1,\, q_1]^{\mathsf{T}} \\ [p_2,\, q_2]^{\mathsf{T}} \\ \cdot \\ \cdot \\ [p_n,\, q_n]^{\mathsf{T}} \end{bmatrix}, \tag{6}$$

  where p_k, q_k are the active and reactive power injected at bus k, respectively.
- $\underline{\underline{\boldsymbol{Y}}} \in \mathbb{R}^{2\times 2^{n\times n}}$ is the admittance matrix of the network, whose elements are represented using two by two matrices:

$$\begin{aligned} \underline{\underline{\boldsymbol{Y}}} &= \begin{bmatrix} \underline{\underline{Y}}_{11} & \underline{\underline{Y}}_{1n} \\ \cdot & \cdot \\ \cdot & \cdot \\ \underline{\underline{Y}}_{n1} & \underline{\underline{Y}}_{nn} \end{bmatrix}, \\ &= \begin{bmatrix} \begin{bmatrix} g_{11} & -b_{11} \\ b_{11} & g_{11} \end{bmatrix} & \begin{bmatrix} g_{1n} & -b_{1n} \\ b_{1n} & g_{1n} \end{bmatrix} \\ \cdot & \cdot \\ \cdot & \cdot \\ \begin{bmatrix} g_{21} & -b_{21} \\ b_{21} & g_{21} \end{bmatrix} & \begin{bmatrix} g_{nn} & -b_{nn} \\ b_{nn} & g_{nn} \end{bmatrix} \end{bmatrix}, \end{aligned} \tag{7}$$

  where g_{hk}, b_{hk} are the real and imaginary parts of the hk-th element of the admittance matrix of the network, respectively.

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B. Time derivative

A dot over a scalar quantity denotes its time derivative:

$$\dot{x} = \frac{d}{dt}\{x\}\,. \tag{8}$$

Regarding objects with bars above or below, anytime the reference frame is fixed, a dot over the object simply denotes the time derivative of its components:

$$\dot{\bar{s}} = \dot{p} + \jmath\,\dot{q}\,. \tag{9}$$

In turn, special care has to be taken when calculating time derivatives of dynamic objects, such as dynamic Clarke vectors (e.g., $\bar{v}$) or its equivalent representation in matrix or vector form (e.g., $\underline{\underline{v}}$ or $\underline{v}$). This is because their total time derivative depends on the motion of the reference frame used to define the components of the object. In this paper, we do not need to evaluate the total time derivative, but the motion of dynamic objects relative to a common stationary reference frame rotating at the fundamental frequency ω_0. Therefore, a dot over a dynamic object denotes its time derivative relative to this reference. For instance, considering the dynamic Clarke vector $\bar{v} = v_{\mathrm{d}} + \jmath\,v_{\mathrm{q}}$:

$$\dot{\bar{v}} = \dot{v}_{\mathrm{d}} + \jmath\,\dot{v}_{\mathrm{q}} + \jmath\,(\omega_{\mathrm{dq}} - \omega_0)\,\bar{v}\,, \tag{10}$$

where ω_{dq} is the angular speed of the dq coordinates of $\bar{v}$. In case it matches the fundamental frequency, i.e., $\omega_{\mathrm{dq}} = \omega_0$, the time derivative of the vector is simply equal to the derivative of its components $\dot{\bar{v}} = \dot{v}_{\mathrm{d}} + \jmath\,\dot{v}_{\mathrm{q}}$.

II. Mathematical proofs

- *Property 1:* Δ of a constant with time is null.

  Proof. Applying the definition of the Δ operator:

$$\Delta\alpha = \alpha^{+} - \alpha^{-}\,. \tag{11}$$

  Since $\alpha^{+} = \alpha^{-} = \alpha$:

$$\Delta\alpha = \alpha - \alpha = 0\,. \tag{12}$$

□

- *Property 2:* Linearity.

  Proof. Applying the definition of the Δ operator:

$$\begin{aligned}\Delta\{\alpha f(t) + \beta g(t)\} = \alpha^{+} f^{+} + \beta^{+} g^{+} \\ - (\alpha^{-} f^{-} + \beta^{-} g^{-})\,.\end{aligned} \tag{13}$$

  Using *Property 1*, (13) yields:

$$\begin{aligned}\Delta\{\alpha f(t) + \beta g(t)\} &= \alpha f^{+} + \beta g^{+} - \alpha f^{-} - \beta g^{-} & (14)\\ &= \alpha(f^{+} - f^{-}) + \beta(g^{+} - g^{-}) & (15)\\ &= \alpha\Delta f(t) + \beta\Delta g(t)\,. & (16)\end{aligned}$$

□

- *Property 3:* Multiplication rule.

  Proof. Applying the definition of the Δ operator:

$$\Delta\{f(t)g(t)\} = f^{+}g^{+} - f^{-}g^{-}\,. \tag{17}$$

A conveniently chosen zero is added to (17) by adding and subtracting the quantity $f^{+}g^{-}$:

$$\begin{aligned}\Delta\{f(t)g(t)\} &= f^{+}g^{+} - f^{+}g^{-} - f^{-}g^{-} + f^{+}g^{-} & (18)\\ &= f^{+}(g^{+} - g^{-}) + g^{-}(f^{+} - f^{-}) & (19)\\ &= f^{+}\Delta g(t) + g^{-}\Delta f(t)\,. & (20)\end{aligned}$$

Starting back from (17), this time the quantity $f^{-}g^{+}$ is added and subtracted. Analogously to the procedure in (18)-(20), it can be shown that:

$$\Delta\{f(t)g(t)\} = g^{+}\Delta f(t) + f^{-}\Delta g(t)\,. \tag{21}$$

Adding (20) and (21):

$$\begin{aligned}2\Delta\{f(t)g(t)\} = f^{+}\Delta g(t) + g^{-}\Delta f(t) \\ + g^{+}\Delta f(t) + f^{-}\Delta g(t)\,,\end{aligned} \tag{22}$$

$$\begin{aligned}\Delta\{f(t)g(t)\} = \Delta f(t)\frac{(g^{+} + g^{-})}{2} \\ + \Delta g(t)\frac{(f^{+} + f^{-})}{2}\,.\end{aligned} \tag{23}$$

Finally, recalling the definition of the instantaneous arithmetic mean:

$$\Delta\{f(t)g(t)\} = \Delta f(t)\tilde{g}(t) + \tilde{f}(t)\Delta g(t)\,. \tag{24}$$

□

- *Property 4:* Division rule.

  Proof. Applying the definition of the Δ operator:

$$\begin{aligned}\Delta\left\{\frac{f(t)}{g(t)}\right\} &= \frac{f^{+}}{g^{+}} - \frac{f^{-}}{g^{-}} & (25)\\ &= \frac{f^{+}g^{-} - f^{-}g^{+}}{g^{+}g^{-}}\,. & (26)\end{aligned}$$

A conveniently chosen zero is added to the numerator of (25) by adding and subtracting the quantity $f^{+}g^{+}$:

$$\begin{aligned}\Delta\left\{\frac{f(t)}{g(t)}\right\} &= \frac{f^{+}g^{-} - f^{+}g^{+} - f^{-}g^{+} + f^{+}g^{+}}{g^{+}g^{-}} & (27)\\ &= \frac{-f^{+}(g^{+} - g^{-}) + g^{+}(f^{+} - f^{-})}{g^{+}g^{-}} & (28)\\ &= \frac{-f^{+}\Delta g(t) + g^{+}\Delta f(t)}{g^{+}g^{-}}\,. & (29)\end{aligned}$$

Starting back from (25), this time the quantity $f^{-}g^{-}$ is added and subtracted. Analogously to the procedure in (27)-(29), it can be shown that:

$$\Delta\left\{\frac{f(t)}{g(t)}\right\} = \frac{g^{-}\Delta f(t) - f^{-}\Delta g(t)}{g^{+}g^{-}}\,. \tag{30}$$

Adding (29) and (30):

$$\begin{aligned}2\Delta\left\{\frac{f(t)}{g(t)}\right\} = \frac{-f^{+}\Delta g(t) + g^{+}\Delta f(t)}{g^{+}g^{-}} \\ + \frac{g^{-}\Delta f(t) - f^{-}\Delta g(t)}{g^{+}g^{-}}\,,\end{aligned} \tag{31}$$

$$\Delta\left\{\frac{f(t)}{g(t)}\right\} = \frac{\Delta f(t)\frac{(g^{+} + g^{-})}{2} - \Delta g(t)\frac{(f^{+} + f^{-})}{2}}{g^{+}g^{-}}\,. \tag{32}$$

Finally, recalling the definitions of the instantaneous arithmetic mean and the instantaneous geometric mean:

$$\Delta \left\{ \frac{f(t)}{g(t)} \right\} = \frac{\Delta f(t) \tilde{g}(t) - \tilde{f}(t) \Delta g(t)}{\hat{g}(t)^2} \,. \tag{33}$$

□

- *Property 5:* Chain rule of the complex exponential function:

  Proof. Applying the definition of the Δ operator:

$$\Delta \exp(\jmath f(t)) = \exp(\jmath f^+) - \exp(\jmath f^-) \,. \tag{34}$$

  Using Euler's formula:

$$\begin{aligned} \Delta \exp(\jmath f(t)) &= \cos(f^+) + \jmath \sin(f^+) \\ &\quad - \cos(f^-) - \jmath \sin(f^-) \end{aligned} \tag{35}$$

$$\begin{aligned} &= \jmath(\sin(f^+) - \sin(f^-)) \\ &\quad + \cos(f^+) - \cos(f^-) \,. \end{aligned} \tag{36}$$

  Consider the following trigonometric identities:

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right) , \tag{37}$$

$$\sin(\alpha) - \sin(\beta) = 2 \cos\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right) . \tag{38}$$

  Using (37) and (38), (36) becomes:

$$\begin{aligned} \Delta \exp(\jmath f(t)) &= 2\jmath \cos\left(\frac{f^+ + f^-}{2}\right) \sin\left(\frac{f^+ - f^-}{2}\right) \\ &\quad -2 \sin\left(\frac{f^+ + f^-}{2}\right) \sin\left(\frac{f^+ - f^-}{2}\right) . \end{aligned} \tag{39}$$

  Recalling the definition of the instantaneous arithmetic mean and the Δ operator:

$$\begin{aligned} \Delta \exp(\jmath f(t)) &= 2\jmath \cos(\tilde{f}(t)) \sin(\Delta f/2) \\ &\quad -2 \sin(\tilde{f}(t)) \sin(\Delta f/2) \end{aligned} \tag{40}$$

$$= 2\jmath \sin(\Delta f/2) \left(\cos(\tilde{f}(t)) + \jmath \sin(\tilde{f}(t)) \right) \tag{41}$$

$$= 2\jmath \sin(\Delta f/2) \exp(\jmath \tilde{f}(t)) \,. \tag{42}$$

  Finally, note that (42) can be equivalently rewritten as:

$$\Delta \exp(\jmath f(t)) = \exp(\jmath \tilde{f}(t)) \frac{\sin(\Delta f(t)/2)}{\Delta f(t)/2} \jmath \Delta f(t) \,. \tag{43}$$

□

III. Matrix form of complex operators

Consider a complex number $\bar{z} = a + \jmath b$. The real part, imaginary part, and conjugate operators applied to $\bar{z}$ are:

$$\Re\{\bar{z}\} = a \,; \quad \Im\{\bar{z}\} = b \,, \tag{44}$$

$$\bar{z}^* = a - \jmath b \,. \tag{45}$$

Using a two dimensional vector form of the complex quantity, i.e., $\underline{z} = [a\ b]^{\mathsf{T}}$. The equivalent $2{\times}2$ matrix form of the operators presented above are:

$$\Re\{\bar{z}\} = a \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} , \tag{46}$$

$$\Im\{\bar{z}\} = b \Leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} , \tag{47}$$

$$\bar{z}^* = a - \jmath b \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix} . \tag{48}$$