Construction of SDE-based wind speed models with exponentially decaying autocorrelation

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Abstract

This paper provides a systematic method to build wind speed models based on stochastic differential equations (SDEs). The resulting models produce stochastic processes with a given probability distribution and exponentially decaying autocorrelation function. The only information needed to build the models is the probability density function of the wind speed and its autocorrelation coefficient. Unlike other methods previously proposed in the literature, the proposed method leads to models able to reproduce an exact exponential autocorrelation even if the probability distribution is not Gaussian. A sufficient condition for the property above is provided. The paper includes the explicit formulation of SDE-based wind speed models obtained from several probability distributions used in the literature to describe different wind speed behaviors. All models are validated through numerical simulations. Finally, the proposed procedure is applied to model the wind speed observed at a meteorological station in New Zealand. A comparison of the statistical properties of the wind speed measurements and of the stochastic process generated by the SDE model is also provided.

Key words: Stochastic differential equations, Wind speed modeling, Stationary process, Regression theorem, Exponential autocorrelation, Non-gaussian processes.

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1 Introduction

1.1 Motivation

Wind speed models are used in the analysis of many aspects related to power systems, for example, in power system economics and operation (e.g., [1–3]), generation capacity reliability evaluation (e.g., [4–6]), and dynamic studies and control of wind turbines (e.g., [7–10]). The types of models traditionally used in the different research fields include time series, four-component composite models, and models based on Kalman filters. Independently of the type of the model, the appropriate characterization of the wind behaviour is a key modeling aspect, since the reliability of the results obtained in the above studies depends on it. In this paper, we develop a novel method based on stochastic differential equations, the regression theorem, and the Fokker-Planck equation, to construct wind speed models.

1.2 State of the art

From a statistical point of view, the wind speed is characterized by its probability distribution and autocorrelation. Therefore, to be adequate, wind speed models should be able to reproduce such characteristics. The type of probability distribution that best describes the wind variability depends on the particular location and on the time frame [11–14]. With regard to the autocorrelation of the wind speed, this has been usually characterized by an exponentially decaying function, either for hourly wind speed measurements in the time frame of hours [15], or for wind speed measurements on a one-second basis in the time frame of minutes [14]. However, other studies have identified scaling properties in the wind speed measurements at different sites where the autocorrelation is better described by means of power-law decaying functions [16, 17]. This paper focuses on the development of wind speed models for locations where the autocorrelation observed in the wind speed is of exponential type. Therefore, the validity of the proposed models is limited to cases for which such a condition is satisfied.

The application of stochastic differential equations (SDEs) to the modeling of stochastic processes occurring in power systems is gaining interest in recent years (e.g, [18–20]). A SDE is composed of two terms: the drift term and the diffusion term. The specific formulation of each term determines the statistical properties of the phenomenon under consideration. With this regard, SDEs have been successfully applied to wind speed fluctuation modeling when such fluctuations show an exponentially autocorrelated Gaussian behaviour [14].

However, the construction of SDEs to model exponentially autocorrelated non-gaussian phenomena, as it can be the case of hourly wind speeds, is still an open task.

In a previous work, [21], we proposed to overcome this difficulty by transforming a well-known SDE widely used to model exponentially autocorrelated Gaussian processes. For that, translation techniques are applied in order to obtain another SDE that reproduce a given non-gaussian probability distribution. The resulting model is able to reproduce such probability distribution but it cannot guarantee a good reproduction of the autocorrelation of the process.

1.3 Contributions

The method proposed in this paper relies on basic stochastic calculus concepts (such as the Regression Theorem) to derive an expression for the drift term of the SDE that ensures an exponentially autocorrelated process. Then, the stationary Fokker-Planck equation is solved to obtain the expression of the diffusion term that guarantee a given probability distribution. Therefore, the models that result from applying the proposed method are able to exactly reproduce both the probability distribution and the exponential autocorrelation for which they are designed.

The proposed method is systematically applied to construct SDE-based models from different probability distributions proposed in the literature to describe the wind speed behaviour. As a result, together with the detailed description and justification of the proposed method, the paper provides a collection of SDE-based models ready to be used in different studies related to wind power. Although the development of the method is motivated by wind speed modeling, the proposed technique is general, and it can applied to model phenomena other than wind speed.

1.4 Paper organization

The remainder of the paper is organized as follows. Section 2 describes and justifies the procedure that leads to the mathematical formulation of the wind speed models. Examples of SDEs that generate exponentially autocorrelated stochastic process for several different probability distribution functions are given in Section 3, while Section 4 illustrates the statistical properties of these examples through numerical simulations. In Section 5, the proposed procedure is applied to construct a wind speed model based on wind speed measurements recorded at a meteorological station located in New Zealand. Finally, Section

6 provides relevant conclusions. In addition, Appendix A provides a brief description of the key theorem on which the developing of the proposed model is based.

2 Proposed Building Method of the SDE Model

A one-dimensional Itô Stochastic Differential Equation (SDE) has the general form

$$\begin{aligned} dx(t) &= a(x(t),t)\cdot dt + b(x(t),t)\cdot dW(t), \quad t\in[0,T], \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where the initial value x_0 can be a deterministic or a random value, and $W(t)$ is a standard Wiener process, also loosely called Brownian motion [22,23]. The integral form of equation (1) is

$$x(t) - x_0 = \int_0^t a(x(u),u)\cdot du + \int_0^t b(x(u),s)\cdot dW(u), \quad t\in[0,T], \tag{2}$$

where the first integral is an ordinary Riemann-Stieltjes integral and the second one is a stochastic integral interpreted in the Itô's sense. The solution of (1) or (2) is a stochastic process so-called diffusion process, and functions $a(x(t),t)$ and $b(x(t),t)$ are referred to as the drift and the diffusion terms of the Itô SDE, respectively. Diffusion processes are continuous-time Markov processes with almost surely continuous sample paths [23].

Our goal is to build a SDE model to generate an exponentially autocorrelated stochastic process with a given probability distribution. In other words, we look for the form of the drift and diffusion terms of equation (1) so that the solution of the resulting SDE is a process with those statistical properties.

Inspired in the approach of [14], our method is based on the relation that the drift and the diffusion terms should satisfy in order to get a given probability distribution. This relation is obtained from the stationary Fokker-Planck equation. For stationary processes, $a(x(t),t) = a(x(t))$, $b(x(t),t) = b(x(t))$, and $p(x(t),t) = p(x(t))$, and the stationary Fokker-Planck equation is

$$0 = -a(x(t))\cdot p(x(t)) + \frac{1}{2}\cdot\frac{\partial}{\partial x(t)}\left[b^2(x(t))\cdot p(x(t))\right] \tag{3}$$

By solving (3) for $a(x(t))$ we obtain

$$a(x(t)) = b(x(t)) \cdot \frac{\partial b(x(t))}{\partial x(t)} + \frac{1}{2} \cdot b^2(x(t)) \cdot \frac{\partial \ln p(x(t))}{\partial x(t)} \tag{4}$$

and, by solving (3) for $b^2(x(t))$ we obtain

$$b^2(x(t)) = \frac{2}{p(x(t))} \cdot \int_{-\infty}^{x(t)} a(z(t)) \cdot p(z(t)) \cdot dz(t) \tag{5}$$

for $p(x(t)) \neq 0$, and $b(x(t)) = 0$ if $p(x(t)) = 0$. Therefore, for a given probability density function $p(x(t))$, if one of the functions $b(x(t))$ or $a(x(t))$ is known, the other function can be obtained by solving (4) or (5), respectively.

In reference [14] the diffusion term $b(x(t))$ is fixed to a constant value according to Kolmogorov's theory of local isotropy [24], and the drift term $a(x(t))$ is obtained by solving (4) for different probability distributions. With this approach, the resulting SDE provides a stochastic process with the given probability distribution, but the empirical exponential decay of the autocorrelation is not guaranteed for non-gaussian processes. We proceed in a different way: first, we obtain a drift term $a(x(t))$ that ensures an exponential autocorrelation function with a given decay rate. Second, we obtain the diffusion term $b(x(t))$ by solving (5) for the given probability density function $p(x(t))$.

To identify the desired drift function, we base on the Regression Theorem (see Appendix A). According to this theorem, an exponentially decaying autocorrelation is obtained if the autocovariance of the stochastic process obeys a linear differential equation of the type of (A.2). With that in mind, a differential equation of the stationary autocovariance of a process modeled with (1) is developed on the basis of the Itô formula. For an arbitrary function $g(\cdot)$ of the stochastic variable $x(t)$ defined by (1), the Itô formula gives the differential of $g(\cdot)$, as follows:

$$\begin{aligned} dg(x(t),t) = \\ & \left[\frac{\partial g(x(t),t)}{\partial t} + a(x(t),t) \cdot \frac{\partial g(x(t),t)}{\partial x(t)} + \frac{1}{2} \cdot b^2(x(t),t) \cdot \frac{\partial^2 g(x(t),t)}{\partial x^2(t)}\right] \cdot dt \\ & + b(x(t),t) \cdot \frac{\partial g(x(t),t)}{\partial x(t)} \cdot dW(t) \end{aligned} \tag{6}$$

where $a(x(t),t)$ and $b(x(t),t)$ are the drift and the diffusion terms of (1), respectively [22, 23]. For our purpose, function $g(\cdot)$ is selected to be

$$g(x(t),t) = g(x(t)) = (x(s) - \mu) \cdot (x(t) - \mu) \tag{7}$$

where $s < t$. The derivatives involved in (6) are as follows:

$$\frac{\partial g(x(t))}{\partial t} = 0 \tag{8}$$

$$\frac{\partial g(x(t))}{\partial x(t)} = x(s) - \mu \tag{9}$$

$$\frac{\partial^2 g(x(t))}{\partial x^2(t)} = 0 \tag{10}$$

Observe that, in the previous derivations, we have used the fact that $x(s)$ is independent of $x(t)$ due to the Markov property [23], and that the chosen function $g(x(t))$ does not explicitly depend on time. From (6) and (8)-(10), the resulting SDE is

$$\begin{aligned} d[(x(s) - \mu) \cdot (x(t) - \mu)] = \\ a(x(t)) \cdot (x(s) - \mu) \cdot dt + b(x(t)) \cdot (x(s) - \mu) \cdot dW(t) \end{aligned} \tag{11}$$

with initial condition $(x(s) - \mu)^2$. The integral form of the previous SDE is

$$\begin{aligned} (x(s) - \mu) \cdot (x(t) - \mu) - (x(s) - \mu)^2 = \\ \int_s^t a(x(u)) \cdot (x(s) - \mu) \cdot du + \int_s^t b(x(u)) \cdot (x(s) - \mu) \cdot dW(u) \end{aligned} \tag{12}$$

where we perform the integration over the interval $[s,t]$. By applying the expectation operator $E[\cdot]$ to equation (12), and taking into account that the expectation of an Itô stochastic integral is zero [25], i.e.,

$$E\left[\int_s^t b(x(u)) \cdot (x(s) - \mu) \cdot dW(u)\right] = 0 \tag{13}$$

we obtain the following expression

$$E\left[(x(s)-\mu)\cdot(x(t)-\mu)\right] - E\left[(x(s)-\mu)^2\right] = \int_s^t E\left[a(x(u))\cdot(x(s)-\mu)\right]\cdot du \qquad (14)$$

where the first term of the right hand side of equation (14) is the autocovariance function $c(s,t)$. The differential form of (14) is

$$\frac{dE\left[(x(s)-\mu)\cdot(x(t)-\mu)\right]}{dt} = E\left[a(x(t))\cdot(x(s)-\mu)\right] \qquad (15)$$

In order to obtain an equation similar to (A.2) it is clear that

$$a(x(t)) = -\alpha\cdot(x(t)-\mu) \qquad (16)$$

and (15) can be expressed as

$$\frac{dc(s,t)}{dt} = -\alpha\cdot c(s,t) \qquad (17)$$

For stationary processes, the autocovariance only depends on the time lag $\tau = t - s$, therefore equation (17) reduces to (A.2), and the autocovariance and the autocorrelation of the stochastic process $x(t)$ follow the decaying exponential expressions (A.3) and (A.4), respectively.

Observe also that as the drift term (16) is linear, the requirement of a linear evolution equation for the mean value expressed in the regression theorem is also satisfied. This can be shown from the integral version of a generic SDE with the computed drift term, i.e.,

$$x(t) - x_0 = \int_0^t -\alpha\cdot(x(u)-\mu)\cdot du + \int_0^t b(x(u))\cdot dW(u) \qquad (18)$$

By applying the expectation operator to equation (18), and taking into account that the expectation of an Itô stochastic integral is zero, we obtain

$$E[x(t)] - E[x_0] = \int_0^t -\alpha\cdot E\left[(x(u)-\mu)\right]\cdot du \qquad (19)$$

and, recovering the differential form,

$$\frac{dE[x(t)]}{dt} = -\alpha \cdot E[x(t)] + \alpha \cdot \mu \quad (20)$$

with initial condition $E[x_0]$. Observe that equation (20) expresses a linear law similar to (A.1).

In summary, to model a stationary stochastic process with given probability distribution function $p(x(t))$ and exponential autocorrelation with a SDE, it is a sufficient condition to define a drift term in the form

$$a(x(t)) = -\alpha \cdot (x(t) - \mu) \quad (21)$$

where μ is the mean of the particular probability distribution $p(x(t))$, and a diffusion term computed by solving

$$b^2(x(t)) = \frac{2}{p(x(t))} \int_{-\infty}^{x(t)} -\alpha \cdot (z(t) - \mu) \cdot p(z(t)) \cdot dz(t) \quad (22)$$

3 Examples

In this section, we apply the proposed method to construct SDE-based wind speed models for different probability distributions that have been proposed in the literature to describe the wind speed variability. In Subsections 3.1 and 3.2 we use the Normal distribution and the Gram-Charlier expansion proposed in [14], respectively to fit wind speed fluctuations around a mean value measured on a one-second basis. In Subsections 3.3-3.10 we use a variety of probability distributions analyzed in [11] to fit hourly mean wind speeds recorded at different meteorological stations. To simplify the notation, the explicit dependency of variable x on time is removed. All models have the following structure:

$$dx = a(x) \cdot dt + b(x) \cdot dW(t) \quad (23)$$

where $a(x)$ and $b(x)$ are defined according to (21) and (22), respectively.

3.1 Normal distribution

The probability density function $p_{\text{N}}(x)$ of the Normal distribution is

$$p_{\mathrm{N}}(x) = \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot \exp\left(-\frac{(x-\mu)^2}{2 \cdot \sigma^2}\right) \tag{24}$$

where μ is the mean, and σ is the standard deviation.

By applying the proposed method, the drift term is

$$a(x) = -\alpha \cdot (x - \mu)\,, \tag{25}$$

and the diffusion term is

$$b(x) = \sqrt{2 \cdot \alpha} \cdot \sigma \tag{26}$$

Observe that, for the normal distribution, the resulting model is the well-known Ornstein-Ulhenbeck process.

3.2 Gram-Charlier III-order expansion

The Gram-Charlier expansions are generally used to describe deviations from the Normal distribution by means of the incorporation of the skewness and kurtosis factors to the distribution. In particular, the Gram-Charlier III-order expansion has the following probability density function:

$$p_{\mathrm{GC}}(x) = \left(1 + \frac{S}{6} \cdot \mathrm{He}_3\left(\frac{x-\mu}{\sigma}\right)\right) \cdot p_{\mathrm{N}}(x) \tag{27}$$

where $p_{\mathrm{N}}(x)$ is the Normal probability density function (24), S is the skewness factor, and

$$\mathrm{He}_3\left(\frac{x-\mu}{\sigma}\right) = \left(\frac{x-\mu}{\sigma}\right)^3 - 3\left(\frac{x-\mu}{\sigma}\right) \tag{28}$$

is the Hermite polynomial of order 3.

For the standard Normal distribution $N(0,1)$ the probability density function $p_{\mathrm{GC}}(x)$ is

$$p_{\mathrm{GC}}(x) = \left(1 + \frac{S}{6} \cdot \left(x^3 - 3 \cdot x\right)\right) \cdot \frac{1}{\sqrt{2 \cdot \pi}} \exp\left(-\frac{1}{2} \cdot x^2\right) \tag{29}$$

By applying the proposed method, the drift term is

$$a(x) = -\alpha \cdot x \tag{30}$$

and the diffusion term is

$$b(x) = \sqrt{\frac{2 \cdot \alpha \cdot (S \cdot x^3 + 6)}{S \cdot x \cdot (x^2 - 3) + 6}} \tag{31}$$

3.3 Three-parameter Beta distribution

The probability density function $p_{\mathrm{B}}(x)$ of the three-parameter Beta distribution is

$$p_{\mathrm{B}}(x) = \begin{cases} \dfrac{1}{\lambda_3 \cdot B(\lambda_1, \lambda_2)} \cdot \left(\dfrac{x}{\lambda_3}\right)^{\lambda_1 - 1} \cdot \left(\dfrac{\lambda_3 - x}{\lambda_3}\right)^{\lambda_2 - 1} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where $B(\cdot, \cdot)$ is the Beta function, λ_1 and λ_2 are shape parameters, and λ_3 is a noncentrality parameter.

By applying the proposed method, the drift term is

$$a(x) = -\alpha \cdot \left(x - \frac{\lambda_1 \cdot \lambda_3}{\lambda_1 + \lambda_2}\right) \tag{32}$$

and the diffusion term is

$$b(x) = \sqrt{\frac{2 \cdot \alpha \cdot (\lambda_3 - x) \cdot x}{\lambda_1 + \lambda_2}} \tag{33}$$

3.4 Two-parameter Gamma distribution

The probability density function $p_{\mathrm{G}}(x)$ of the two-parameter Gamma distribution is

$$p_{\mathrm{G}}(x) = \begin{cases} \dfrac{1}{{\lambda_2}^{\lambda_1} \cdot \Gamma\left(\lambda_1\right)} \cdot x^{\lambda_1 - 1} \cdot \exp\left(-\dfrac{x}{\lambda_2}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function, λ_1 is a shape parameter, and λ_2 is a scale parameter.

By applying the proposed method, the drift term is

$$a(x) = -\alpha \cdot (x - \lambda_1 \cdot \lambda_2) \tag{34}$$

and the diffusion term is

$$b(x) = \sqrt{2 \cdot \alpha \cdot \lambda_2 \cdot x} \tag{35}$$

3.5 Three-parameter Generalized Gamma distribution

The probability density function $p_{\mathrm{GG}}(x)$ of the three-parameter Generalized Gamma distribution is

$$p_{\mathrm{GG}}(x) = \begin{cases} \dfrac{1}{\lambda_2 \cdot \Gamma\left(\lambda_1\right)} \cdot \lambda_3 \cdot \left(\dfrac{x}{\lambda_2}\right)^{\lambda_1 \cdot \lambda_3 - 1} \cdot \exp\left(-\left(\dfrac{x}{\lambda_2}\right)^{\lambda_3}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function, λ_1 and λ_3 are shape parameters, and λ_2 is a scale parameter.

By applying the proposed method, the drift term is

$$a(x) = -\alpha \cdot \left(x - \frac{\lambda_2 \cdot \Gamma\left(\lambda_1 + \dfrac{1}{\lambda_3}\right)}{\Gamma(\lambda_1)} \right) \tag{36}$$

and the diffusion term is

$$b(x) = \sqrt{b_1(x) \cdot b_2(x)} \tag{37}$$

with

$$b_1(x) = 2 \cdot \alpha \cdot \lambda_2 \cdot x \cdot \left(\frac{x}{\lambda_2}\right)^{-\lambda_1 \cdot \lambda_3} \cdot \exp\left(\left(\frac{x}{\lambda_2}\right)^{\lambda_3}\right) \tag{38}$$

and

$$b_2(x) = \frac{\Gamma(\lambda_1) \cdot \Gamma\left(\lambda_1 + \frac{1}{\lambda_3}, \left(\frac{x}{\lambda_2}\right)^{\lambda_3}\right) - \Gamma\left(\lambda_1 + \frac{1}{\lambda_3}\right) \cdot \Gamma\left(\lambda_1, \left(\frac{x}{\lambda_2}\right)^{\lambda_3}\right)}{\lambda_3 \cdot \Gamma(\lambda_1)} \tag{39}$$

where $\Gamma(\cdot, \cdot)$ is the Incomplete Gamma function.

3.6 Two-parameter Inverse Gaussian distribution

The probability density function $p_{\mathrm{IG}}(x)$ of the two-parameter Inverse Gaussian distribution is

$$p_{\mathrm{IG}}(x) = \begin{cases} \frac{1}{\sqrt{2 \cdot \pi}} \cdot \sqrt{\frac{\lambda}{x^3}} \cdot \exp\left(-\frac{\lambda (x-\mu)^2}{2 \cdot \mu^2 \cdot x}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where μ is the mean, and λ is a scale parameter.

By applying the proposed method, the drift term is

$$a(x) = -\alpha \cdot (x - \mu) \tag{40}$$

and the diffusion term is

$$b(x) = \sqrt{\frac{2 \cdot \sqrt{2 \cdot \pi} \cdot \alpha \cdot \mu \cdot \exp\left(\frac{\lambda \cdot (x+\mu)^2}{2 \cdot \mu^2 \cdot x}\right) \cdot \operatorname{erfc}\left(\frac{\sqrt{\frac{\lambda}{x}} \cdot (x+\mu)}{\sqrt{2} \cdot \mu}\right)}{\sqrt{\frac{\lambda}{x^3}}}} \tag{41}$$

where erfc(·) is the Complementary Error function.

3.7 Two-parameter Lognormal distribution

The probability density function $p_{\mathrm{LN}}(x)$ of the two-parameter Lognormal distribution is

$$p_{\mathrm{LN}}(x) = \begin{cases} \dfrac{1}{\sqrt{2\cdot\pi}\cdot\sigma\cdot x}\cdot\exp\left(-\dfrac{(\log(x)-\mu)^2}{2\cdot\sigma^2}\right) & \text{if } x>0 \\ 0 & \text{if } x\leq 0 \end{cases}$$

where μ and σ are the mean and the standard deviation of the natural logarithm of variable x, respectively.

By applying the proposed method, the drift term is

$$a(x) = -\alpha\cdot\left(x-\exp\left(\mu+\frac{\sigma^2}{2}\right)\right) \tag{42}$$

and the diffusion term is

$$b(x) = \sqrt{b_1(x)\cdot b_2(x)} \tag{43}$$

with

$$b_1(x) = \sqrt{2\cdot\pi}\cdot\alpha\cdot\sigma\cdot x\cdot\exp\left(\mu+\frac{\sigma^2}{2}+\frac{(\log(x)-\mu)^2}{2\cdot\sigma^2}\right) \tag{44}$$

and

$$b_2(x) = \mathrm{erf}\left(\frac{\mu+\sigma^2-\log(x)}{\sqrt{2}\cdot\sigma}\right)-\mathrm{erf}\left(\frac{\mu-\log(x)}{\sqrt{2}\cdot\sigma}\right) \tag{45}$$

where erf(·) is the Error function.

3.8 One-parameter Rayleigh distribution

The probability density function $p_{\mathrm{R}}(x)$ of the one-parameter Rayleigh distribution is

$$p_{\mathrm{R}}(x) = \begin{cases} \dfrac{x}{\lambda^2} \cdot \exp\left(-\dfrac{x^2}{2 \cdot \lambda^2}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where λ is a scale parameter.

By applying the proposed method, the drift term is

$$a(x) = -\alpha \cdot \left(x - \sqrt{\frac{\pi}{2}} \cdot \lambda\right) \tag{46}$$

and the diffusion term is

$$b(x) = \sqrt{\frac{\alpha \cdot \lambda^2}{x} \cdot \left(2 \cdot x + \sqrt{2 \cdot \pi} \cdot \lambda \cdot \left(\exp\left(\frac{x^2}{2 \cdot \lambda^2}\right) \operatorname{erfc}\left(\frac{x}{\sqrt{2} \cdot \lambda}\right) - 1\right)\right)} \tag{47}$$

where $\operatorname{erfc}(\cdot)$ is the Complementary Error function.

3.9 Two-parameter Truncated Normal distribution

The probability density function $p_{\mathrm{TN}}(x)$ of the two-parameter Truncated Normal distribution is

$$p_{\mathrm{TN}}(x) = \begin{cases} \sqrt{\dfrac{2}{\pi}} \cdot \dfrac{\exp\left(-\dfrac{(x-\mu)^2}{2 \cdot \sigma^2}\right)}{\sigma \cdot \left(1 + \operatorname{erf}\left(\dfrac{\mu}{\sqrt{2} \cdot \sigma}\right)\right)} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where $\operatorname{erf}(\cdot)$ is the Error function, and μ and σ are, respectively, the mean and the standard deviation of the Normal distribution before truncation.

By applying the proposed method, the drift term is

$$a(x) = -\alpha \cdot \left(x - \mu - \frac{\sigma \cdot \sqrt{\frac{2}{\pi}} \cdot \exp\left(-\frac{\mu^2}{2 \cdot \sigma^2}\right)}{1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2} \cdot \sigma}\right)} \right) \tag{48}$$

and the diffusion term is

$$b(x) = \sqrt{2 \cdot \alpha \cdot \sigma^2 \cdot \left(1 + \frac{\exp\left(\frac{(x - 2 \cdot \mu) \cdot x}{2 \cdot \sigma^2}\right) \left(\operatorname{erfc}\left(\frac{\mu - x}{\sqrt{2} \cdot \sigma}\right) - 2\right)}{1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2} \cdot \sigma}\right)} \right)} \tag{49}$$

where $\operatorname{erfc}(\cdot)$ is the Complementary Error function.

3.10 Two-parameter Weibull distribution

The probability density function $p_{\mathrm{W}}(x)$ of the two-parameter Weibull distribution is

$$p_{\mathrm{W}}(x) = \begin{cases} \frac{\lambda_1}{\lambda_2} \cdot \left(\frac{x}{\lambda_2}\right)^{\lambda_1 - 1} \cdot \exp\left(-\left(\frac{x}{\lambda_2}\right)^{\lambda_1}\right) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where λ_1 is a shape parameter and λ_2 is a scale parameter.

By applying the proposed method, the drift term is

$$a(x) = -\alpha \cdot \left(x - \lambda_2 \cdot \Gamma\left(1 + \frac{1}{\lambda_1}\right)\right) \tag{50}$$

and the diffusion term is

$$b(x) = \sqrt{b_1(x) \cdot b_2(x)} \tag{51}$$

with

$$b_1(x) = 2 \cdot \alpha \cdot \frac{\lambda_2}{{\lambda_1}^2} \cdot x \cdot \left(\frac{\lambda_2}{x}\right)^{\lambda_1} \tag{52}$$

and

$$b_2(x) = \lambda_1 \cdot \exp\left(\left(\frac{x}{\lambda_2}\right)^{\lambda_1}\right) \cdot \Gamma\left(1 + \frac{1}{\lambda_1}, \left(\frac{x}{\lambda_2}\right)^{\lambda_1}\right) - \Gamma\left(\frac{1}{\lambda_1}\right) \tag{53}$$

where $\Gamma(\cdot)$ is the Gamma function, and $\Gamma(\cdot, \cdot)$ is the Incomplete Gamma function.

4 Numerical Results

In this section, we test the statistical properties of the processes generated by the SDE-based wind speed models developed in Section 3. The values of the parameters of the different SDEs have been taken from references [11] and [14]. In particular, all parameters of the models developed in Subsections 3.1 and 3.2 are taken from [14] and correspond to the analysis of wind speed fluctuations around a mean value measured on a one-second basis. The parameters related to the probability distributions used to develop the models of Subsections 3.3-3.10 are taken from [11] and are the result of the analysis of hourly-mean wind speed data. Specifically, we have taken the values of the parameters corresponding to the application of the maximum likelihood estimation method to the data recorded at La Palma meteorological station. Since reference [11] does not include any study related to the autocorrelation of wind speeds we have chosen an autocorrelation coefficient of 0.25, which is a reasonable value according to the wind speed autocorrelation studies performed by using hourly wind speed data in [15, 21]. Table 1 summarizes the data used in the simulations, classified according to the probability density function used to construct the SDE-based model.

The generation of the stochastic processes modeled by SDEs implies the numerical integration of these equations. For that, we used the multiprocessor stochastic integration tools available in the software Dome [26]. Specifically, we applied the implicit Milstein integration scheme in [21]. Other stochastic integration schemes can be found in [27].

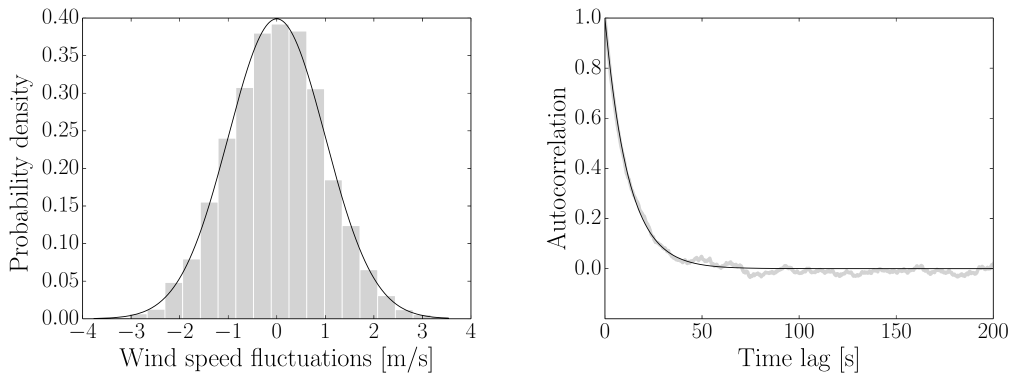
To obtain the statistical properties of the processes generated by the SDE-based models, 2000 trajectories were simulated. In order to illustrate the exponential decay of the autocorrelation function, a time frame of 200 seconds was used for the simulations of the models developed in Subsections 3.1 and

p_{N}	p_{GC}	p_{B}	p_{G}	p_{GG}
$\mu = 0.0$	$\mu = 0.0$	$\lambda_1 = 3.671$	$\lambda_1 = 4.383$	$\lambda_1 = 2.817$
$\sigma = 1.0$	$\sigma = 1.0$	$\lambda_2 = 16.729$	$\lambda_2 = 1.06$	$\lambda_2 = 3.95$
–	$S = -0.94$	$\lambda_3 = 25.788$	–	$\lambda_3 = 1.812$
$\alpha = 0.083$	$\alpha = 0.029$	$\alpha = 0.25$	$\alpha = 0.25$	$\alpha = 0.25$
p_{IG}	p_{LN}	p_{R}	p_{TN}	p_{W}
$\mu = 4.644$	$\mu = 1.417$	$\lambda = 3.605$	$\mu = 4.48$	$\lambda_1 = 2.343$
$\lambda = 14.748$	$\sigma = 0.519$	–	$\sigma = 2.272$	$\lambda_2 = 5.244$
$\alpha = 0.25$	$\alpha = 0.25$	$\alpha = 0.25$	$\alpha = 0.25$	$\alpha = 0.25$

Table 1
Parameters of the simulated SDE models.

3.2, whereas a time frame of 24 hours was used for the simulations of the models of Subsections 3.3-3.10.

To illustrate the ability of the developed models to reproduce the statistical properties for which they are designed, we compare the histograms and autocorrelations computed from the trajectories generated by the models to the corresponding probability density and decaying exponential autocorrelation functions. Figures 1-10 depict the results of such comparisons. In all figures, values computed from the processes generated by SDE-based models are represented in gray, whereas theoretical values are represented in black.

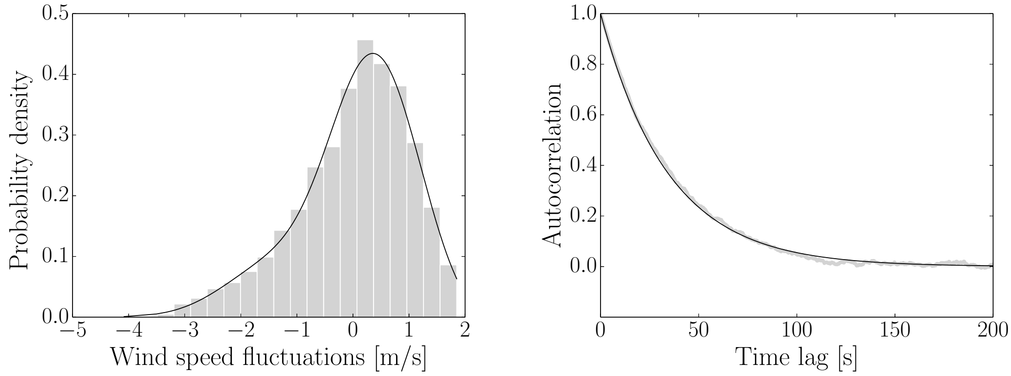


(a) PDF　　(b) Autocorrelation

Fig. 1. Normal distribution. Model (25)-(26).

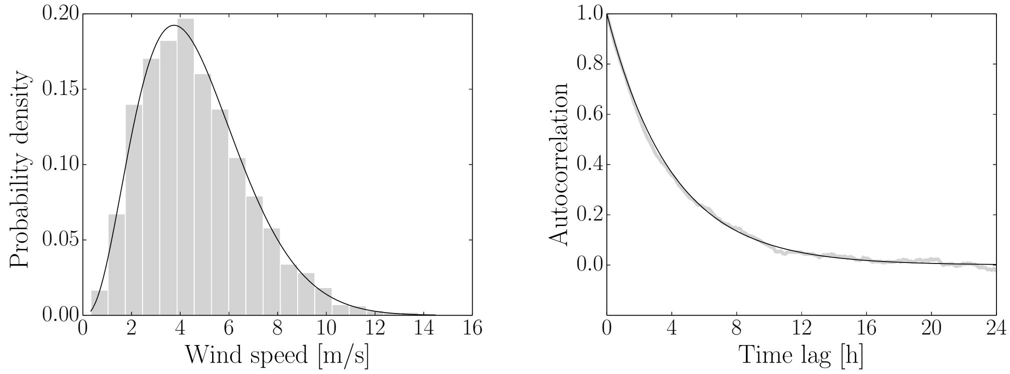
5 Case Study

In this section, we consider wind speed measurements collected at Baring Head meteorological station, located in the Wellington region of New Zealand. The



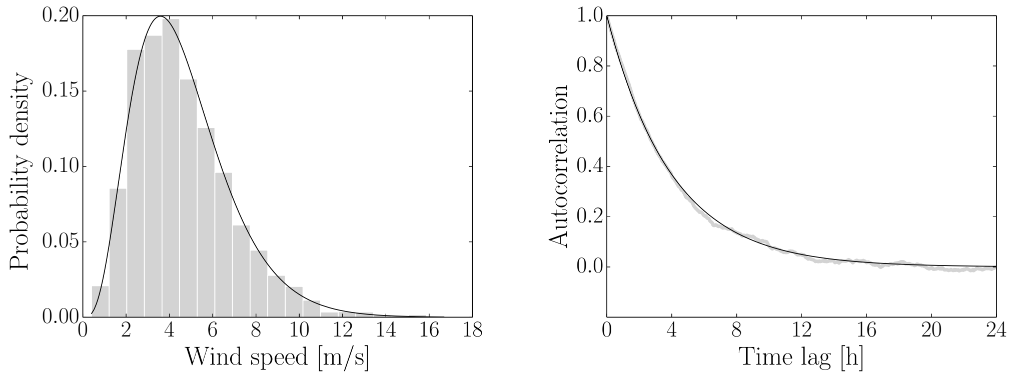
(a) PDF (b) Autocorrelation

Fig. 2. Gram-Charlier expansion. Model (30)-(31).



(a) PDF (b) Autocorrelation

Fig. 3. Three-parameter Beta distribution. Model (32)-(33).

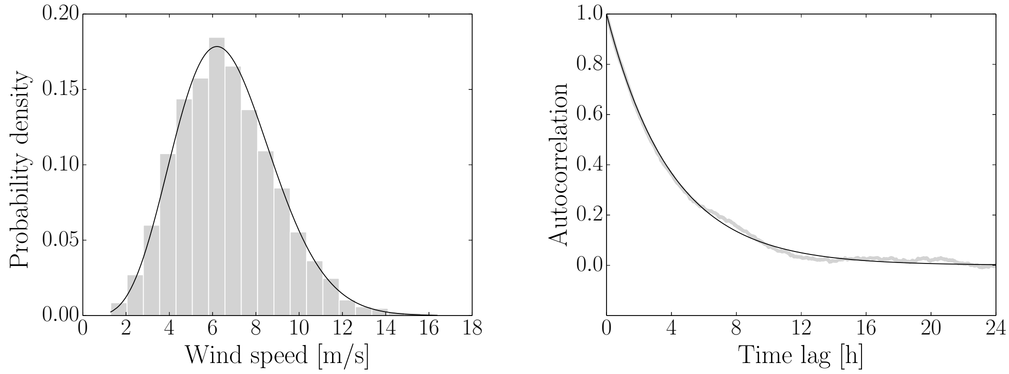


(a) PDF (b) Autocorrelation

Fig. 4. Two-parameter Gamma distribution. Model (34)-(35).

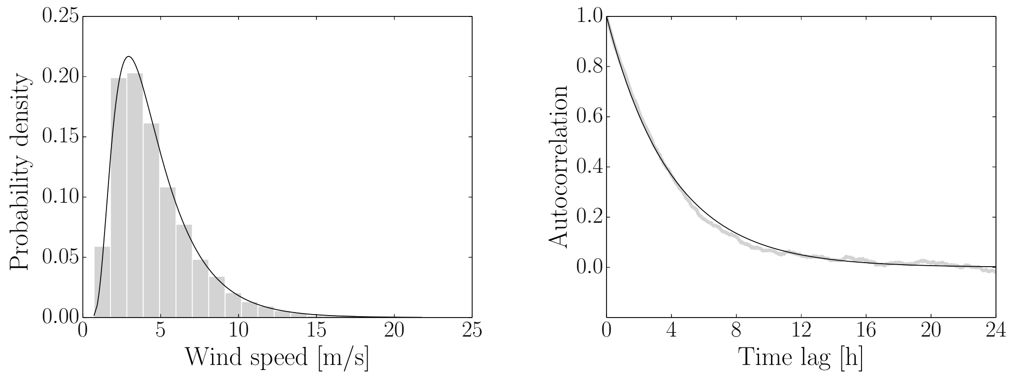
data set consists of hourly mean values of the wind speed recorded for the whole year 2014, i.e., it contains 8760 values. This data set is available in [28].

In order to construct a wind speed model for this site, the probability distribution and the autocorrelation of the wind speed are analyzed based on the available data set. Figure 11.(a) shows a table that contains the values of the



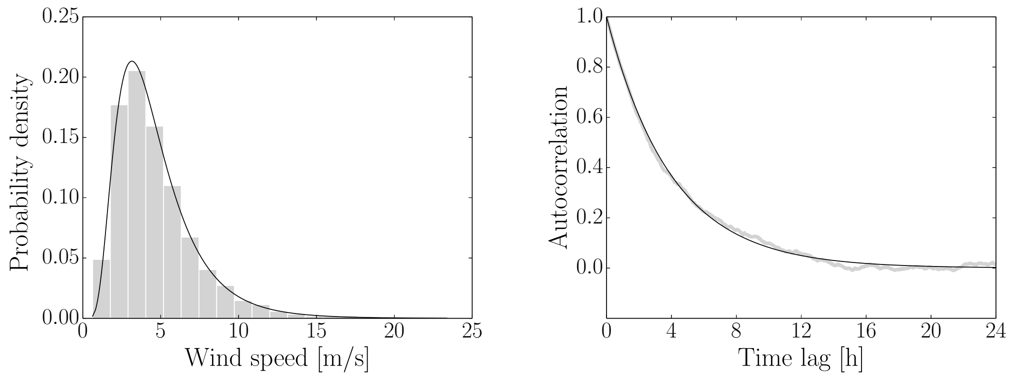
(a) PDF (b) Autocorrelation

Fig. 5. Three-parameter Generalized Gamma distribution. Model (36)-(39).



(a) PDF (b) Autocorrelation

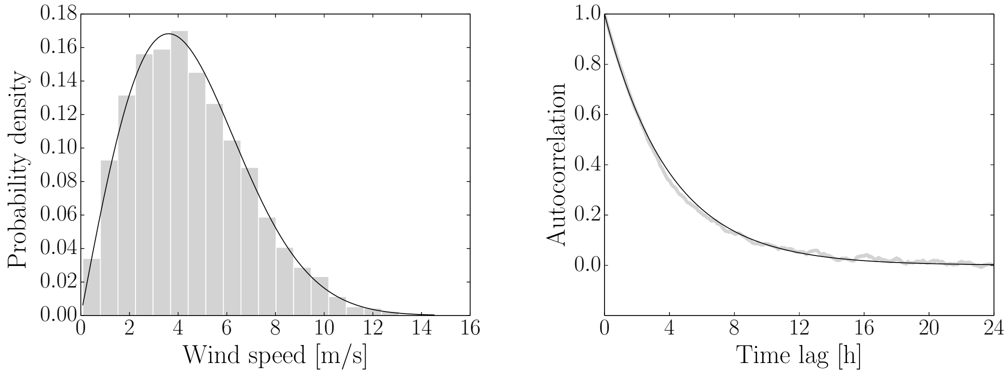
Fig. 6. Two-parameter Inverse Gaussian distribution. Model (40)-(41).



(a) PDF (b) Autocorrelation

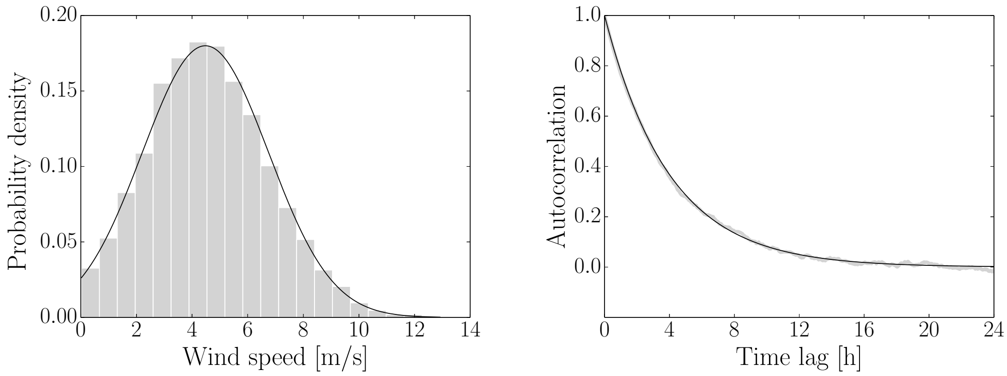
Fig. 7. Two-parameter Lognormal distribution. Model (42)-(45).

negative log likelihood function obtained when each probability density function considered in Section 3 for hourly mean wind speed values is fitted to the histogram of the data. It can be observed that the probability density function of the three-parameter Generalized Gamma distribution (p_{GG}) represents the best fit according to the value of the negative log likelihood function. Figure 11.(b) depicts the normalized histogram of the data set and the probability



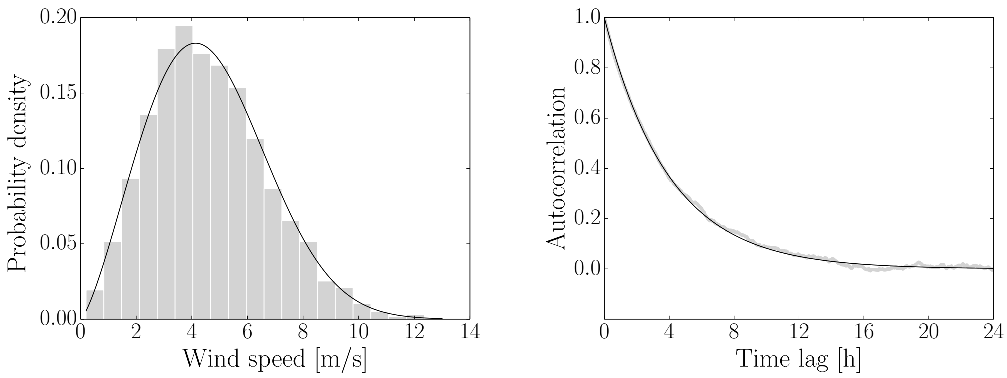
(a) PDF (b) Autocorrelation

Fig. 8. One-parameter Rayleigh distribution. Model (46)-(47).



(a) PDF (b) Autocorrelation

Fig. 9. Two-parameter Truncated Normal distribution. Model (48)-(49).



(a) PDF (b) Autocorrelation

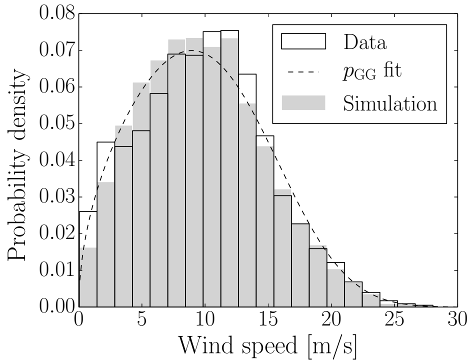
Fig. 10. Two-parameter Weibull distribution. Model (50)-(53).

density function fit. The parameters of this probability distribution function are $\lambda_1 = 0.4603$, $\lambda_3 = 3.2992$, and $\lambda_2 = 15.6672$.

Figure 12.(a) represents the analysis of the autocorrelation of the wind speed data set for time lags up to 240 hours (10 days). The solid black line is the autocorrelation computed from data, while the dashed and the dotted lines

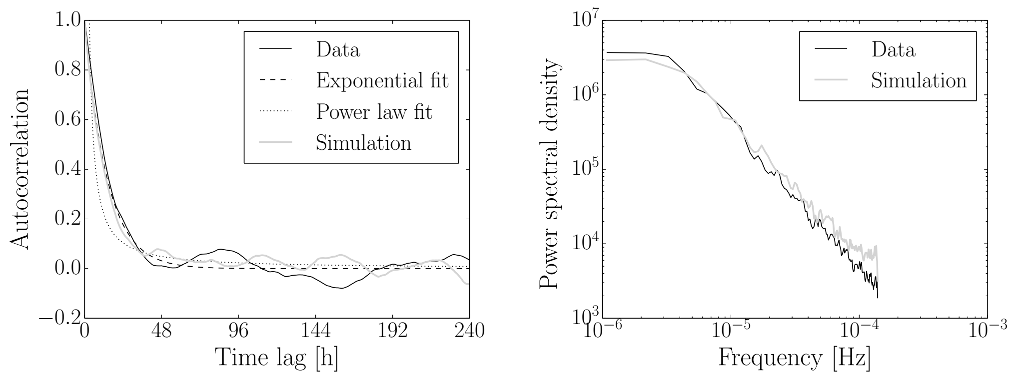
p_{B}	26520.45
p_{G}	26934.30
p_{GG}	26405.15
p_{IG}	29302.87
p_{LN}	27953.50
p_{R}	26548.48
p_{TN}	27439.81
p_{W}	26546.42

(a)



(b)

Fig. 11. (a) Negative log likelihood value of the PDFs parameter estimation; (b) Generalized Gamma PDF fit to the data histogram and histogram of the simulated process.



(a) (b)

Fig. 12. (a) Autocorrelation analysis of data and autocorrelation of the simulated process; (b) Power spectral density of data and of the simulated process.

are the exponential (A.4) and power law ($k \cdot \tau^{-\beta}$) fits to this autocorrelation, respectively. It is apparent that, for the considered data set, the exponential function constitutes a better approximation to the autocorrelation of the wind speed than the power law function. Therefore, the procedure proposed in this paper to model the wind speed applies. The parameter of the exponential fit in this case is $\alpha = 0.0722$.

According to the previous statistical analysis of the data set, the wind speed is modeled by means of a SDE where the drift and the diffusion terms are defined by equations (36)-(39), particularized for the values of parameters λ_1, λ_2, λ_3, and α specified above. In order to carry out a direct comparison with the statistical properties of the data set, a single simulation of the SDE model is performed. In this simulation the SDE is integrated by using a time step of one hour for a total simulation time of 8760 hours. Figures 11.(b) and 12.(a) include, respectively, the histogram and the autocorrelation corresponding to the values obtained in this simulation. These statistical properties are similar to those observed in the data set. Finally, Figure 12.(b) shows the log-log plot of the power spectral density computed from both the data set and the simulated values. It can be observed the similarity of both results.

6 Conclusions

In this paper, we develop a systematic method to construct wind speed models based on stochastic differential equations. We apply a novel, analytically exact approach to define the formulation of the drift and diffusion terms of a stochastic differential equation in order to reproduce the given stationary probability distribution and exponential autocorrelation characterizing the wind speed. This new approach accurately reproduces both the probability distribution and the autocorrelation of the wind speed, as opposed to existing methods that are approximated. The application of the proposed method is straightforward and can be carried out systematically. Proof of that is the collection of models developed in the paper for different probability distributions proposed in the literature to describe the wind speed behaviour. The analysis of the numerical simulation of all models demonstrates their ability to generate stochastic processes with the required statistical properties. Finally, the proposed method is general and can be applied to model any stationary process with exponential autocorrelation. Future work will focus on the definition of SDE-based models for processes with autocorrelation other than exponential as, for example, power-law or sinusoidal.

Acknowledgments

The second author is partly funded by the Science Foundation Ireland, under Grant No. SFI/09/SRC/E1780. The opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the Science Foundation Ireland. The second author has also benefit from the financial support of EC Marie Skłodowska-Curie Career Integration Grant No. PCIG14-GA-2013-630811.

A Regression Theorem

In the theory of stochastic processes, the regression theorem states that if the mean value of a Markov process obeys linear evolution equations of the type

$$\frac{dE[x(t)]}{dt} = -\alpha \cdot E[x(t)] \tag{A.1}$$

then, in the stationary state, the autocovariance function $c(\tau)$ can be obtained by solving

$$\frac{dc(\tau)}{d\tau} = -\alpha \cdot c(\tau) \tag{A.2}$$

with initial condition $c(0) = \sigma^2$, where σ^2 is the variance of the process [22]. The result of solving (A.2) is

$$c(\tau) = \sigma^2 \cdot e^{-\alpha \cdot \tau} \tag{A.3}$$

showing that the autocovariance function of such processes is an exponential decaying function. As a consequence, the autocorrelation $r(\tau)$ is

$$r(\tau) = e^{-\alpha \cdot \tau} \tag{A.4}$$

which is also an exponentially decaying function.

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