Voltage differences as functions of a matrix eigensystem

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Abstract We firstly consider an electrical network that consists of multiple subnetworks. We refer to some basic definitions which relate net current flows and voltages, the edges and nodes respectively of each subnetwork, and define matrix $\mathbf{G}$, as the direct sum of the admittance matrices that correspond to each subnetwork. Then, we prove that any voltage difference at two arbitrary nodes can be explicitly written as a function only of the eigensystem of $\mathbf{G}$, namely its eigenvalues and their corresponding eigenvectors. Next, we consider a DC/AC circuit network and by using duality properties of a matrix pencil related to the network, we obtain expressions for voltage differences as functions of the pencil's eigenvalues and their corresponding eigenvectors. To validate our findings, we present illustrative numerical examples.

Keywords: net current flows; voltages; eigensystem; admittance matrix; network.

1 Introduction

Several authors have studied static electrical networks where the nodes are the voltages and the branches are the current flows, see for example [1, 2, 3, 4]. In these articles, it can be observed that the eigensystem of the admittance matrix that forms the network can actually be directly related to the nodal voltages and branch net current flows. The term 'static electrical network' refers to a steady-state condition where the voltages and currents do not vary with time. This condition is often assumed in electrical network analysis to simplify calculations and focus on the network's behavior under constant operating conditions.

In this article we consider an electrical network, and prove formulas where any voltage difference between arbitrary nodes can be solved by only using the eigensystem of the matrix and the pencil that descibes the network.

In addition, by proving closed-form expressions of solutions of net current flows only related to eigenvalues and eigenvectors, we provide new insights on the structure of the network, see [5], which can lead to new approaches in power system optimization problems, such as real-time topology optimization or long-term transmission expansion planning, see [6], [7].

Ohm's law linearly relates the current flowing through an edge in a circuit with the voltage difference between the nodes that the edge connects. Specifically,

$$I_{kj} = Y_{kj}\Delta V_{kj}, \quad k, j = 1, 2, ..., N, \tag{1}$$

and

$$\sum_{j=1}^{N} I_{kj} = F_k, \quad k = 1, 2, ..., N. \tag{2}$$

Where $I_{kj} \in \mathbb{C}$ is the current flowing from the k-th to the j-th node in a network of N nodes; $\Delta V_{kj} = V_k - V_j$ with $\Delta V_{kj}, V_k, V_j \in \mathbb{C}$, is the voltage difference between the k-th

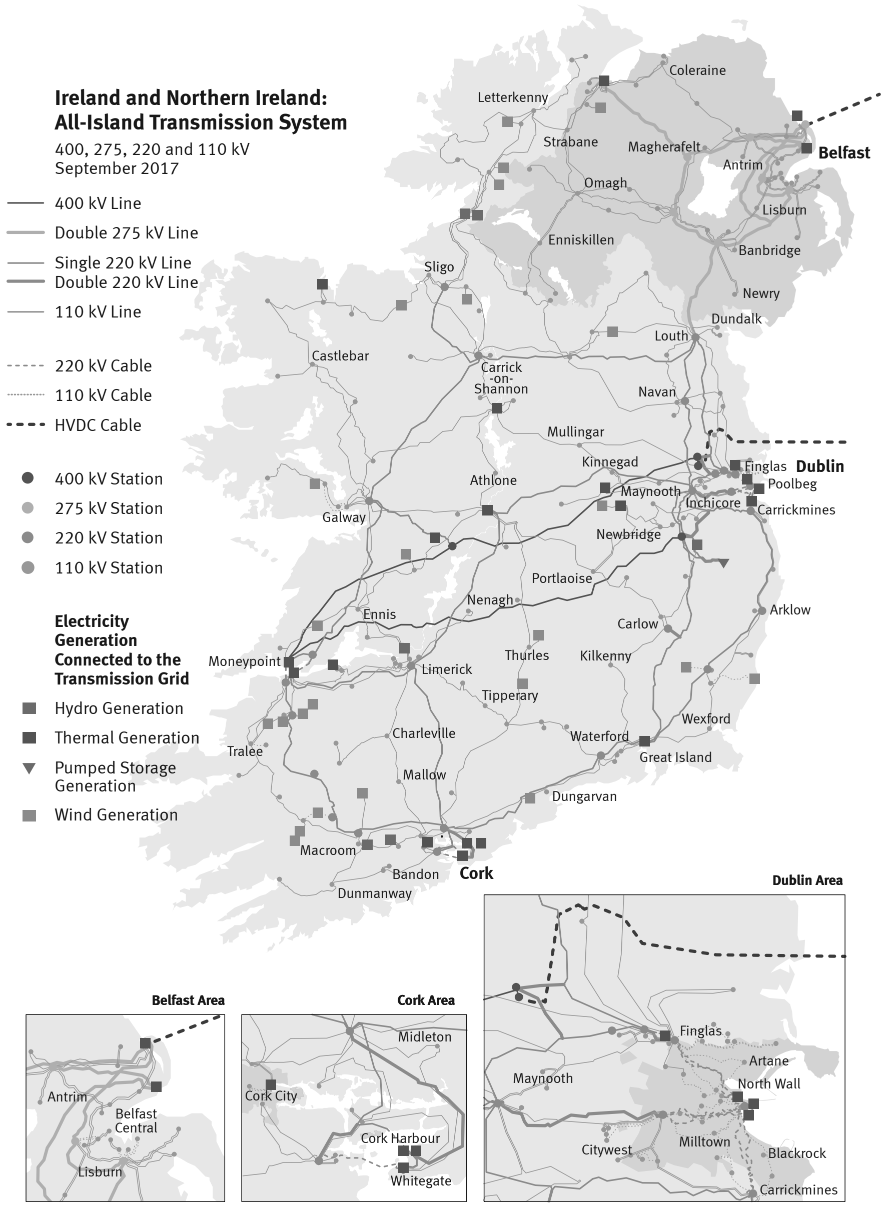

Figure 1: Example of real-world power network: Map of the all-island Irish transmission system [8].

and j-th node; Y_{kj} is admittance; and $F_k \in \mathbb{C}$ are complex-valued net current flows. By replacing (1) into (2) we arrive at:

$$\sum_{j=1}^{N} Y_{kj} \Delta V_{kj} = F_k, \quad \forall k = 1, 2, ..., N,$$

or, equivalently,

$$\sum_{j=1}^{N} Y_{kj}(V_k - V_j) = F_k, \quad \forall k = 1, 2, ..., N,$$

or, equivalently,

$$\sum_{j=1}^{N} Y_{kj} V_k - \sum_{j=1}^{N} Y_{kj} V_j = F_k, \quad \forall k = 1, 2, ..., N. \tag{3}$$

Next, we provide the following definitions.

Definition 1.1. In the article, we denote:

- δ_{ij} the Kronecker delta, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$;
- $\bar{u}$ the complex conjugate of u, and with * the conjugate transpose tensor;
- $[a_{ij}]_{i=1,2,...,n}^{j=1,2,...,m}$ the elements of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix};$$

- $\mathbf{B}_{n_1} \oplus \mathbf{B}_{n_2} \oplus \cdots \oplus \mathbf{B}_{n_r}$ the direct sum which represents the block diagonal matrix

$$\text{blockdiag} \begin{bmatrix} \mathbf{B}_{n_1} & \mathbf{B}_{n_2} & \dots & \mathbf{B}_{n_r} \end{bmatrix}, \quad \mathbf{B}_{n_i} \in \mathbb{C}^{n_i \times ni}, \quad \forall i = 1, 2, ..., r.$$

Lemma 1.1. By using (3) we can relate the currents and voltages of the network as follows:

$$\mathbf{G}\boldsymbol{V} = \mathbf{F}. \tag{4}$$

Whereby considering that the network contains p subnetworks:

$$\mathbf{G} = \mathbf{G}_1 \oplus \mathbf{G}_2 \oplus \cdots \oplus \mathbf{G}_p , \tag{5}$$

with

$$\mathbf{G}_i = \begin{bmatrix} \sum_{l=1}^{N_i} Y_{1l}^{(i)} - Y_{11}^{(i)} & -Y_{12}^{(i)} & \dots & -Y_{1N_i}^{(i)} \\ -Y_{21}^{(i)} & \sum_{l=1}^{N_i} Y_{2l}^{(i)} - Y_{22}^{(i)} & \dots & -Y_{2N_i}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ -Y_{N_i1}^{(i)} & -Y_{N_i2}^{(i)} & \dots & \sum_{l=1}^{N_i} Y_{N_il}^{(i)} - Y_{N_iN_i}^{(i)} \end{bmatrix} \tag{6}$$

and

$$\mathbf{V} = \begin{bmatrix} V_1^{(1)} \\ V_2^{(1)} \\ \vdots \\ V_{N_1}^{(1)} \\ \vdots \\ V_1^{(p)} \\ V_2^{(p)} \\ \vdots \\ V_{N_p}^{(p)} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ \vdots \\ F_{N_1}^{(1)} \\ \vdots \\ F_1^{(p)} \\ F_2^{(p)} \\ \vdots \\ F_{N_p}^{(p)} \end{bmatrix}.$$

Proof By using (3) we get:

$$\begin{bmatrix} (\sum_{l=1}^{N_1} Y_{1l}^{(1)})V_1^{(1)} - \sum_{l=1}^{N_1} Y_{1l}^{(1)} V_l^{(1)} \\ (\sum_{l=1}^{N_1} Y_{2l}^{(1)})V_2^{(1)} - \sum_{l=1}^{N_1} Y_{2l}^{(1)} V_l^{(1)} \\ \vdots \\ (\sum_{l=1}^{N_1} Y_{N_1 l}^{(1)})V_{N_1}^{(1)} - \sum_{l=1}^{N_1} Y_{N_1 l}^{(1)} V_l^{(1)} \\ \vdots \\ (\sum_{l=1}^{N_p} Y_{1l}^{(p)})V_1^{(p)} - \sum_{l=1}^{N_p} Y_{1l}^{(p)} V_l^{(p)} \\ (\sum_{l=1}^{N_p} Y_{2l}^{(p)})V_2^{(p)} - \sum_{l=1}^{N_p} Y_{2l}^{(p)} V_l^{(p)} \\ \vdots \\ (\sum_{l=1}^{N_p} Y_{N_p l}^{(p)})V_{N_p}^{(p)} - \sum_{l=1}^{N_p} Y_{N_p l}^{(p)} V_l^{(p)} \end{bmatrix} = \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ \vdots \\ F_{N_1}^{(1)} \\ \vdots \\ F_1^{(p)} \\ F_2^{(p)} \\ \vdots \\ F_{N_p}^{(p)} \end{bmatrix}.$$

From the above system if we consider the i-th block, $\forall i = 1, 2, ..., p$:

$$\begin{bmatrix} (\sum_{l=1}^{N_i} Y_{1l}^{(i)})V_1^{(i)} - \sum_{l=1}^{N_i} Y_{1l}^{(i)} V_l^{(i)} \\ (\sum_{l=1}^{N_i} Y_{2l}^{(i)})V_2^{(i)} - \sum_{l=1}^{N_i} Y_{2l}^{(i)} V_l^{(i)} \\ \vdots \\ (\sum_{l=1}^{N_i} Y_{N_i l}^{(i)})V_{N_i}^{(i)} - \sum_{l=1}^{N_i} Y_{N_i l}^{(i)} V_l^{(i)} \end{bmatrix} = \begin{bmatrix} F_1^{(i)} \\ F_2^{(i)} \\ \vdots \\ F_{N_i}^{(i)} \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} (\sum_{l=1}^{N_i} Y_{1l}^{(i)})V_1^{(i)} - Y_{11}^{(i)} V_1^{(i)} - Y_{12}^{(i)} V_2^{(i)} - \cdots - Y_{1N_i}^{(i)} V_{N_i}^{(i)} \\ (\sum_{l=1}^{N_i} Y_{2l}^{(i)})V_2^{(i)} - Y_{21}^{(i)} V_1^{(i)} - Y_{22}^{(i)} V_2^{(i)} - \cdots - Y_{2N_i}^{(i)} V_{N_i}^{(i)} \\ \vdots \\ (\sum_{l=1}^{N_i} Y_{N_i l}^{(i)})V_{N_i}^{(i)} - Y_{N_i 1}^{(i)} V_1^{(i)} - Y_{N_i 2}^{(i)} V_2^{(i)} - ... - Y_{N_i N_i}^{(i)} V_{N_i}^{(i)} \end{bmatrix} = \begin{bmatrix} F_1^{(i)} \\ F_2^{(i)} \\ \vdots \\ F_{N_i}^{(i)} \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} \sum_{l=1}^{N_i} Y_{1l}^{(i)} - Y_{11}^{(i)} & -Y_{12}^{(i)} & \dots & -Y_{1N_i}^{(i)} \\ -Y_{21}^{(i)} & \sum_{l=1}^{N_i} Y_{2l}^{(i)} - Y_{22}^{(i)} & \dots & -Y_{2N_i}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ -Y_{N_i1}^{(i)} & -Y_{N_i2}^{(i)} & \dots & \sum_{l=1}^{N_i} Y_{N_il}^{(i)} - Y_{N_iN_i}^{(i)} \end{bmatrix} \begin{bmatrix} V_1^{(i)} \\ V_2^{(i)} \\ \vdots \\ V_{N_i}^{(i)} \end{bmatrix} = \begin{bmatrix} F_1^{(i)} \\ F_2^{(i)} \\ \vdots \\ F_{N_i}^{(i)} \end{bmatrix},$$

and hence we arrive at (4). The proof is completed.

The rest of this paper is organised as follows. The main results and proof are provided in Section 2. Section 3 provides examples that illustrate the main findings of our work. Finally, the paper is concluded in Section 4.

2 Main Results

We consider a network that contains p subnetworks. An example of such network is illustrated in Fig. 2. We prove the following theorem.

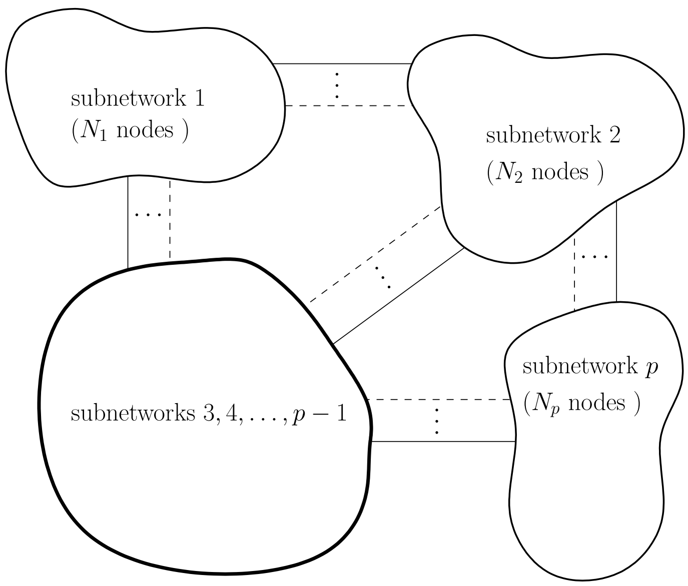

Figure 2: Example of network containing p subnetworks. The i-th subnetwork, $i = 1, 2, ..., p$, has N_i nodes.

Theorem 2.1. Consider a DC/AC circuit network of p subnetworks. The i-th subnetwork, $i = 1, 2, ..., p$, has N_i nodes. The current $I_{kj}^{(i)}$, $k, j = 1, 2, ..., N_i$ is flowing from k to j, the admittance in subnetwork i is $Y_{kj}^{(i)} = Y_{jk}^{(i)}$ and the complex-valued net current flow at each bus in subnetwork i is $F_k^{(i)}$ with $\sum_{k=1}^{N_i} F_k^{(i)} = 0$, $\forall i = 1, 2, ..., p$. Then, the voltage difference $\Delta V_{mn}^{(i)} = V_m^{(i)} - V_n^{(i)}$ in subnetwork i is given by:

$$\Delta V_{mn}^{(i)} = \sum_{j=2}^{N_i} \left[\frac{1}{\lambda_j^{(i)}} (u_{mj}^{(i)} - u_{nj}^{(i)}) \sum_{k=1}^{N_i} \bar{u}_{kj}^{(i)} F_k^{(i)} \right] . \tag{7}$$

Where $\lambda_k^{(i)}$, $k = 2, 3, ..., N_i$, $i = 1, 2, ..., p$, are the eigenvalues of the matrix

$$\mathbf{G} = \mathbf{G}_1 \oplus \mathbf{G}_2 \oplus \cdots \oplus \mathbf{G}_p , \tag{8}$$

where $\mathbf{G}_i = [g_{kj}^{(i)}]_{k=1,2,...,N}^{j=1,2,...,N}$ and:

$$g_{kj}^{(i)} = [\delta_{kj} \sum_{l=1}^{N} Y_{kl}^{(i)} + (\delta_{kj} - 1) Y_{kj}^{(i)}]_{k=1,2,...,N}^{j=1,2,...,N} . \tag{9}$$

The eigenvalues $\lambda_k^{(i)}$ may be either real or complex and can include the zero eigenvalue, which confirms the inclusion of the ground node. . With

$$\begin{bmatrix} u_{1k}^{(i)} \\ u_{2k}^{(i)} \\ \vdots \\ u_{Nk}^{(i)} \end{bmatrix}$$

we denote an eigenvector of the eigenvalue $\lambda_k^{(i)}$.

Proof. By using (3) and Lemma 1.1 we relate the currents and voltages of the network as follows:

$$\mathbf{G}\boldsymbol{V} = \mathbf{F} .$$

Where $\mathbf{G}$, $\boldsymbol{V}$, $\mathbf{F}$ are as defined in Lemma 1.1. We observe that if $g_{kj}^{(i)}$, $k, j = 1, 2, ..., N_i$ is an element of $\mathbf{G}_i$, then for $k = j$, $g_{kk}^{(i)} = \sum_{l=1}^{N_i} Y_{kl}^{(i)} - Y_{kk}^{(i)}$ and for $k \neq j$, $g_{kj}^{(i)} = -Y_{kj}^{(i)}$.

Note that the rows of $\mathbf{G}$ sum to zero, i.e. the matrix has the zero eigenvalue [9]. The algebraic multiplicity of the zero eigenvalue is the number of connected components in the network, i.e. the number of subnetworks of a network in which any two vertices of each subnetwork are connected to each other by paths, and which are connected to no additional vertices in the network. In our case we have p subnetworks which means that the algebraic multiplicity of the zero eigenvalue is p.

The eigenvalues of $\mathbf{G}$ have algebraic multiplicity equal to geometric, see [10] and the canonical form of $\mathbf{G}$ can be written as, see [11]:

$$\mathbf{G} = \mathbf{P}\mathbf{D}\mathbf{P}^* .$$

Where $\mathbf{P}$ is a matrix with columns the $N = N_1 + N_2 + \ldots + N_p$ eigenvectors of $\mathbf{G}$; $\mathbf{P}^*$ is the conjugate transpose of $\mathbf{P}$ with $\mathbf{P}\mathbf{P}^* = \mathbf{I}_N$; and $\mathbf{D}$ is the diagonal matrix with non-zero elements the eigenvalues of $\mathbf{G}$. By applying the above expression into (4) we get

$$\mathbf{P}\mathbf{D}\mathbf{P}^*\boldsymbol{V} = \mathbf{F}$$

and since $\mathbf{P}^*$ is the inverse of $\mathbf{P}$ we have

$$\mathbf{D}\mathbf{P}^*\mathbf{V} = \mathbf{P}^*\mathbf{F}\ ,$$

or, equivalently,

$$\begin{bmatrix} 0 \\ \lambda_2^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k2}^{(1)} V_k^{(1)} \\ \lambda_3^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k3}^{(1)} V_k^{(1)} \\ \vdots \\ \lambda_{N_1}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{kN_1}^{(1)} V_k^{(1)} \\ \vdots \\ 0 \\ \lambda_2^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k2}^{(p)} V_k^{(p)} \\ \lambda_3^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k3}^{(p)} V_k^{(p)} \\ \vdots \\ \lambda_{N_p}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{kN_p}^{(p)} V_k^{(p)} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{N_1} \bar{u}_{k1}^{(1)} F_k^{(1)} \\ \sum_{k=1}^{N_1} \bar{u}_{k2}^{(1)} F_k^{(1)} \\ \sum_{k=1}^{N_1} \bar{u}_{k3}^{(1)} F_k^{(1)} \\ \vdots \\ \sum_{k=1}^{N_1} \bar{u}_{kN_1}^{(1)} F_k^{(1)} \\ \vdots \\ \sum_{k=1}^{N_p} \bar{u}_{k1}^{(p)} F_k^{(p)} \\ \sum_{k=1}^{N_p} \bar{u}_{k2}^{(p)} F_k^{(p)} \\ \sum_{k=1}^{N_p} \bar{u}_{k3}^{(p)} F_k^{(p)} \\ \vdots \\ \sum_{k=1}^{N_p} \bar{u}_{kN_p}^{(p)} F_k^{(p)} \end{bmatrix}.$$

Obviously, $\sum_{k=1}^{N_i} \bar{u}_{k1}^{(i)} F_k^{(i)} = 0,\ \forall i = 1, 2, ..., p$. By ignoring the rows with zeros we can rewrite the above expression in the following form:

$$\begin{bmatrix} \sum_{k=1}^{N_1} \bar{u}_{k1}^{(1)} V_k^{(1)} \\ \lambda_2^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k2}^{(1)} V_k^{(1)} \\ \lambda_3^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k3}^{(1)} V_k^{(1)} \\ \vdots \\ \lambda_{N_1}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{kN_1}^{(1)} V_k^{(1)} \\ \vdots \\ \sum_{k=1}^{N_p} \bar{u}_{k1}^{(p)} V_k^{(p)} \\ \lambda_2^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k2}^{(p)} V_k^{(p)} \\ \lambda_3^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k3}^{(p)} V_k^{(p)} \\ \vdots \\ \lambda_{N_p}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{kN_p}^{(p)} V_k^{(p)} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{N_1} \bar{u}_{k1}^{(1)} V_k^{(1)} \\ \sum_{k=1}^{N_1} \bar{u}_{k2}^{(1)} F_k^{(1)} \\ \sum_{k=1}^{N_1} \bar{u}_{k3}^{(1)} F_k^{(1)} \\ \vdots \\ \sum_{k=1}^{N_1} \bar{u}_{kN_1}^{(1)} F_k^{(1)} \\ \vdots \\ \sum_{k=1}^{N_p} \bar{u}_{k1}^{(p)} V_k^{(p)} \\ \sum_{k=1}^{N_p} \bar{u}_{k2}^{(p)} F_k^{(p)} \\ \sum_{k=1}^{N_p} \bar{u}_{k3}^{(p)} F_k^{(p)} \\ \vdots \\ \sum_{k=1}^{N_p} \bar{u}_{kN_p}^{(p)} F_k^{(p)} \end{bmatrix}.$$

The left column on the above expression can be written as

$$\boldsymbol{\Lambda}\mathbf{P}^*\boldsymbol{V},$$

where

$$\mathbf{\Lambda} = \mathbf{\Lambda}_1 \oplus \mathbf{\Lambda}_2 \oplus \cdots \oplus \mathbf{\Lambda}_p \,,$$

and

$$\mathbf{\Lambda}_i = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \lambda_2^{(i)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{N_i}^{(i)} \end{bmatrix}, \quad \forall i = 1, 2, ..., p.$$

Hence

$$\boldsymbol{V} = \mathbf{P}\mathbf{\Lambda}^{-1} \begin{bmatrix} \sum_{k=1}^{N_1} \bar{u}_{k1}^{(1)} V_k^{(1)} \\ \sum_{k=1}^{N_1} \bar{u}_{k2}^{(1)} F_k^{(1)} \\ \sum_{k=1}^{N_1} \bar{u}_{k3}^{(1)} F_k^{(1)} \\ \vdots \\ \sum_{k=1}^{N_1} \bar{u}_{kN_1}^{(1)} F_k^{(1)} \\ \vdots \\ \sum_{k=1}^{N_p} \bar{u}_{k1}^{(p)} V_k^{(p)} \\ \sum_{k=1}^{N_p} \bar{u}_{k2}^{(p)} F_k^{(p)} \\ \sum_{k=1}^{N_p} \bar{u}_{k3}^{(p)} F_k^{(p)} \\ \vdots \\ \sum_{k=1}^{N_p} \bar{u}_{kN_p}^{(p)} F_k^{(p)} \end{bmatrix},$$

or, equivalently,

$$\boldsymbol{V} = \begin{bmatrix} u_{11}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k1}^{(1)} V_k + \frac{1}{\lambda_2} u_{12}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k2}^{(1)} F_k + \cdots + \frac{1}{\lambda_{N_1}} u_{1N_1}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{kN_1}^{(1)} F_k^{(1)} \\ u_{21}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k1}^{(1)} V_k^{(1)} + \frac{1}{\lambda_2} u_{22}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k2}^{(1)} F_k^{(1)} + \cdots + \frac{1}{\lambda_{N_1}} u_{2N_1}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{kN_1}^{(1)} F_k^{(1)} \\ \vdots \\ u_{N_1 1}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k1}^{(1)} V_k^{(1)} + \frac{1}{\lambda_2} u_{N_1 2}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{k2}^{(1)} F_k^{(1)} + \cdots + \frac{1}{\lambda_{N_1}} u_{N_1 N_1}^{(1)} \sum_{k=1}^{N_1} \bar{u}_{kN_1}^{(1)} F_k^{(1)} \\ \vdots \\ u_{11}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k1}^{(p)} V_k + \frac{1}{\lambda_2} u_{12}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k2}^{(p)} F_k + \cdots + \frac{1}{\lambda_{N_p}} u_{1N_p}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{kN_p}^{(p)} F_k^{(p)} \\ u_{21}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k1}^{(p)} V_k^{(p)} + \frac{1}{\lambda_2} u_{22}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k2}^{(p)} F_k^{(p)} + \cdots + \frac{1}{\lambda_{N_p}} u_{2N_p}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{kN_p}^{(p)} F_k^{(p)} \\ \vdots \\ u_{N_p 1}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k1}^{(p)} V_k^{(p)} + \frac{1}{\lambda_2} u_{N_p 2}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{k2}^{(p)} F_k^{(p)} + \cdots + \frac{1}{\lambda_{N_p}} u_{N_p N_p}^{(p)} \sum_{k=1}^{N_p} \bar{u}_{kN_p}^{(p)} F_k^{(p)} \end{bmatrix}.$$

Let $\mathbf{1}_i$ be a column that contains exactly i ones (for example $\mathbf{1}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{1}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, etc.) and let $\mathbf{0}_i$ be the zero column $i \times 1$. Then, since the non-zero elements of every row of $\mathbf{G}$

sum to zero, the p vectors

$$c_{N_1}\begin{bmatrix}\mathbf{1}_{N_1}\\ \mathbf{0}_{N_2}\\ \vdots\\ \mathbf{0}_{N_p}\end{bmatrix}, c_{N_2}\begin{bmatrix}\mathbf{0}_{N_1}\\ \mathbf{1}_{N_2}\\ \vdots\\ \mathbf{0}_{N_p}\end{bmatrix}, \ldots, c_{N_p}\begin{bmatrix}\mathbf{0}_{N_1}\\ \mathbf{0}_{N_2}\\ \vdots\\ \mathbf{1}_{N_p}\end{bmatrix},$$

will be p linear independent eigenvectors of the zero eigenvalue of $\mathbf{G}$ which has algebraic multiplicity p. Where $c_{N_i} \in \mathbb{C}$, $\forall i = 1, 2, ..., p$. By setting $b_j^{(i)} = \sum_{k=1}^{N_i} \bar{u}_{kj}^{(i)} F_k^{(i)}$, $i = 1, 2, ..., N_i$ and taking into account the above note we get:

$$\boldsymbol{V} = \begin{bmatrix} c_{N_1}\bar{c}_{N_1}\sum_{k=1}^{N_1} V_k^{(1)} + \sum_{j=2}^{N_1}\frac{1}{\lambda_j^{(1)}} b_j^{(1)} u_{1j}^{(1)} \\ c_{N_1}\bar{c}_{N_1}\sum_{k=1}^{N_1} V_k^{(1)} + \sum_{j=2}^{N_1}\frac{1}{\lambda_j^{(1)}} b_j^{(1)} u_{2j}^{(1)} \\ \vdots \\ c_{N_1}\bar{c}_{N_1}\sum_{k=1}^{N_1} V_k^{(1)} + \sum_{j=2}^{N_1}\frac{1}{\lambda_j^{(1)}} b_j^{(1)} u_{N_1 j}^{(1)} \\ \vdots \\ c_{N_p}\bar{c}_{N_p}\sum_{k=1}^{N_p} V_k^{(p)} + \sum_{j=2}^{N_p}\frac{1}{\lambda_j^{(p)}} b_j^{(p)} u_{N_p j}^{(p)} \\ c_{N_p}\bar{c}_{N_p}\sum_{k=1}^{N_p} V_k^{(p)} + \sum_{j=2}^{N_p}\frac{1}{\lambda_j^{(p)}} b_j^{(p)} u_{N_p j}^{(p)} \\ \vdots \\ c_{N_p}\bar{c}_{N_p}\sum_{k=1}^{N_p} V_k^{(p)} + \sum_{j=2}^{N_p}\frac{1}{\lambda_j^{(p)}} b_j^{(p)} u_{N_p j}^{(p)} \end{bmatrix}.$$

Let $V_m^{(i)}$, $V_n^{(i)}$ be two arbitrary nodal voltages in subnetwork i, $i = 1, 2, ..., p$, i.e.

$$V_m^{(i)} = c_{N_i}\bar{c}_{N_i}\sum_{k=1}^{N_i} V_k^{(i)} + \sum_{j=2}^{N_i}\frac{1}{\lambda_j^{(i)}} b_j^{(i)} u_{mj}^{(i)}$$

$$V_n^{(i)} = c_{N_i}\bar{c}_{N_i}\sum_{k=1}^{N_i} V_k^{(i)} + \sum_{j=2}^{N_i}\frac{1}{\lambda_j^{(i)}} b_j^{(i)} u_{nj}^{(i)}.$$

Then, the difference between these two arbitrary nodal voltages is given by

$$V_m^{(i)} - V_n^{(i)} = c_{N_i}\bar{c}_{N_i}\sum_{k=1}^{N_i} V_k^{(i)} + \sum_{j=2}^{N_i}\frac{1}{\lambda_j^{(i)}} b_j^{(i)} u_{mj}^{(i)} - c_{N_i}\bar{c}_{N_i}\sum_{k=1}^{N_i} V_k^{(i)} - \sum_{j=2}^{N_i}\frac{1}{\lambda_j^{(i)}} b_j^{(i)} u_{nj}^{(i)},$$

or, equivalently,

$$V_m^{(i)} - V_n^{(i)} = \sum_{j=2}^{N_i}\frac{u_{mj}^{(i)}}{\lambda_j^{(i)}} b_j^{(i)} - \sum_{j=2}^{N_i}\frac{u_{nj}^{(i)}}{\lambda_j^{(i)}} b_j^{(i)},$$

or, equivalently,

$$V_m^{(i)} - V_n^{(i)} = \sum_{j=2}^{N_i}\left[\frac{1}{\lambda_j^{(i)}}(u_{mj}^{(i)} - u_{nj}^{(i)})\sum_{k=1}^{N_i}\bar{u}_{kj}^{(i)} F_k^{(i)}\right].$$

The proof is completed.

In Theorem 2.1 we obtained the voltage difference $V_m^{(i)} - V_n^{(i)}$ as a function of the elements of $\mathbf{F}$ and spectrum of $\mathbf{G}$, or, equivalently, the eigenvalues and eigenvectors of the pencil $s\mathbf{I}_N - \mathbf{G}$. It is worth noting that the matrix $\mathbf{G}$ as used in Theorem 2.1 is symmetric and can be written as $\mathbf{G} = \mathbf{PDP}^*$, with $\mathbf{PP}^* = \mathbf{I}_N$ and $\mathbf{D}$ diagonal with non-zero elements the eigenvalues of $\mathbf{G}$. In the next theorem we will generalize the result in Theorem 2.1 for any matrix $\mathbf{G} \in \mathbb{C}^{N\times N}$. To obtain this result we will use matrix pencil theory. A pencil $s\mathbf{E} - \mathbf{A}$ is regular when its determinant is equal to a polynomial of a degree equal or less than N. Then there exist regular square matrices $\mathbf{P}$, $\mathbf{Q}$ such that:

$$\begin{aligned} \mathbf{PEQ} &= \mathbf{I}_p \oplus \mathbf{H}_q\,, \\ \mathbf{PAQ} &= \mathbf{J}_p \oplus \mathbf{I}_q\,. \end{aligned} \tag{10}$$

Where $p + q = N$, $\mathbf{I}_p$ the identity $p \times p$ matrix with p the sum of algebraic multiplicities of the finite eigenvalues of the pencil, $\mathbf{H}_q$ a $q \times q$ nilpotent matrix with index q_* with q the algebraic multiplicity of the infinite eigenvalue, and $\mathbf{J}_p$ the $p \times p$ Jordan matrix constructed by the finite eigenvalues of the pencil and their algebraic multiplicity. Furthermore

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_p \\ \mathbf{P}_q \end{bmatrix}, \; \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_p & \mathbf{Q}_q \end{bmatrix}, \tag{11}$$

with $\mathbf{P}_p$, $\mathbf{P}_q$ being $p \times n$, $q \times n$ matrices respectively, and $\mathbf{Q}_p$, $\mathbf{Q}_q$ being $n \times p$, $n \times q$ matrices respectively. The matrices $\mathbf{P}_p$, $\mathbf{Q}_p$ are constructed by left, right respectively linear independent eigenvectors of the finite eigenvalues while matrices $\mathbf{P}_q$, $\mathbf{Q}_q$ are constructed by left, right respectively linear independent eigenvectors of the infinite eigenvalues. We state the following theorem:

Theorem 2.2. We consider a DC/AC circuit network with N nodes modelled as the system of algebraic variables (4). Where $\mathbf{G} \in \mathbb{C}^{N\times N}$ is the admittance matrix having its rows sum to zero. Let $\mathbf{J}_p$ be the Jordan matrix of the finite eigenvalues of the pencil $s\mathbf{G} - \mathbf{I}_N$, and $\mathbf{Q}_p$ the matrix constructed by the $N-1$ corresponding eigenvectors. Then the i-th element of $\mathbf{V}$ is given by

$$V_i = f_i + c,$$

and a random voltage difference will be given by

$$V_i - V_j = f_i - f_j. \tag{12}$$

Where $c \in \mathbb{R}$ constant, f_i is the i-th element of the column $\mathbf{f}$ defined as:

$$\mathbf{f} := \mathbf{Q}_p \mathbf{J}_p \tilde{\mathbf{F}}_p,$$

and

$$\mathbf{Q}^{-1}\mathbf{F} = \tilde{\mathbf{F}} = \begin{bmatrix} \tilde{\mathbf{F}}_p \\ \tilde{\mathbf{F}}_q \end{bmatrix}.$$

The matrix $\mathbf{Q}$ is constructed by $\mathbf{Q}_p$ and $\mathbf{1}_m$.

Proof. The matrix $\mathbf{G}$ has the zero eigenvalue with multiplicity one. From duality theory, see [12], the pencil $s\mathbf{G} - \mathbf{I}_N$ will have an infinite eigenvalue with algebraic multiplicity $q = 1$, and p non-zero finite eigenvalues. We set $\mathbf{V} = \mathbf{Q}\tilde{\mathbf{V}}$ and $\mathbf{F} = \mathbf{Q}\tilde{\mathbf{F}}$ with

$$\tilde{\mathbf{V}} = \begin{bmatrix} \tilde{\mathbf{V}}_p \\ \tilde{\mathbf{V}}_q \end{bmatrix}, \quad \tilde{\mathbf{F}} = \begin{bmatrix} \tilde{\mathbf{F}}_p \\ \tilde{\mathbf{F}}_q \end{bmatrix}.$$

Where $\tilde{\mathbf{V}}_p$, $\tilde{\mathbf{F}}_p$ are $p \times 1$ matrices and $\tilde{\mathbf{V}}_q$, $\tilde{\mathbf{F}}_q$ are $q \times 1$. Then $\mathbf{GV} = \mathbf{F}$ can be written as

$$\mathbf{GQ}\tilde{\mathbf{V}} = \mathbf{Q}\tilde{\mathbf{F}},$$

whereby multiplying with $\mathbf{P}$ we get

$$\mathbf{PGQ}\tilde{\mathbf{V}} = \mathbf{PQ}\tilde{\mathbf{F}},$$

or, equivalently by using (10),

$$(\mathbf{I}_p \oplus \mathbf{H}_q) \begin{bmatrix} \tilde{\mathbf{V}}_p \\ \tilde{\mathbf{V}}_q \end{bmatrix} = (\mathbf{J}_p \oplus \mathbf{I}_q) \begin{bmatrix} \tilde{\mathbf{F}}_p \\ \tilde{\mathbf{F}}_q \end{bmatrix},$$

or, equivalently, since $q = 1$

$$(\mathbf{I}_p \oplus 0) \begin{bmatrix} \tilde{\mathbf{V}}_p \\ \tilde{\mathrm{V}}_q \end{bmatrix} = (\mathbf{J}_p \oplus 1) \begin{bmatrix} \tilde{\mathbf{F}}_p \\ \tilde{F}_q \end{bmatrix},$$

or, equivalently

$$\tilde{\mathbf{V}}_p = \mathbf{J}_p \tilde{\mathbf{F}}_p,$$

and

$$0 = \tilde{F}_q.$$

It is obvious that $\tilde{\mathrm{V}}_q$ can not be determined from the second subsystem and we can set $\tilde{\mathrm{V}}_q = c$, $c \in \mathbb{R}$ constant, while $\tilde{\mathbf{V}}_p$ is given from the first subsystem. We have $\mathbf{V} = \mathbf{Q}\tilde{\mathbf{V}}$ whereby using the notation in (11) we get

$$\mathbf{V} = \mathbf{Q}_p \tilde{\mathbf{V}}_p + \mathbf{Q}_q \tilde{\mathrm{V}}_q$$

or, equivalently, since $\mathbf{Q}_q = \mathbf{1}_N$,

$$\mathbf{V} = \mathbf{Q}_p \mathbf{J}_p \tilde{\mathbf{F}}_p + c\mathbf{1}_N.$$

The proof is completed.

Remark 2.1. In the case of multiple networks instead of one network, as in Theorem 2.1, the matrix $\mathbf{G}$ will have the zero eigenvalue with algebraic multiplicity greater than 1. In this case the number of independent subsystems will be equal to the geometric multiplicity of the zero eigenvalue. The constant c used in the proof of Theorem 2.1 will be replaced by a vector $\tilde{V}_q$ which will have all elements as unknown constants. The number of these constants will be equal to the geometric multiplicity of the zero eigenvalue of $\mathbf{G}$.

Our procedure can be extended to applications in Mechanics and Engineering, such as discrete models for force-based elasticity and plasticity, see [13, 14, 15], growth grain problems, see [16, 17, 18], and gas networks, see [19]. These applications leverage the same mathematical frameworks to describe physical phenomena, demonstrating the versatility and broad applicability of our approach. Future research will explore these extensions in more detail.

3 Examples

In this section we illustrate the main results provided in Theorem 2.1 and Theorem 2.2 through simple numerical examples that are relevant to the analysis of electrical power networks.

(a) Series admittance line model: We assume a single electrical line connecting two nodes 1 and 2 ($N = N_i = 2$). The line is modeled as a series admittance $Y_{12} = Y_{21} = Y$ (the superscript i is not necessary here and thus is omitted for simplicity). In the analysis of electrical power networks, this model is commonly used to accurately represent AC lines up to 100 km long. From (6), (5), we have that matrix $\mathbf{G}$ takes the form:

$$\mathbf{G} = \begin{bmatrix} Y & -Y \\ -Y & Y \end{bmatrix}.$$

The eigenvalues of $\mathbf{G}$ are the solutions of $\det(s\mathbf{I}_2 - \mathbf{G}) = 0$, where

$$\det(s\mathbf{I}_2 - \mathbf{G}) = \det\left(\begin{bmatrix} s - Y & Y \\ Y & s - Y \end{bmatrix}\right) = s\,(s - 2Y)\,.$$

Hence, the eigenvalues of $\mathbf{G}$ are $\lambda_1 = 2Y$ and $\lambda_2 = 0$. Matrix $\mathbf{P}$ is then:

$$\mathbf{P} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

where the elements of $\mathbf{P}$ have been normalized so that $\mathbf{P}\mathbf{P}^* = \mathbf{I}_2$.

Then, the main result of Theorem 2.1, i.e. equation (7), takes the following form:

$$\Delta V_{12} = V_1 - V_2 = \frac{1}{\lambda_1}(u_{11} - u_{21})(\bar{u}_{11}F_1 + \bar{u}_{21}F_2)\,,$$

or, equivalently,

$$\Delta V_{12} = \frac{\sqrt{2}}{2Y}\left(\frac{\sqrt{2}}{2}F_1 - \frac{\sqrt{2}}{2}F_2\right)\,,$$

or, equivalently,

$$\Delta V_{12} = \frac{1}{2Y} \left(F_1 - F_2 \right) \,. \tag{13}$$

The last equation holds, since from (3) we have that $F_1 = Y(V_1 - V_2)$, $F_2 = Y(V_2 - V_1)$, and by considering the difference $F_1 - F_2$ we arrive to (13).

Moreover, the eigenvalues of the pencil $s\mathbf{G} - \mathbf{I}_2$ are $\hat{\lambda}_1 = \frac{1}{\lambda_1} = \frac{1}{2Y}$, $\hat{\lambda}_2 = \frac{1}{\lambda_2} \to \infty$, with the associated $\mathbf{Q}$ and J_p being:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_p & \mathbf{Q}_q \end{bmatrix} = \begin{bmatrix} b & 1 \\ -b & 1 \end{bmatrix} \,, \quad b \in \mathbb{C} \,,$$

and $J_p = [1/2Y]$. From Theorem 2.2 we then have that:

$$\tilde{\mathbf{F}} = \mathbf{Q}^{-1}\mathbf{F} = \begin{bmatrix} \frac{1}{2b} & -\frac{1}{2b} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} Y(V_1 - V_2) \\ Y(V_2 - V_1) \end{bmatrix} \,,$$

or, equivalently:

$$\tilde{\mathbf{F}} = \begin{bmatrix} \tilde{F}_p \\ \tilde{F}_q \end{bmatrix} = \begin{bmatrix} \frac{1}{b} Y(V_1 - V_2) \\ 0 \end{bmatrix} \,.$$

From Theorem 2.2 we also have that:

$$\begin{aligned} \mathbf{f} := \mathbf{Q}_p J_p \tilde{F}_p &= \begin{bmatrix} b \\ -b \end{bmatrix} \frac{1}{2Y} \frac{Y}{b} (V_1 - V_2), \\ &= \begin{bmatrix} \frac{1}{2}(V_1 - V_2) \\ -\frac{1}{2}(V_1 - V_2) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \,, \end{aligned}$$

and thus:

$$f_1 - f_2 = V_1 - V_2 \,. \tag{14}$$

which is true from (12).

Figure 3 illustrates ΔV_{12} as a function of the line's R/X ratio, where $R + jX = 1/Y$. For the sake of example, we have assumed in Fig. 3 that $|Y| = 70.7$ S, and we have considered six different values for F_1.

(b) Single-phase load: We assume an electrical load with admittance Y connected between node 1 and the ground (node 2). In this case, we have that $V_2 = 0$ and thus (13) becomes:

$$V_1 = \frac{1}{2Y} \left(F_1 - F_2 \right) \,, \tag{15}$$

while (12) becomes $f_1 - f_2 = V_1$.

(c) Short circuit: Assuming that there is a short-circuit between nodes 1 and 2, i.e. $\Delta V_{12} = 0$, then (13) and (14) give the trivial results $F_1 = F_2$, $f_1 = f_2$, and $V_1 = V_2$, which obviously hold in this case.

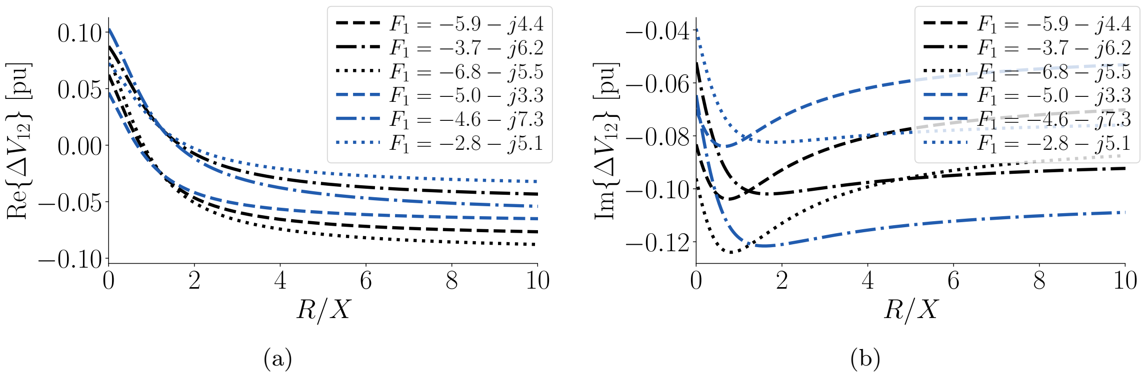

Figure 3: ΔV_{12} as a function of the electrical line's R/X ratio, for different values of F_1.

(d) Three-bus network: We assume a network comprising three nodes. In this case, matrix $\mathbf{G}$ is defined as:

$$\mathbf{G} = \begin{bmatrix} Y_{12}+Y_{13} & -Y_{12} & -Y_{13} \\ -Y_{21} & Y_{22}+Y_{23} & -Y_{23} \\ -Y_{31} & -Y_{32} & Y_{31}+Y_{32} \end{bmatrix},$$

whereby taking into account that $Y_{12}=Y_{21}$, $Y_{13}=Y_{31}$, $Y_{23}=Y_{32}$, we have:

$$\mathbf{G} = \begin{bmatrix} Y_{12}+Y_{13} & -Y_{12} & -Y_{13} \\ -Y_{12} & Y_{22}+Y_{23} & -Y_{23} \\ -Y_{13} & -Y_{23} & Y_{23}+Y_{13} \end{bmatrix}.$$

The eigenvalues of $\mathbf{G}$ are the solutions of $\det(s\mathbf{I}_3-\mathbf{G})=0$, where

$$\begin{aligned}\det(s\mathbf{I}_3-\mathbf{G}) =& \det\left(\begin{bmatrix} s-Y_{12}-Y_{13} & Y_{12} & Y_{13} \\ Y_{12} & s-Y_{22}-Y_{23} & Y_{23} \\ Y_{13} & Y_{23} & s-Y_{13}-Y_{23} \end{bmatrix}\right) \\ =& s[s^2-2(Y_{12}+Y_{13}+Y_{23})s+3(Y_{12}Y_{13}+Y_{12}Y_{23}+Y_{13}Y_{23})].\end{aligned}$$

For example, assuming that $Y_{12}=5j$ pu, $Y_{13}=9j$ pu, $Y_{23}=9j$ pu, we get that the eigenvalues of $\mathbf{G}$ are $\lambda_1=19j$, $\lambda_2=27j$, $\lambda_3=0$. The eigenvectors associated to the non-zero eigenvalues λ_1 and λ_2 are:

$$\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0.7071 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = \begin{bmatrix} -0.4082 \\ -0.4082 \\ 0.8164 \end{bmatrix},$$

where $\sqrt{u_{11}^2+u_{21}^2+u_{31}^2}=1$ and $\sqrt{u_{12}^2+u_{22}^2+u_{32}^2}=1$. In this case, from Theorem 2.1 and (7) we get:

$$\Delta V_{12} = \sum_{j=1}^{2}\left[\frac{1}{\lambda_j}(u_{1j}-u_{2j})\sum_{k=1}^{3}\bar{u}_{kj}F_k\right]$$

or, equivalently:

$$\begin{aligned}
\Delta V_{12} =& \frac{1}{\lambda_1}(u_{11} - u_{21})(\bar{u}_{11}F_1 + \bar{u}_{21}F_2 + \bar{u}_{31}F_3) \\
&+ \frac{1}{\lambda_2}(u_{12} - u_{22})(\bar{u}_{12}F_1 + \bar{u}_{22}F_2 + \bar{u}_{32}F_3) \\
=& \left(\frac{\bar{u}_{11}(u_{11} - u_{21})}{\lambda_1} + \frac{\bar{u}_{12}(u_{12} - u_{22})}{\lambda_2}\right) F_1 \\
&+ \left(\frac{\bar{u}_{21}(u_{11} - u_{21})}{\lambda_1} + \frac{\bar{u}_{22}(u_{12} - u_{22})}{\lambda_2}\right) F_2 \\
&+ \frac{\bar{u}_{31}(u_{11} - u_{21})}{\lambda_1} + \left(\frac{\bar{u}_{32}(u_{12} - u_{22})}{\lambda_2}\right) F_3
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
\Delta V_{13} =& \left(\frac{\bar{u}_{11}(u_{11} - u_{31})}{\lambda_1} + \frac{\bar{u}_{12}(u_{12} - u_{32})}{\lambda_2}\right) F_1 \\
&+ \left(\frac{\bar{u}_{21}(u_{11} - u_{31})}{\lambda_1} + \frac{\bar{u}_{22}(u_{12} - u_{32})}{\lambda_2}\right) F_2 \\
&+ \frac{\bar{u}_{31}(u_{11} - u_{31})}{\lambda_1} + \left(\frac{\bar{u}_{32}(u_{12} - u_{32})}{\lambda_2}\right) F_3
\end{aligned}$$

or, equivalently, by substituting numerical values:

$$\begin{aligned}
\Delta V_{12} =& -0.0526jF_1 + 0.0526jF_2 + 0F_3\,, \\
\Delta V_{13} =& -0.0448jF_1 - 0.0078jF_2 + 0.037jF_3\,.
\end{aligned}$$

From (3) we have that:

$$\sum_{j=1}^{3} Y_{1j}\Delta V_{1j} = Y_{12}\Delta V_{12} + Y_{13}\Delta V_{13} = F_1\,,$$

or, equivalently:

$$\frac{2}{3}F_1 - \frac{1}{3}F_2 - \frac{1}{3}F_3 = F_1,$$

or, equivalently:

$$\sum_{j=1}^{3} F_j = 0\,,$$

which always holds.

The eigenvalues of the pencil $s\mathbf{G}-\mathbf{I}_3$ are $\hat{\lambda}_1 = \frac{1}{\lambda_1} = \frac{1}{19j}$, $\hat{\lambda}_2 = \frac{1}{\lambda_2} = \frac{1}{27j}$, $\hat{\lambda}_3 = \frac{1}{\lambda_3} \to \infty$, with the associated $\mathbf{Q}$ and $\mathbf{J}_p$ being:

$$\mathbf{Q} = \begin{bmatrix}\mathbf{Q}_p & \mathbf{Q}_q\end{bmatrix} = \begin{bmatrix} b & c & 1 \\ -b & c & 1 \\ 0 & -2c & 1 \end{bmatrix}, \quad b, c \in \mathbb{C},$$
$$\mathbf{J}_p = \begin{bmatrix} \frac{1}{19j} & 0 \\ 0 & \frac{1}{27j} \end{bmatrix}.$$

From Theorem 2.2 we have that:

$$\tilde{\mathbf{F}} = \mathbf{Q}^{-1}\mathbf{F} = \begin{bmatrix} \frac{1}{2b} & -\frac{1}{2b} & 0 \\ \frac{1}{6c} & \frac{1}{6c} & -\frac{1}{3c} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} Y_{12}\Delta V_{12} + Y_{13}\Delta V_{13} \\ -Y_{12}\Delta V_{12} + Y_{23}\Delta V_{23} \\ -Y_{13}\Delta V_{13} - Y_{23}\Delta V_{23} \end{bmatrix},$$

or, equivalently:

$$\tilde{\mathbf{F}} = \begin{bmatrix} \frac{1}{b}Y_{12}\Delta V_{12} + \frac{1}{2b}Y_{13}\Delta V_{13} - \frac{1}{2b}Y_{23}\Delta V_{23} \\ \frac{1}{2c}Y_{13}\Delta V_{13} + \frac{1}{2c}Y_{23}\Delta V_{23} \\ 0 \end{bmatrix}.$$

From Theorem 2.2 we also have that:

$$\begin{aligned}
\mathbf{f} :=& \mathbf{Q}_p\mathbf{J}_p\tilde{\mathbf{F}}_p \\
=& \begin{bmatrix} b & c \\ -b & c \\ 0 & -2c \end{bmatrix} \begin{bmatrix} \frac{1}{19j} & 0 \\ 0 & \frac{1}{27j} \end{bmatrix} \begin{bmatrix} \frac{1}{b}Y_{12}\Delta V_{12} + \frac{1}{2b}Y_{13}\Delta V_{13} - \frac{1}{2b}Y_{23}\Delta V_{23} \\ \frac{1}{2c}Y_{13}\Delta V_{13} + \frac{1}{2c}Y_{23}\Delta V_{23} \end{bmatrix} \\
=& \begin{bmatrix} \frac{b}{19j} & \frac{c}{27j} \\ \frac{-b}{19j} & \frac{c}{27j} \\ 0 & \frac{-2c}{27j} \end{bmatrix} \begin{bmatrix} \frac{5j}{b}\Delta V_{12} + \frac{9j}{2c}\Delta V_{13} - \frac{9j}{2c}\Delta V_{23} \\ \frac{9j}{2c}\Delta V_{13} + \frac{9j}{2c}\Delta V_{23} \end{bmatrix} \\
=& \begin{bmatrix} \frac{5}{19}\Delta V_{12} + \frac{46}{114}\Delta V_{13} - \frac{8}{114}\Delta V_{23} \\ -\frac{5}{19}\Delta V_{12} - \frac{8}{114}\Delta V_{13} + \frac{46}{114}\Delta V_{23} \\ -\frac{1}{3}\Delta V_{13} - \frac{1}{3}\Delta V_{23} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},
\end{aligned}$$

and thus:

$$\begin{aligned}
f_1 - f_2 &= \frac{60}{114}\Delta V_{12} + \frac{54}{114}\Delta V_{13} - \frac{54}{114}\Delta V_{23}, \\
&= \Delta V_{12} + \frac{54}{114}(-\Delta V_{12} + \Delta V_{13} - \Delta V_{23}) \\
&= \Delta V_{12} = V_1 - V_2,
\end{aligned}$$

where we have made use of Kirchhoff's voltage law, i.e. $-\Delta V_{12} + \Delta V_{13} - \Delta V_{23} = 0$. Similarly we get:

$$\begin{aligned}
f_1 - f_3 &= V_1 - V_3, \\
f_2 - f_3 &= V_2 - V_3.
\end{aligned}$$

In order to further illustrate the theoretical findings of the paper, we assume a variation of the admittance Y_{13} in the range $0j$ to $20j$ (the values of the admittances Y_{12} and Y_{23} remain unchanged, i.e. $Y_{12} = 5j$ pu and $Y_{23} = 9j$ pu). For each value in this range, the eigenvalues and eigenvectors of $\mathbf{G}$ are first calculated. Then, the voltage differences ΔV_{12}, ΔV_{13}, ΔV_{23} are determined by applying Theorem 2.1. Assuming for the sake of example that $F_1 = 0.3$, $F_2 = 0.2$, $F_3 = -0.5$, Fig. 4 shows the system node voltage differences as functions of the imaginary parts of the finite eigenvalues of $\mathbf{G}$. Eigenvalue real parts are omitted because they are always zero in this numerical example.

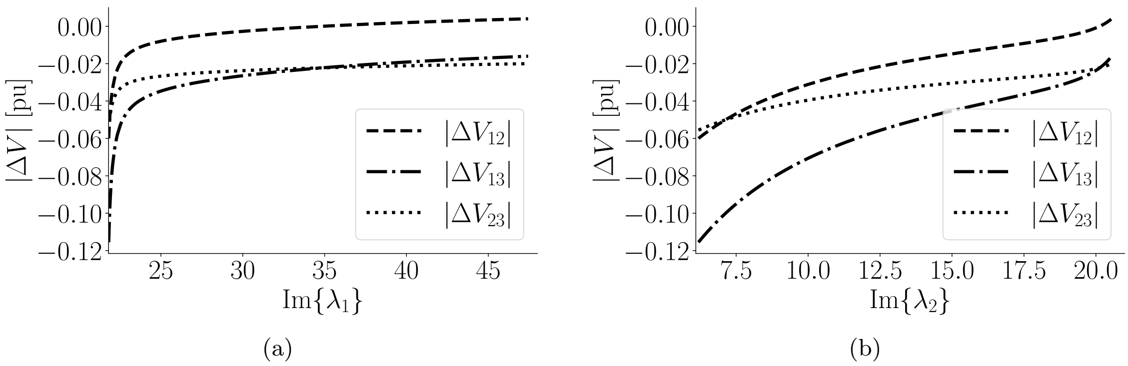

Figure 4: Voltage differences as functions of the finite eigenvalues of $\mathbf{G}$.

(e) As a last example we consider the application of Theorem 2.1 to the IEEE 14-bus benchmark system. Detailed data of the test system can be found in [20]. For the purposes of this example, the system's ground is treated as a distinct node in the definition of its admittance matrix, thus resulting in matrix $\mathbf{G}$ having dimensions 15×15.

The eigenvalues of matrix $\mathbf{G}$ are as follows:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \\ \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \\ \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{15} \end{bmatrix} = \begin{bmatrix} 16.94 - 59.72j \\ 13.31 - 42.03j \\ 7.13 - 34.3j \\ 8.97 - 22.37j \\ 3.87 - 22.45j \\ 4.86 - 17.43j \\ 6.71 - 11.02j \\ 3.35 - 11.19j \\ 4.02 - 9.86j \\ 0 \\ 0.27j \\ 0.36 - 2.19j \\ 0.46 - 2.82j \\ 1.18 - 4.79j \\ 2.39 - 5.11j \end{bmatrix}.$$

Finally, considering the power flow solution of the test system to determine the net bus current flows F_k, application of Theorem 2.1 yields the following voltage differences between bus 1 and buses 2-14:

$$\begin{bmatrix} \Delta V_{1,2} \\ \Delta V_{1,3} \\ \Delta V_{1,4} \\ \Delta V_{1,5} \\ \Delta V_{1,6} \\ \Delta V_{1,7} \\ \Delta V_{1,8} \\ \Delta V_{1,9} \\ \Delta V_{1,10} \\ \Delta V_{1,11} \\ \Delta V_{1,12} \\ \Delta V_{1,13} \\ \Delta V_{1,14} \end{bmatrix} = \begin{bmatrix} 0.07 + 0.13j \\ 0.19 + 0.31j \\ 0.17 + 0.25j \\ 0.14 + 0.21j \\ 0.17 + 0.37j \\ 0.18 + 0.33j \\ 0.14 + 0.35j \\ 0.22 + 0.36j \\ 0.23 + 0.37j \\ 0.21 + 0.37j \\ 0.21 + 0.38j \\ 0.22 + 0.38j \\ 0.26 + 0.38j \end{bmatrix}.$$

4 Conclusions

This work has indeed established groundbreaking relationships between net current flows and voltages in static electrical networks, with implications that extend far beyond traditional boundaries. The newly formulated relationships not only provide deeper insights into network structures but also offer significant computational advantages for solving power and optimal load flow problems. Moreover, they could potentially augment existing optimization methods for network topology.

Looking ahead, future research avenues are abundant. Expanding into dynamical networks presents an exciting frontier, where the principles elucidated here can be applied to dynamic systems, unlocking a wealth of opportunities for understanding and optimizing their behavior over time. Furthermore, exploring applications in Mechanics and Engineering, particularly in discrete models for force-based elasticity and plasticity, promises to extend the reach of this work beyond traditional electrical networks. Leveraging the principles established here, novel approaches to modeling and optimizing mechanical systems can be developed, driving innovation in fields such as structural engineering and materials science.

Additionally, the applicability of these formulations to energy systems, such as gas networks, presents yet another avenue for exploration. By adapting the insights gained from electrical networks to the realm of gas networks, it is possible to develop more efficient and robust solutions for energy distribution and management. This not only has implications for improving the resilience and sustainability of energy infrastructures but also opens doors to optimizing resource utilization and mitigating environmental impacts.

In conclusion, this work has the potential to significantly impact various disciplines, from dynamical systems to Mechanics and Engineering applications, and energy systems such as gas networks. By harnessing the full potential of these new formulations, we can drive innovation and advance the state of the art across multiple domains.

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