Synchronous Machines

POWER SYSTEM MODELLING AND CONTROL (EEEN40550)

Prof. Federico Milano

Email: federico.milano@ucd.ie

Tel.: 01 716 1844

Room 157a – Engineering & Materials Science Centre

School of Electrical & Electronic Engineering

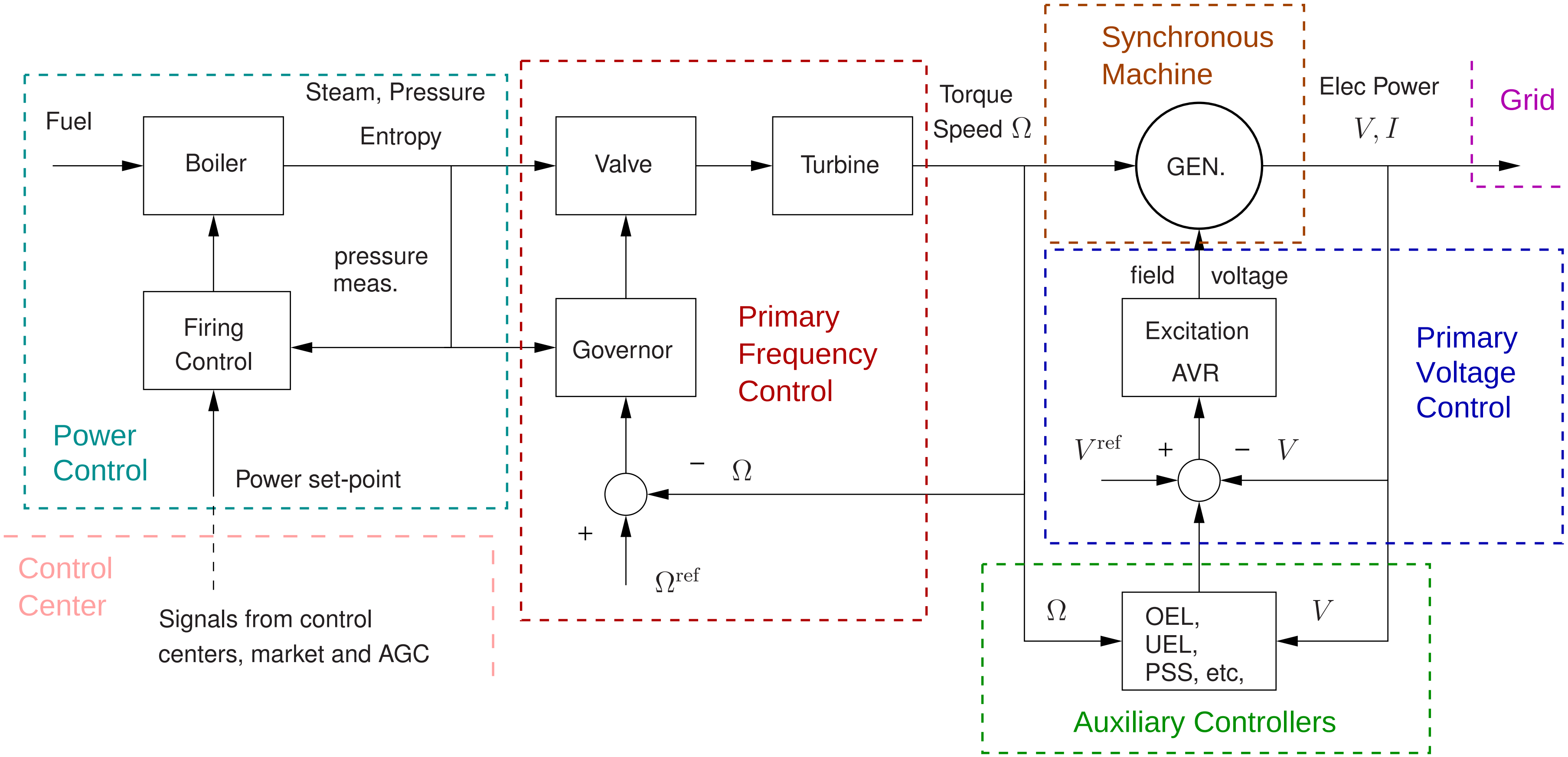
University College Dublin

Dublin, Ireland

Synchronous Machine Overview

- Generator Overview
- Model of the Synchronous Machine
- Models for Stability Analysis and Control:
  - Subtransient Model
  - Transient Model
  - Electro-Mechanical Model
- Steady-State Model

Generator Components



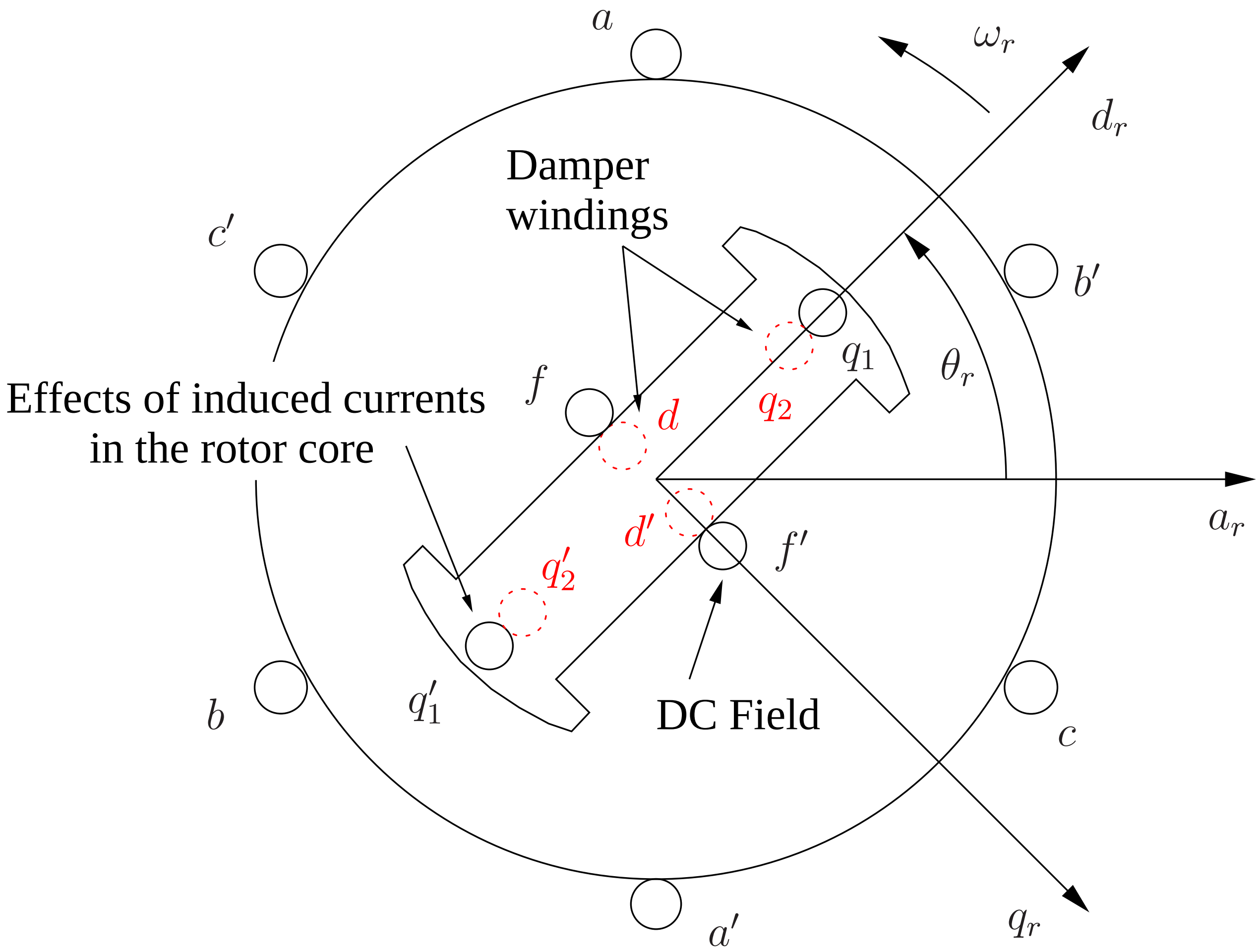
3-Phase Synchronous Machines

- Complete dynamic equations of 3-phase synchronous machines
- Characterisation of machine inductances
- Why transforming machine variables?
- Park's transformation
- Transformed (d-q-0) circuit equations
- Steady-state open circuit operation

Synchronous Machine Types (I)

- Round Rotor
  - Solid state rotor with (generally) only one pair of poles ($\Omega_n = 3{,}000$ rpm at 50Hz). This is used in steam or gas turbine groups.
- Salient Pole Rotor
  - Laminated rotor with several pairs of poles ($\Omega_n \ll 3{,}000$ rpm at 50 Hz). This is used in big hydro plants coupled with Kaplan's turbines.
- Permanent Magnet/Brushless Rotor
  - Mainly used for motors or for "small" generators (~ 1 MVA). Since there is no need for brushes, the machine is compact and robust. However, no reactive power regulation is possible.

Synchronous Machine Scheme



Synchronous Machine Windings

- $a - a', b - b', c - c'$ $\longrightarrow$ stator windings (ac)
- $f - f'$ $\longrightarrow$ field winding (dc)
- $q_1 - q_1'$ $\longrightarrow$ effect of induced currents in the rotor core
- $d - d', q_2 - q_2'$ $\longrightarrow$ damper windings (fictitious!)

Circuit Equations for 3-Phase Machine

- 3 stator coils (a, b, c)
- 2 rotor coils in direct axis:
  - Field winding
  - Direct-axis damper
- 1 rotor coil in quadrature (damper)
- The machine is represented by 6 coils
- There is no mutual coupling between the circuits in the direct axis and those in the quadrature axis
- Rotor-stator and stator-stator inductances are a function of θ_r

Stator Flux Linkage Equations

$$
\begin{aligned}
\psi_a &= + L_{aa} i_a - L_{ab} i_b - L_{ac} i_c \\
&\quad - L_{af} i_f - L_{ad} i_{d1} - L_{aq} i_{q1} \\
\psi_b &= - L_{ab} i_a + L_{bb} i_b - L_{bc} i_c \\
&\quad - L_{bf} i_f - L_{bd} i_{d1} - L_{bq} i_{q1} \\
\psi_c &= - L_{ac} i_a - L_{cb} i_b + L_{cc} i_c \\
&\quad - L_{cf} i_f - L_{cd} i_{d1} - L_{cq} i_{q1}
\end{aligned}
$$

In Matrix Form

$$\begin{bmatrix} \psi_a \\ \psi_b \\ \psi_c \end{bmatrix} = \begin{bmatrix} l_{aa} & -l_{ab} & -l_{ac} \\ -l_{ab} & l_{bb} & -l_{bc} \\ -l_{ac} & -l_{cb} & l_{cc} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} - \begin{bmatrix} l_{af} & l_{ad} & l_{aq} \\ l_{bf} & l_{bd} & l_{bq} \\ l_{cf} & l_{cd} & l_{cq} \end{bmatrix} \begin{bmatrix} i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

$$\begin{bmatrix} -\psi_a \\ -\psi_b \\ -\psi_c \end{bmatrix} = - \begin{bmatrix} l_{aa} & -l_{ab} & -l_{ac} \\ -l_{ba} & l_{bb} & -l_{bc} \\ -l_{ca} & -l_{cb} & l_{cc} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \begin{bmatrix} l_{af} & l_{ad} & l_{aq} \\ l_{bf} & l_{bd} & l_{bq} \\ l_{cf} & l_{cd} & l_{cq} \end{bmatrix} \begin{bmatrix} i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

$$\begin{bmatrix} \psi_s \end{bmatrix} = - \begin{bmatrix} L_s \end{bmatrix} \begin{bmatrix} i_s \end{bmatrix} + \begin{bmatrix} L_{sr} \end{bmatrix} \begin{bmatrix} i_r \end{bmatrix}$$

Rotor Flux Linkage Equations

$$
\begin{aligned}
\psi_f =& - L_{af} i_a - L_{bf} i_b - L_{cf} i_c \\
& + L_{ff} i_f + L_{fd} i_{d1} \\
\psi_{d1} =& - L_{ad} i_a - L_{bd} i_b - L_{cd} i_c \\
& + L_{fd} i_f + L_{dd} i_{d1} \\
\psi_{q1} =& - L_{aq} i_a - L_{bq} i_b - L_{cq} i_c \\
& + L_{qq} i_{q1}
\end{aligned}
$$

In Matrix Form

$$\begin{bmatrix} \psi_f \\ \psi_{d1} \\ \psi_{q1} \end{bmatrix} = -\begin{bmatrix} l_{fa} & l_{fb} & l_{fc} \\ l_{da} & l_{db} & l_{dc} \\ l_{qa} & l_{qb} & l_{qc} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \begin{bmatrix} l_{ff} & l_{fd} & 0 \\ l_{df} & l_{dd} & 0 \\ 0 & 0 & l_{qq} \end{bmatrix} \begin{bmatrix} i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \psi_r \end{bmatrix} = -\begin{bmatrix} L_{rs} \end{bmatrix} \begin{bmatrix} i_s \end{bmatrix} + \begin{bmatrix} L_r \end{bmatrix} \begin{bmatrix} i_r \end{bmatrix}$$

Stator Voltage Equations

- $$e_a = -\frac{d\psi_a}{dt} - R_a i_a$$
- $$e_b = -\frac{d\psi_b}{dt} - R_b i_b$$
- $$e_c = -\frac{d\psi_c}{dt} - R_c i_c$$

In Matrix Form

$$\begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} -\psi_a \\ -\psi_b \\ -\psi_c \end{bmatrix} - \begin{bmatrix} R_a & 0 & 0 \\ 0 & R_b & 0 \\ 0 & 0 & R_c \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

$$\Downarrow$$

$$\left[e_s \right] = \frac{d}{dt} \left[\psi_s \right] + \left[R_s \right] \left[i_s \right]$$

Rotor Voltage Equations

- $$e_f = +\frac{d\psi_f}{dt} + R_f i_f$$
- $$e_{d1} = +\frac{d\psi_{d1}}{dt} + R_{d1} i_{d1}$$
- $$e_{q1} = +\frac{d\psi_{q1}}{dt} + R_{q1} i_{q1}$$

In Matrix Form

$$\begin{bmatrix} e_f \\ e_{d1} \\ e_{q1} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \psi_f \\ \psi_{d1} \\ \psi_{q1} \end{bmatrix} + \begin{bmatrix} R_f & 0 & 0 \\ 0 & R_{d1} & 0 \\ 0 & 0 & R_{q1} \end{bmatrix}\begin{bmatrix} i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

$$\Downarrow$$

$$\left[e_r\right] = \frac{d}{dt}\left[\psi_r\right] + \left[R_r\right]\left[i_r\right]$$

General Equations in Matrix Form (I)

- Magnetical Equations:

$$\left[\psi\right] = \left[L\right]\left[i\right]$$

- Electrical Equations

$$\left[e\right] = \frac{d}{dt}\left[\psi\right] + \left[R\right]\left[i\right]$$

- Where:

$$\left[e\right] = \left[e_a, e_b, e_c, e_f, e_{d1}, e_{q1}\right]^T$$

$$\left[i\right] = \left[i_a, i_b, i_c, i_f, i_{d1}, i_{q1}\right]^T$$

$$\left[\psi\right] = \left[-\psi_a, -\psi_b, -\psi_c, \psi_f, \psi_{d1}, \psi_{q1}\right]^T$$

General Equations in Matrix Form (II)

$$\left[L\right] = \begin{bmatrix} -l_{aa} & l_{ab} & l_{ac} & l_{af} & l_{ad} & l_{aq} \\ l_{ab} & -l_{bb} & l_{bc} & l_{bf} & l_{bd} & l_{bq} \\ l_{ca} & l_{cb} & -l_{cc} & l_{cf} & l_{cd} & l_{cq} \\ -l_{fa} & -l_{fb} & -l_{fc} & l_{ff} & l_{fd} & 0 \\ -l_{da} & -l_{db} & -l_{dc} & l_{df} & l_{dd} & 0 \\ -l_{qa} & -l_{qb} & -l_{qc} & 0 & 0 & l_{qq} \end{bmatrix}$$

$$\left[L\right] = \begin{bmatrix} -[L_s] & [L_{sr}] \\ -[L_{rs}] & [L_r] \end{bmatrix}$$

General Equations in Matrix Form (III)

$$[R] = \begin{bmatrix} -R_a & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_b & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_c & 0 & 0 & 0 \\ 0 & 0 & 0 & R_f & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{d1} & 0 \\ 0 & 0 & 0 & 0 & 0 & R_{q1} \end{bmatrix}$$

$$[R] = \begin{bmatrix} -[R_s] & [0] \\ [0] & [R_r] \end{bmatrix}$$

Partitioned Matrix Equations

- Flux Linkage:

$$\left[\psi_s\right] = -\left[L_s\right]\left[i_s\right] + \left[L_{sr}\right]\left[i_r\right]$$

$$\left[\psi_r\right] = -\left[L_{rs}\right]\left[i_s\right] + \left[L_r\right]\left[i_r\right]$$

- Voltage Equations:

$$\left[e_s\right] = \frac{d}{dt}\left[\psi_s\right] + \left[R_s\right]\left[i_s\right]$$

$$\left[e_r\right] = \frac{d}{dt}\left[\psi_r\right] + \left[R_r\right]\left[i_r\right]$$

Self & Mutual Inductances

- We make the following assumptions:

1. There is no saturation $\longrightarrow$The system is linear
2. Stator surface is smooth $\longrightarrow$We ignore the tooth ripple
3. We have a sinusoidally distributed MMF in the air gap. The winding and pole shape are such that there are no space harmonics. In this case only fundamental frequency EMF will be induced in the windings at steady-state.

Stator Self Inductance

- For the round rotor, the self inductance is constant i.e. not a function of θ_r.

  Hence:

  $$l_{aa} = l_{bb} = l_{cc} = \text{constant}$$

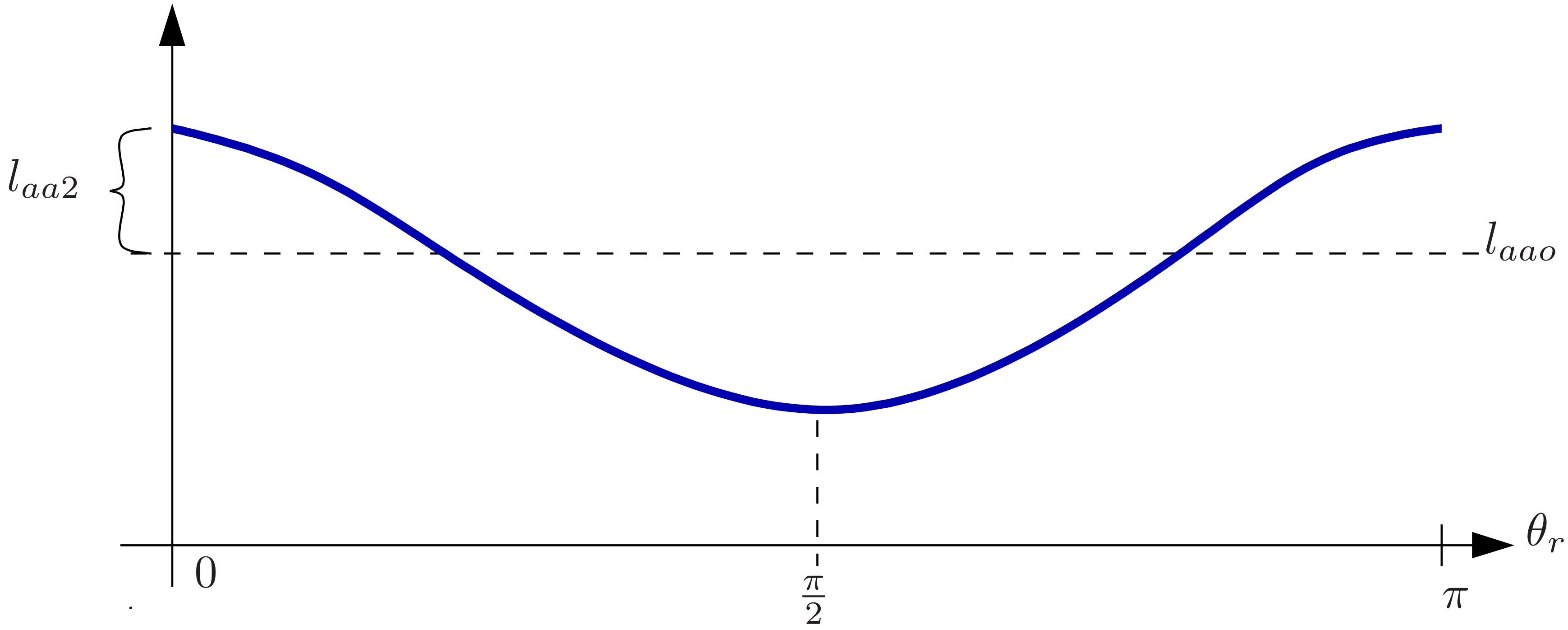
- For the salient pole rotor, there is a constant (average) and a variable part of the self inductance dependent on θ_r.

  Hence:

  $$\begin{aligned} l_{aa} &= l_{aao} + l_{aa2}\cos(2\theta_r) \\ l_{bb} &= l_{aao} + l_{aa2}\cos(2(\theta_r - \frac{2\pi}{3})) \\ l_{cc} &= l_{aao} + l_{aa2}\cos(2(\theta_r + \frac{2\pi}{3})) \end{aligned}$$

Stator Self Inductance of Salient Pole Rotors

- Variation of $l_{aa}(\theta_r)$ in arbitrary units:



$$l_{aa}(\theta_r) = l_{aao} + l_{aa2}\cos(2\theta_r)$$

Stator Mutual Inductances (I)

- For the round rotor case, the stator-stator mutual inductances are constant, i.e. not a function of θ_r.

  Hence:

$$l_{ab} = l_{bc} = l_{ca} = l_{abo} = \text{constant}$$

- Observe that for symmetry:

$$l_{ab} = l_{ba}, \qquad l_{bc} = l_{cb}, \qquad l_{ca} = l_{ac}$$

Stator Mutual Inductances (II)

- For the salient pole rotor, the stator-stator mutual inductances is a constant plus a sinusoidal function:

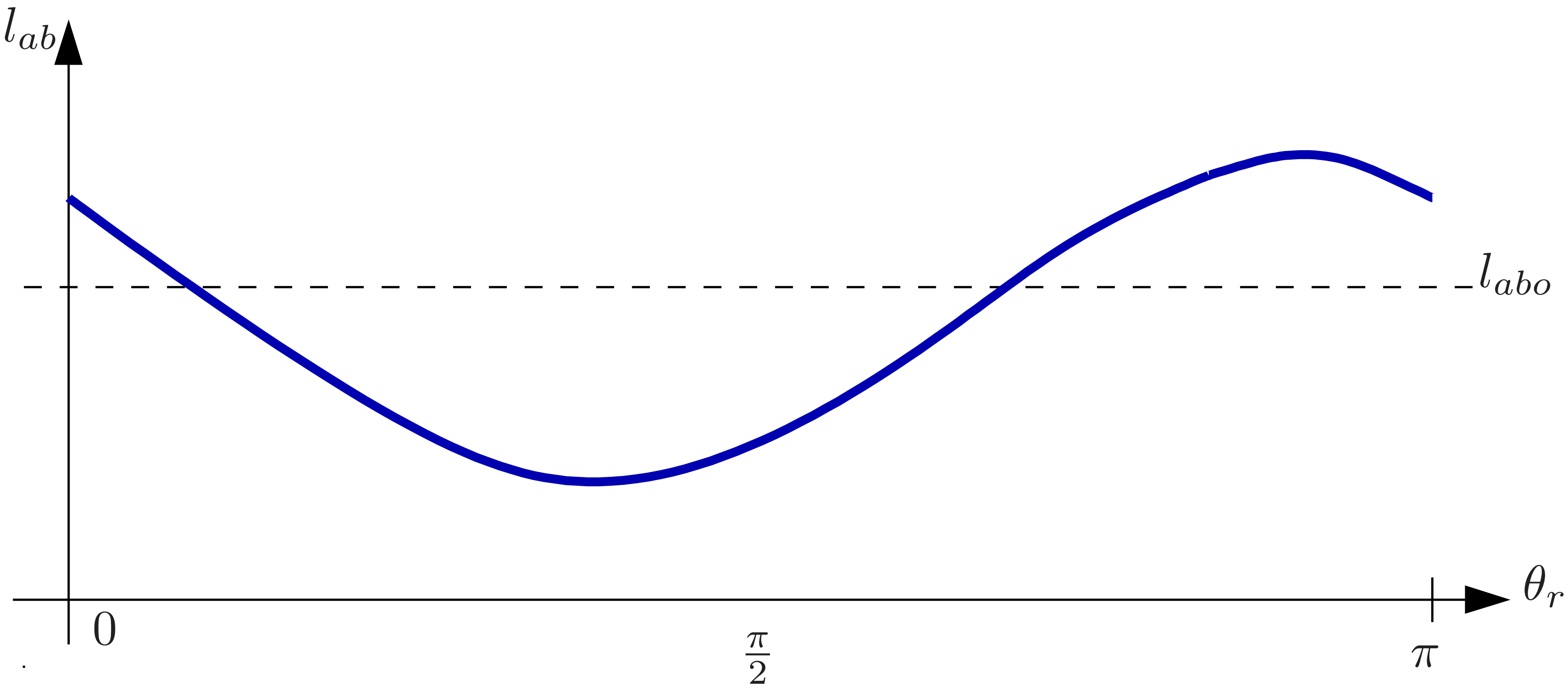
  Hence:

$$
\begin{aligned}
l_{ab} &= l_{abo} + l_{ab2}\cos(2(\theta_r + \frac{\pi}{6})) \\
l_{bc} &= l_{abo} + l_{ab2}\cos(2(\theta_r - \frac{\pi}{2})) \\
l_{ca} &= l_{abo} + l_{ab2}\cos(2(\theta_r + \frac{5\pi}{6}))
\end{aligned}
$$

- Note that the argument of the cosine depends on $2\theta_r$, not θ_r.

Stator Mutual Inductances (II)

- Variation of $l_{ab}(\theta_r)$:



$$l_{ab}(\theta_r) = l_{abo} + l_{ab2}\cos(2(\theta_r + \frac{\pi}{6}))$$

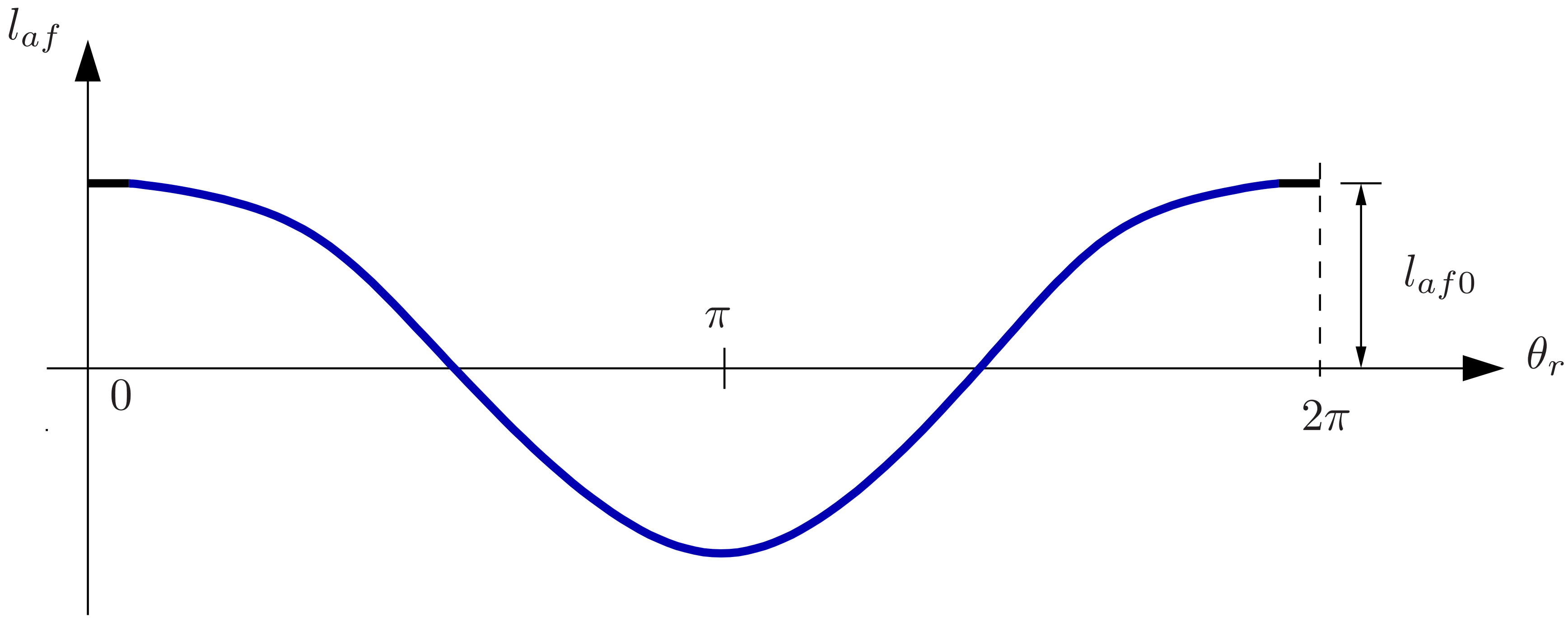
Mutual Inductance Between Rotor and Stator (I)

- Stator to rotor mutuals vary with rotor position
- d-axis rotor circuits (field and armortisseurs) vary as $\cos(\theta_r)$
- q-axis rotor circuits vary as $\sin(\theta_r)$
- Hence:

$$\begin{aligned} l_{af} &= l_{af0}\cos(\theta_r) \\ l_{bf} &= l_{af0}\cos(\theta_r - \frac{2\pi}{3}) \\ l_{cf} &= l_{af0}\cos(\theta_r + \frac{2\pi}{3}) \end{aligned}$$

Mutual Inductance Between Rotor and Stator (II)

- Variation of $l_{af}(\theta_r)$:



$$l_{af}(\theta_r) = l_{af0}\ \cos\theta_r$$

Mutual Inductance Between Rotor and Stator (III)

- For the damper winding we have:

$$
\begin{aligned}
l_{ad} &= l_{ad0}\cos(\theta_r) \\
l_{bd} &= l_{ad0}\cos(\theta_r - \frac{2\pi}{3}) \\
l_{cd} &= l_{ad0}\cos(\theta_r + \frac{2\pi}{3})
\end{aligned}
$$

- In the quadrature axis, we have:

$$
\begin{aligned}
l_{aq} &= l_{aq0}\cos(\theta_r + \frac{\pi}{2}) = -l_{aq0}\sin(\theta_r) \\
l_{bq} &= -l_{aq0}\sin(\theta_r - \frac{2\pi}{3}) \\
l_{cq} &= -l_{aq0}\sin(\theta_r + \frac{2\pi}{3})
\end{aligned}
$$

Rotor Self and Mutual Inductances

- For both the round rotor and the salient pole case, rotor self-inductances are all constant:

$$
\begin{aligned}
l_{ff} &= l_{ff0} = \text{constant} \\
l_{dd} &= l_{dd0} = \text{constant} \\
l_{qq} &= l_{qq0} = \text{constant}
\end{aligned}
$$

- For both the round and the salient pole rotor mutual inductances are:

$$
l_{fq} = l_{dq} = 0
$$

  i.e., there is no mutual coupling between d- and q-axis circuits

- Moreover, we have:

$$
l_{fd} = l_{fd0} = \text{constant}
$$

Inductance Matrices (I)

- Round Rotor:

$$[L_s] = \begin{bmatrix} l_{aao} & -l_{abo} & -l_{abo} \\ -l_{abo} & l_{aao} & -l_{abo} \\ -l_{abo} & -l_{abo} & l_{aao} \end{bmatrix}$$

- Salient pole rotor:

$$[L_s] = \begin{bmatrix} l_{aao} + l_{aa2}\cos 2(\theta_r) & -l_{abo} - l_{ab2}\cos 2(\theta_r + \frac{\pi}{6}) & -l_{abo} - l_{ab2}\cos 2(\theta_r + \frac{5\pi}{6}) \\ -l_{abo} - l_{ab2}\cos 2(\theta_r + \frac{\pi}{6}) & l_{aao} + l_{aa2}\cos 2(\theta_r - \frac{2\pi}{3}) & -l_{abo} - l_{ab2}\cos 2(\theta_r - \frac{\pi}{2}) \\ -l_{abo} - l_{ab2}\cos 2(\theta_r + \frac{5\pi}{6}) & -l_{abo} - l_{ab2}\cos 2(\theta_r - \frac{\pi}{2}) & l_{aao} + l_{aa2}\cos 2(\theta_r + \frac{2\pi}{3}) \end{bmatrix}$$

Inductance Matrices (II)

- Rotor-rotor inductances are all constant:

$$\left[L_R\right] = \begin{bmatrix} l_{ff0} & l_{fd0} & 0 \\ l_{fd0} & l_{dd0} & 0 \\ 0 & 0 & l_{qq0} \end{bmatrix}$$

- Stator and rotor inductances are the same in the round rotor and salient pole cases.

$$\left[L_{sr}\right] = \begin{bmatrix} l_{af0}\cos(\theta_r) & l_{ad0}\cos(\theta_r) & -l_{aq0}\sin(\theta_r) \\ l_{af0}\cos(\theta_r - \frac{2\pi}{3}) & l_{ad0}\cos(\theta_r - \frac{2\pi}{3}) & -l_{aq0}\sin(\theta_r - \frac{2\pi}{3}) \\ l_{af0}\cos(\theta_r + \frac{2\pi}{3}) & l_{ad0}\cos(\theta_r + \frac{2\pi}{3}) & -l_{aq0}\sin(\theta_r + \frac{2\pi}{3}) \end{bmatrix}$$

- Moreover, due to symmetry: $[L_{sr}] = [L_{rs}]^T$

Variable Transformation (I)

- Looking at the flux linkage equations, we see a possible transformation:

$$\left[\psi_r\right] = -\left[L_{rs}\right]\left[i_s\right] + \left[L_r\right]\left[i_r\right]$$

$$\Downarrow$$

$$\begin{bmatrix}\psi_f \\ \psi_{d1} \\ \psi_{q1}\end{bmatrix} = \begin{bmatrix} l_{af0}\cos(\theta_r) & l_{ad0}\cos(\theta_r) & -l_{aq0}\sin(\theta_r) \\ l_{af0}\cos(\theta_r - \frac{2\pi}{3}) & l_{ad0}\cos(\theta_r - \frac{2\pi}{3}) & -l_{aq0}\sin(\theta_r - \frac{2\pi}{3}) \\ l_{af0}\cos(\theta_r + \frac{2\pi}{3}) & l_{ad0}\cos(\theta_r + \frac{2\pi}{3}) & -l_{aq0}\sin(\theta_r + \frac{2\pi}{3}) \end{bmatrix}^T \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

$$+ \begin{bmatrix} l_{ff0} & l_{fd0} & 0 \\ l_{fd0} & l_{dd0} & 0 \\ 0 & 0 & l_{qq0} \end{bmatrix} \begin{bmatrix} i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

Variable Transformation (II)

- The rotor flux linkages are then:

$$\psi_f = -\,l_{af0}[i_a \cos(\theta_r) + i_b \cos(\theta_r - \frac{2\pi}{3}) + i_c \cos(\theta_r + \frac{2\pi}{3})] + l_{ff0} i_f + l_{fd0} i_{d1}$$

$$\psi_{d1} = -\,l_{ad0}[i_a \cos(\theta_r) + i_b \cos(\theta_r - \frac{2\pi}{3}) + i_c \cos(\theta_r + \frac{2\pi}{3})] + l_{fd0} i_f + l_{dd0} i_{d1}$$

$$\psi_{q1} = -\,l_{aq0}[i_a \sin(\theta_r) + i_b \sin(\theta_r - \frac{2\pi}{3}) + i_c \sin(\theta_r + \frac{2\pi}{3})] + l_{qq0} i_{q1}$$

Variable Transformation (III)

- If we define a new variable proportional to:

$$i_a \cos(\theta_r) + i_b \cos(\theta_r - \frac{2\pi}{3}) + i_c \cos(\theta_r + \frac{2\pi}{3})$$

- then the dependence on θ_r of the flux linkage could be eliminated from the direct-axis flux linkage equations.

- Let:

$$i_d = K(i_a \cos(\theta_r) + i_b \cos(\theta_r - \frac{2\pi}{3}) + i_c \cos(\theta_r + \frac{2\pi}{3}))$$

- i_d is the *direct axis* quantity.

Variable Transformation (IV)

- Similarly, if we define a new current proportional to:

$$-(i_a \sin(\theta_r) + i_b \sin(\theta_r - \frac{2\pi}{3}) + i_c \sin(\theta_r + \frac{2\pi}{3}))$$

  the dependence on θ_r is eliminated.

- Let $i_q = -K(i_a \sin(\theta_r) + i_b \sin(\theta_r - \frac{2\pi}{3}) + i_c \sin(\theta_r + \frac{2\pi}{3}))$
- i_q is the *quadrature axis* quantity.

Zero Sequence Variables

- We have introduced i_d and i_q in place of i_a, i_b and i_c.
- To retain the complete information of a, b, c quantities, we introduce a third variable.
- A possible choice is as follows:

$$i_o = \frac{1}{3}(i_a + i_b + i_c)$$

- i_o is called the zero-sequence current and has no mutual coupling to any circuit on the d and q axes.
- The zero sequence is null for balanced conditions.

Variable Transformation (V)

- For balanced 3-phase steady-state operation:

$$i_a = I_M \cos(\omega t)$$
$$i_b = I_M \cos(\omega t - \frac{2\pi}{3})$$
$$i_c = I_M \cos(\omega t + \frac{2\pi}{3})$$

- Substituting into the expression of i_d:

$$i_d = K\frac{3}{2}I_M cos(\theta_r - \omega t)$$

- Hence the maximum value for i_d is: $i_d = K\frac{3}{2}I_M$

  so we select: $K = \frac{2}{3}$

- The constant K is arbitrary, but we choose it to make i_d and i_a numerically equivalent.

Physical Interpretation

- i_d is a fictitious current that can be interpreted as the instantaneous current in a winding that is rotating with the rotor and is symmetrical to the direct axis.
- i_d produces the same MMF on the direct axis as does the 3-phase currents in the real stator windings.
- i_q has the same interpretation but in the quadrature axis.
- i_o is the homopolar current.

Steady State

- For balanced operation:

$$i_d = I_M \cos(\theta_r - \omega t)$$
$$i_q = -I_M \sin(\theta_r - \omega t)$$

where ω is the frequency of the current and θ_r is the angle of the rotor:

$$\theta_r = \omega_r t + \theta_{r0}$$

- For steady state synchronous speed operation $\Rightarrow$ $\omega = \omega_r$, hence:

$$i_d = I_M \cos \theta_{r0}$$
$$i_q = -I_M \sin \theta_{r0}$$

Park's Transformation

- The transformation we have described is known as

Park's Transformation

- For example:

$$\psi_f = -\frac{3}{2}L_{af}i_d + L_{ff}i_f + L_{fd}i_{d1}$$
$$\psi_{d1} = -\frac{3}{2}L_{ad}i_d + L_{fd}i_f + L_{dd}i_{d1}$$
$$\psi_{q1} = -\frac{3}{2}L_{aq}i_q + L_{qq}i_{q1}$$

Park's Transformation

- Projecting the phase quantities (abc) onto the $(dq0)$ axis, we obtain:

$$f_d = \frac{2}{3}(f_a \cos\theta_r + f_b \cos(\theta_r - 120^\circ) + f_c \cos(\theta_r + 120^\circ))$$

$$f_q = -\frac{2}{3}(f_a \sin\theta_r + f_b \sin(\theta_r - 120^\circ) + f_c \sin(\theta_r + 120^\circ))$$

$$f_o = \frac{1}{3}(f_a + f_b + f_c)$$

Inverse Park's Transformation

- The inverse Park's Transformation $(dq0)$ to (abc) is:

$$f_a = f_d \cos\theta_r - f_q \sin\theta_r + f_o$$

$$f_b = f_d \cos(\theta_r - 120^\circ) - f_q \sin(\theta_r - 120^\circ) + f_o$$

$$f_c = f_d \cos(\theta_r + 120^\circ) - f_q \sin(\theta_r - 120^\circ) + f_o$$

Park's Transformation Matrix

- In shorthand notation:

$$\left[f_{dq}\right] = \begin{bmatrix} f_d \\ f_q \\ f_o \end{bmatrix}; \qquad \left[f_s\right] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix};$$

- hence:

$$\left[f_{dq}\right] = \left[P\right]\left[f_s\right]$$

- where:

$$\left[P\right] = \frac{2}{3}\begin{bmatrix} \cos\theta_r & \cos(\theta_r - 120^\circ) & \cos(\theta_r + 120^\circ) \\ -\sin\theta_r & -\sin(\theta_r - 120^\circ) & -\sin(\theta_r + 120^\circ \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Inverse Park's Transformation Matrix

- In shorthand notation:

$$\left[f_s\right] = \left[P\right]^{-1} \left[f_{dq}\right]$$

- where:

$$\left[P\right]^{-1} = \begin{bmatrix} \cos\theta_r & -\sin\theta_r & 1 \\ \cos(\theta_r - 120^\circ) & -\sin(\theta_r - 120^\circ) & 1 \\ \cos(\theta_r + 120^\circ) & -\sin(\theta_r + 120^\circ) & 1 \end{bmatrix}$$

Alternative Park's Transformation Matrices

- Sometimes it is required that:

$$\left[\hat{P}\right] = \left[[\hat{P}]^{-1}\right]^{T}$$

- To this aim we can define:

$$\left[\hat{P}\right] = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\theta_r & \cos(\theta_r - 120^\circ) & \cos(\theta_r + 120^\circ) \\ -\sin\theta_r & -\sin(\theta_r - 120^\circ) & -\sin(\theta_r + 120^\circ \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

- Using this formulation, the formal expression of instantaneous power is conserved:

$$\begin{aligned} P(t) =& [V_s]^T [I_s] = [[\hat{P}]^{-1} [V_s]]^T [[\hat{P}]^{-1} [I_s]] \\ =& [V_s]^T \{[\hat{P}]^{-1}\}^T [\hat{P}]^{-1} [I_s] = [V_{pq}][I_{pq}] \qquad \# \end{aligned}$$

Derivatives of the Transformed Variables

- We define the time derivative of the transformed variables:

$$
\begin{aligned}
& [P]\frac{d}{dt}[f_s(t)] \\
= \;& [P]\frac{d}{dt}\{[P]^{-1}[f_{dq}(t)]\} \\
= \;& \{[P]\frac{d}{dt}[P]^{-1}\}[f_{dq}(t)] + [P][P]^{-1}\frac{d}{dt}[f_{dq}(t)]
\end{aligned}
$$

- where $[P][P]^{-1} = [I_3]$

Derivatives of the Transformed Variables

- and where: $[P]\frac{d}{dt}[P]^{-1}$ =

$$\frac{2}{3}\begin{bmatrix} \cos\theta_r & \cos(\theta_r-120^\circ) & \cos(\theta_r+120^\circ) \\ -\sin\theta_r & -\sin(\theta_r-120^\circ) & -\sin(\theta_r+120^\circ \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \omega_r \begin{bmatrix} -\sin\theta_r & -\cos\theta_r & 0 \\ -\sin(\theta_r-120^\circ) & -\cos(\theta_r-120^\circ) & 0 \\ -\sin(\theta_r+120^\circ) & -\cos(\theta_r+120^\circ) & 0 \end{bmatrix}$$

$$= \quad \frac{2}{3}\omega_r \begin{bmatrix} 0 & -\frac{3}{2} & 0 \\ \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad = \quad \omega_r \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad = \quad \omega_r \left[P_\omega\right]$$

where

$$\frac{d\theta_r}{dt} = \omega_r$$

Derivatives of the Transformed Variables

- Therefore:

$$[P]\frac{d}{dt}[f_s(t)] = \frac{d}{dt}[f_{dq}(t)] + \omega_r[P_\omega][f_{dq}(t)]$$

- In scalar form:

$$\dot{f}_d(t) = \frac{d}{dt}f_d + \omega_r f_q$$
$$\dot{f}_q(t) = \frac{d}{dt}f_q - \omega_r f_d$$
$$\dot{f}_o(t) = \frac{d}{dt}f_o$$

Rotor Equations

$$\begin{bmatrix} \psi_f \\ \psi_{d1} \\ \psi_{q1} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}L_{af} & 0 & L_{af} & L_{fd} & 0 \\ -\frac{3}{2}L_{ad} & 0 & L_{fd} & L_{dd} & 0 \\ 0 & -\frac{3}{2}L_{aq} & 0 & 0 & L_{qq} \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

$$\begin{bmatrix} e_f \\ e_{d1} \\ e_{q1} \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \psi_f \\ \psi_{d1} \\ \psi_{q1} \end{bmatrix} + \begin{bmatrix} R_f & 0 & 0 \\ 0 & R_d & 0 \\ 0 & 0 & R_q \end{bmatrix} \begin{bmatrix} i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

Stator Equations

- We now transform the stator flux linkages and currents to the (dqo) frame.

$$\begin{bmatrix} \psi_a \\ \psi_b \\ \psi_c \end{bmatrix} = \begin{bmatrix} -[L_S] [L_{SR}] \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

Expanding the Flux Linkage Equations

$$
\begin{aligned}
\psi_a = & -[L_{aao} + L_{aa2}\cos 2\theta r]i_a \\
& +[L_{abo} + L_{aa2}\cos 2(\theta r + \frac{\pi}{6})]i_b \\
& +[L_{abo} + L_{aa2}\cos 2(\theta r + \frac{5\pi}{6})]i_c \\
& +L_{af}\cos\theta_r i_f \\
& +L_{ad}\cos\theta_r i_{d1} \\
& +L_{aq}\sin\theta_r i_{q1}
\end{aligned}
$$

Applying the Inverse Transformation

$$
\begin{aligned}
&\psi_d \cos\theta_r - \psi_q \sin\theta_r + \psi_o = \\
&-[L_{abo} + L_{aa2}\cos 2\theta_r](i_d \cos\theta_r - i_q \sin\theta_r + i_o) \\
&+[L_{abo} + L_{aa2}\cos 2(\theta_r + \frac{\pi}{6})](i_d \cos(\theta_r - \frac{2\pi}{3}) - i_q \sin(\theta_r - \frac{2\pi}{3}) + i_o) \\
&+[L_{abo} + L_{aa2}\cos 2(\theta_r + \frac{5\pi}{6})](i_d \cos(\theta_r + \frac{2\pi}{3}) - i_q \sin(\theta_r + \frac{2\pi}{3}) + i_o) \\
&+L_{af}\cos\theta_r i_f + L_{ad}\cos\theta_r i_{d1} - L_{aq}\sin\theta_r i_{q1}
\end{aligned}
$$

Some Identities

- We now use the identities:

$$\cos(\theta_r - \frac{2\pi}{3}) - \cos(\theta_r + \frac{2\pi}{3}) = -\cos\theta_r$$

$$\sin(\theta_r - \frac{2\pi}{3}) - \sin(\theta_r + \frac{2\pi}{3}) = -\sin\theta_r$$

$$\cos(\theta_r - \frac{2\pi}{3}) - \cos(\theta_r + \frac{2\pi}{3}) = \sqrt{3}\sin\theta_r$$

$$\sin(\theta_r - \frac{2\pi}{3}) - \sin(\theta_r + \frac{2\pi}{3}) = -\sqrt{3}\cos\theta_r$$

Stator Equations (contd.)

$$
\begin{aligned}
&\psi_d \cos\theta_r - \psi_q \sin\theta_r + \psi_o = \\
&-L_{aao} i_d \cos\theta_r + L_{aao} i_q \sin\theta_r - L_{abo} i_d \cos\theta_r + L_{abo} i_q \sin\theta_r \\
&-L_{aa2} i_d [\cos 2\theta_r \cos\theta_r - \cos 2(\theta_r + \frac{\pi}{6}) \cos(\theta_r - \frac{2\pi}{3}) - \cos 2(\theta_r + \frac{5\pi}{6}) \cos(\theta_r + \frac{2\pi}{3})] \\
&-L_{aa2} i_q [\cos 2\theta_r \sin\theta_r - \cos 2(\theta_r + \frac{\pi}{6}) \sin(\theta_r - \frac{2\pi}{3}) - \cos 2(\theta_r + \frac{5\pi}{6}) \sin(\theta_r + \frac{2\pi}{3})] \\
&+(-L_{aao} + 2L_{abo}) i_o \\
&+L_{af} \cos\theta_r i_f + L_{ad} \cos\theta_r i_{d1} + L_{aq} \sin\theta_r i_{q1}
\end{aligned}
$$

Stator Equations (contd.)

- The first term in square brackets can be simplified as:

$$\cos 2\theta_r \cos\theta_r - \cos 2(\theta_r + \frac{\pi}{6})\cos(\theta_r - \frac{2\pi}{3}) - \cos 2(\theta_r + \frac{5\pi}{6})\cos(\theta_r + \frac{2\pi}{3})$$

$$= \quad \frac{3}{2}\cos 2\theta_r \cos\theta_r + \frac{3}{2}\sin 2\theta_r \sin\theta_r$$

$$= \quad \frac{3}{2}\cos\theta_r$$

- The second term in square brackets can be simplified as:

$$\cos 2\theta_r \sin\theta_r - \cos 2(\theta_r + \frac{\pi}{6})\sin(\theta_r - \frac{2\pi}{3}) - \cos 2(\theta_r + \frac{5\pi}{6})\sin(\theta_r + \frac{2\pi}{3})$$

$$= \quad \frac{3}{2}\cos 2\theta_r \sin\theta_r + \frac{3}{2}\sin 2\theta_r \cos\theta_r$$

$$= \quad -\frac{3}{2}\sin\theta_r$$

Stator Equations (contd.)

- Substituting, we find:

$$\psi_d \cos\theta_r - \psi_q \sin\theta_r + \psi_o =$$

$$[L_{af}i_f + L_{ad}i_{d1} - (L_{aao} + L_{abo} + \frac{3}{2}L_{aa2})i_d] \cos\theta_r$$

$$+[-L_{aq}i_{q1} + (L_{aao} + L_{abo} - \frac{3}{2}L_{aa2})i_q] \sin\theta_r$$

$$-(L_{aao} - 2L_{abo})i_o$$

- Equating similar terms in each side, we obtain:

$$\psi_d = -(L_{aao} + L_{abo} + \frac{3}{2}L_{aa2})i_d + L_{af}i_f + L_{ad}i_{d1}$$

$$\psi_q = -(L_{aao} + L_{abo} - \frac{3}{2}L_{aa2})i_q + L_{aq}i_{q1}$$

$$\psi_o = -(L_{aao} - 2L_{abo})i_o$$

Definition of Inductances

- We now define the inductances:

$$L_d = L_{aao} + L_{abo} + \frac{3}{2} L_{aa2}$$
$$L_q = L_{aao} + L_{abo} - \frac{3}{2} L_{aa2}$$
$$L_o = L_{aao} - 2L_{abo}$$

- The stator flux linkage equations can be rewritten as:

$$\psi_d = -L_d i_d + L_{af} i_f + L_{ad} i_{d1}$$
$$\psi_q = -L_q i_q + L_{aq} i_{q1}$$
$$\psi_o = -L_o i_o$$

Round Rotor Machine

- For round rotor machines:

$$L_{aa2} = 0$$

- Hence, the inductances become:

$$L_d = L_q = L_{aao} + L_{abo}$$

and

$$L_o = L_{aa2} - 2L_{abo}$$

- Note that: $L_d > L_q > L_o$

Stator Voltage Equations

- The transformation is done as follows.
- We begin with the voltage equations:

$$\begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \psi_a \\ \psi_b \\ \psi_c \end{bmatrix} - \begin{bmatrix} R_a & 0 & 0 \\ 0 & R_b & 0 \\ 0 & 0 & R_c \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

- We now expand the equation $e_a = \frac{d}{dt}\psi_a - R_a i_a$
  and use the inverse transformation of variables.

Stator Voltage Equations (contd.)

- This gives:

$$
\begin{aligned}
& e_d \cos\theta_r - e_q \sin\theta_r + e_o \\
= \quad & \frac{d}{dt}(\psi_d \cos\theta_r - \psi_q \sin\theta_r + \psi_o) - R_a(i_d \cos\theta_r - i_q \sin\theta_r + i_o) \\
= \quad & -\omega_r \psi_d \sin\theta_r + \frac{d}{dt}\psi_d \cos\theta_r - \omega_r \psi_q \cos\theta_r - \frac{d}{dt}\psi_q \sin\theta_r + \frac{d\psi_o}{dt} \\
& - R_a(i_d \cos\theta_r - i_q \sin\theta_r + i_o)
\end{aligned}
$$

Stator Voltage Equations (contd.)

- Equating similar terms on each side gives:

$$e_d = \frac{d}{dt}\psi_d - \omega_r \psi_q - R_a i_d$$

$$e_q = \frac{d}{dt}\psi_q + \omega_r \psi_d - R_a i_q$$

$$e_o = \frac{d}{dt}\psi_o - R_a i_o$$

Summary of Transformed Equations - Rotor

$$\begin{bmatrix} \psi_f \\ \psi_{d1} \\ \psi_{q1} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}L_{af} & 0 & 0 & L_{ff} & L_{fd} & 0 \\ -\frac{3}{2}L_{ad} & 0 & 0 & L_{fd} & L_{dd} & 0 \\ 0 & -\frac{3}{2}L_{aq} & 0 & 0 & 0 & L_{qq} \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \\ i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

$$\begin{bmatrix} e_f \\ e_{d1} \\ e_{q1} \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \psi_f \\ \psi_{d1} \\ \psi_{q1} \end{bmatrix} + \begin{bmatrix} R_f & 0 & 0 \\ 0 & R_d & 0 \\ 0 & 0 & R_q \end{bmatrix} \begin{bmatrix} i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

Summary of Transformed Equations - Stator

$$\begin{bmatrix} \psi_d \\ \psi_q \\ \psi_o \end{bmatrix} = \begin{bmatrix} -L_d & 0 & 0 & \frac{3}{2}L_{af} & \frac{3}{2}L_{ad} & 0 \\ 0 & -L_q & 0 & 0 & 0 & \frac{3}{2}L_{aq} \\ 0 & 0 & -L_o & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \\ i_f \\ i_{d1} \\ i_{q1} \end{bmatrix}$$

$$\begin{bmatrix} e_d \\ e_q \\ e_o \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \psi_d \\ \psi_q \\ \psi_o \end{bmatrix} + \omega_r \begin{bmatrix} -\psi_q \\ \psi_d \\ 0 \end{bmatrix} - \begin{bmatrix} R_a & 0 & 0 \\ 0 & R_a & 0 \\ 0 & 0 & R_a \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix}$$

Mechanical Equations

- Mechanical equations are:

$$\frac{d}{dt}\theta_r = \omega_r$$

$$J\frac{d}{dt}\omega_r = T_m - T_e$$

- In the equation above, ω_r is the mechanical rotor speed.

  If the number of pairs of poles $p = 1$, mechanical and electrical rotor speeds are equal.

  Otherwise:

$$\omega_{r,\text{elec}} = p \cdot \omega_r$$

Electrical Torque (I)

- In general, the electrical torque of a machine is given by:

$$T_e = -p\frac{1}{2}[I]^T \frac{d}{d\theta_r}[L(\theta_r)][I]$$

- Imposing the structure of the synchronous machine, we obtain:

$$T_e = -p\frac{1}{2}\left[[I_s]^T[-I_r]^T\right] \frac{d}{d\theta_r}\begin{bmatrix} -[L_s] & [L_{sr}] \\ -[L_{sr}]^T & [L_r] \end{bmatrix}\begin{bmatrix} [I_s] \\ [I_r] \end{bmatrix}$$

- Observe that $[L_r]$ does not depend on θ_r. Hence:

$$T_e = +\frac{p}{2}\{+[I_s]^T[L_{s,\theta r}][I_s] + [I_s]^T[L_{sr,\theta r}][I_r] + [I_r]^T[L_{sr,\theta r}]^T[I_s]\}$$

Electrical Torque (II)

- Observe that, since $T_e \in \mathbb{R}$:

$$[I_r]^T [L_{sr,\theta r}]^T [I_s] = ([I_r]^T [L_{sr,\theta r}]^T [I_s])^T$$

- Moreover:

$$([I_r]^T [L_{sr,\theta r}]^T [I_s])^T = [I_s]^T [L_{sr,\theta r}][I_r]$$

- Finally, we obtain:

$$T_e = +\frac{p}{2}[I_s]^T [L_{s,\theta r}][I_s] + p[I_s]^T [L_{sr,\theta r}][I_r]$$

Electrical Power and Torque (I)

- The instantaneous 3-phase power output of the stator is:

$$P(t) = e_a i_a + e_b i_b + e_c i_c$$

- Substituting in terms of dqo components we have:

$$P(t) = \frac{3}{2}(e_d i_d + e_q i_q + 2e_o i_o)$$

- Under balanced conditions:

$$P(t) = \frac{3}{2}(e_d i_d + e_q i_q)$$

Electrical Power and Torque (II)

- The electromagnetic torque T_e can be determined using:

$$\begin{cases} e_d = \frac{d}{dt}\psi_d - \psi_q\omega_r - R_a i_d \\ e_q = \frac{d}{dt}\psi_q + \psi_d\omega_r - R_a i_q \end{cases}$$

- and substituting e_d and e_q in $P(t)$:

$$\begin{aligned} P(t) = \quad & \tfrac{3}{2}[(i_d \tfrac{d}{dt}\psi_d + i_q \tfrac{d}{dt}\psi_q) && \leftarrow \text{Rate of change of armature magnetic energy} \\ & +(\psi_d i_q - \psi_q i_d)\omega_r && \leftarrow \text{Power transferred across the air gap} \\ & -(i_d^2 + i_q^2)R_a] && \leftarrow \text{Armature losses} \end{aligned}$$

Electrical Power and Torque (III)

- The air-gap torque T_e is obtained by dividing the power transferred across the air-gap by the rotor speed:

$$
\begin{aligned}
T_e &= \frac{3}{2}(\psi_d i_q - \psi_q i_d)\frac{\omega_r}{\omega_{r,\mathrm{mec}}} \\
&= \frac{3}{2}(\psi_d i_q - \psi_q i_d)p
\end{aligned}
$$

- where p is the number of pairs of field poles.

Reducing Rotor Quantities to the Stator

- Let's define

$$
\begin{aligned}
v_j^s &= v_j \left(\sqrt{\frac{3}{2}} \frac{N_s}{N_j} \right) \\
i_j^s &= i_j \left(\frac{N_j}{N_s} \frac{1}{\sqrt{\frac{3}{2}}} \right) \\
\psi_j^s &= \psi_j \left(\sqrt{\frac{3}{2}} \frac{N_s}{N_j} \right)
\end{aligned}
$$

- where:
  - j is the index of the j-th rotor winding;
  - N_s is the number of turns of stator windings; and
  - N_j is the number of turns of the j-th rotor winding.

Summary of Reduced Equations - Rotor

$$\begin{bmatrix} \psi_f^s \\ \psi_{d1}^s \\ \psi_{q1}^s \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}\hat{L}_{af} & 0 & 0 & \frac{3}{2}\hat{L}_{ff} & \frac{3}{2}\hat{L}_{fd} & 0 \\ -\frac{3}{2}\hat{L}_{ad} & 0 & 0 & \frac{3}{2}\hat{L}_{fd} & \frac{3}{2}\hat{L}_{dd} & 0 \\ 0 & -\frac{3}{2}\hat{L}_{aq} & 0 & 0 & 0 & \frac{3}{2}\hat{L}_{qq} \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \\ i_f^s \\ i_{d1}^s \\ i_{q1}^s \end{bmatrix}$$

$$\begin{bmatrix} e_f^s \\ e_{d1}^s \\ e_{q1}^s \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \psi_f^s \\ \psi_{d1}^s \\ \psi_{q1}^s \end{bmatrix} + \begin{bmatrix} \frac{3}{2}\hat{R}_f & 0 & 0 \\ 0 & \frac{3}{2}\hat{R}_d & 0 \\ 0 & 0 & \frac{3}{2}\hat{R}_q \end{bmatrix} \begin{bmatrix} i_f^s \\ i_{d1}^s \\ i_{q1}^s \end{bmatrix}$$

- where $\hat{L}_{ff} = \frac{N_s^2}{N_f^2} L_{ff}$, $\hat{L}_{dd} = \frac{N_s^2}{N_{d1}^2} L_{dd}$ etc.

Summary of Reduced Equations - Stator

$$\begin{bmatrix} \psi_d \\ \psi_q \\ \psi_o \end{bmatrix} = \begin{bmatrix} -L_d & 0 & 0 & \frac{3}{2}\hat{L}_{af} & \frac{3}{2}\hat{L}_{ad} & 0 \\ 0 & -L_q & 0 & 0 & 0 & \frac{3}{2}\hat{L}_{aq} \\ 0 & 0 & -L_o & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \\ i_f^s \\ i_{d1}^s \\ i_{q1}^s \end{bmatrix}$$

$$\begin{bmatrix} e_d \\ e_q \\ e_o \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \psi_d \\ \psi_q \\ \psi_o \end{bmatrix} + \omega_r \begin{bmatrix} -\psi_q \\ \psi_d \\ 0 \end{bmatrix} - \begin{bmatrix} R_a & 0 & 0 \\ 0 & R_a & 0 \\ 0 & 0 & R_a \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix}$$

Per Unit System for the Stator Quantities

- $E_{s,\text{base}}$ = peak value of rated line-to-neutral voltage [V]
- $I_{s,\text{base}}$ = peak value of rated line current [A]
- f_{base} = f_n = rated frequency [Hz]

⇒ Derived quantities:

$$\omega_n = \omega_{\text{base}} = 2\pi f_{\text{base}} \qquad \omega_{m,\text{base}} = \omega_{mn} = \frac{1}{p}\omega_{\text{base}}$$

$$Z_{s,\text{base}} = Z_n = \frac{E_{s,\text{base}}}{I_{s,\text{base}}} [\omega] \qquad L_{s,\text{base}} = L_n = \frac{Z_n}{\omega_n} \text{ [H]}$$

Other Derived Per Unit Quantities

- Flux base: $\Psi_{s,\text{base}} = \Psi_n = L_{s,\text{base}} I_{s,\text{base}} = \dfrac{E_{s,\text{base}}}{\omega_{\text{base}}}$ [Wb·t]
- 3-phase power base: $S_{\text{base}} = S_n = 3\dfrac{E_{s,\text{base}}}{\sqrt{2}}\dfrac{I_{s,\text{base}}}{\sqrt{2}} = \dfrac{3}{2}E_{s,\text{base}} I_{s,\text{base}}$ [VA]
- Torque base: $T_{\text{base}} = T_n = \dfrac{S_n}{\omega_{mn}} = \dfrac{3}{2} p \Psi_{s,\text{base}} I_{s,\text{base}}$ [Nm]

Summary of Per Unit Equations - Rotor

$$\begin{bmatrix} \psi_f^s \\ \psi_{d1}^s \\ \psi_{q1}^s \end{bmatrix} = \begin{bmatrix} -x_{af} & 0 & 0 & x_{ff}^s & x_{fd}^s & 0 \\ -x_{ad} & 0 & 0 & x_{fd}^s & x_{dd}^s & 0 \\ 0 & -x_{aq} & 0 & 0 & 0 & x_{qq}^s \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \\ i_f^s \\ i_{d1}^s \\ i_{q1}^s \end{bmatrix}$$

$$\begin{bmatrix} e_f^s \\ e_{d1}^s \\ e_{q1}^s \end{bmatrix} = \frac{1}{\omega_n}\frac{d}{dt} \begin{bmatrix} \psi_f^s \\ \psi_{d1}^s \\ \psi_{q1}^s \end{bmatrix} + \begin{bmatrix} r_f^s & 0 & 0 \\ 0 & r_{d1}^s & 0 \\ 0 & 0 & r_{q1}^s \end{bmatrix} \begin{bmatrix} i_f^s \\ i_{d1}^s \\ i_{q1}^s \end{bmatrix}$$

- where $x_{af}^s = \frac{3}{2}\hat{L}_{af}^s / L_n$, etc. $\quad r_f^s = \frac{3}{2}\frac{\hat{R}_f}{Z_n}$, etc.

Summary of Per Unit Equations - Stator

$$\begin{bmatrix} \psi_d \\ \psi_q \\ \psi_o \end{bmatrix} = \begin{bmatrix} -x_d & 0 & 0 & x^s_{af} & x^s_{ad} & 0 \\ 0 & -x_q & 0 & 0 & 0 & \frac{3}{2}x^s_{aq} \\ 0 & 0 & -x_o & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \\ i^s_f \\ i^s_{d1} \\ i^s_{q1} \end{bmatrix}$$

$$\begin{bmatrix} e_d \\ e_q \\ e_o \end{bmatrix} = \frac{1}{\omega_n}\frac{d}{dt} \begin{bmatrix} \psi_d \\ \psi_q \\ \psi_o \end{bmatrix} + \omega \begin{bmatrix} -\psi_q \\ \psi_d \\ 0 \end{bmatrix} - \begin{bmatrix} r_a & 0 & 0 \\ 0 & r_a & 0 \\ 0 & 0 & r_o \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix}$$

where $\omega = \frac{\omega_r}{\omega_n}$

Mechanical Equations in Per Unit (I)

- Torque equation in per unit:

$$\tau_e = \psi_d i_q - \psi_q i_d$$

- Mechanical equation:

$$\begin{aligned} T_m &= T_e + J\frac{d}{dt}\omega_{r,\mathrm{mec}} \\ \Rightarrow \quad \frac{T_m}{T_n} &= \frac{T_e}{T_n} + J\frac{\omega_{mn}}{T_n}\frac{d}{dt}\omega \\ \Rightarrow \quad \tau_m &= \tau_e + J\frac{\omega_{mn}^2}{T_n\omega_{mn}}\frac{d}{dt}\omega \end{aligned}$$

Mechanical Equations in Per Unit (II)

- The quantity $J\frac{\omega_{mn}^2}{S_n}$ is called $M = \text{start} - \text{up time}$ (observe that $T_n \cdot \omega_{mn} = S_n$)
- It is often defined $H = \text{inertia constant}$ as:

$$2H = M$$

- hence:

$$\tau_m = \tau_e + 2H\frac{d}{dt}\omega$$

- If considering damping:

$$\tau_m = \tau_e + D(\omega - \omega_s) + 2H\frac{d}{dt}\omega$$

Leakage Reactance

- Let's define:

$$x_d = x_\ell + x_{md}$$

$$x_q = x_\ell + x_{mq}$$

where x_ℓ is the leakage reactance.

- hence

$$\psi_d = -x_\ell i_d + \psi_{md}$$

$$\psi_q = -x_\ell i_q + \psi_{mq}$$

where:

$$\psi_{md} = -x_{md} i_d + x_{af} i_f + x_{ad} i_{d1}$$

$$\psi_{mq} = -x_{mq} i_q + x_{aq} i_{q1}$$

- hence: $\tau_\ell = \psi_d i_q - \psi_q i_d = \psi_{md} i_q - \psi_{mq} i_d$

  Note that flux leakage does not contribute to the air-gap torque!

Steady-State Conditions

- In steady-state, we put all $\frac{d}{dt}\psi_j$ terms to zero
- Therefore all amortisseur currents are zero: $i_{d1} = i_{q1}$ = 0
- Then:

$$
\begin{aligned}
e_d &= -\omega_r \psi_q - r_a i_d \\
e_q &= \omega_r \psi_d - r_a i_q \\
e_{fd}^s &= r_{fd}^s i_f^s \\
\psi_d &= -x_d i_d + x_{ad}^s i_f^s \\
\psi_q &= -x_q i_q \\
\psi_f^s &= x_{ff}^s i_f^s - x_{ad}^s i_d \\
\psi_{d1} &= x_{fd}^s i_f^s - x_{ad}^s i_d \\
\psi_{q1} &= -x_{aq}^s i_q
\end{aligned}
$$

Steady-state Field Current

- According to the previous equations, the steady-state field current is:

$$i_f^s = \frac{\psi_d + x_d i_d}{x_{ad}^s}$$

- Then, substituting ψ_d in terms of e_d and i_q in terms of ψ_d and e_q:

$$i_f^s = \frac{e_q + r_a i_q + \omega x_d i_d}{\omega x_{ad}^s}$$

- Finally, observing that in steady state $\omega = \omega_s = 1$ pu, one has:

$$i_f^s = \frac{e_q + r_a i_q + x_d i_d}{x_{ad}^s}$$

Phasor Representation (Steady-State)

- Given:

$$
\begin{aligned}
e_a =& e_T \cos(\omega_s t + \alpha) \\
e_b =& e_T \cos(\omega_s t - \frac{2\pi}{3} + \alpha) \\
e_c =& e_T \cos(\omega_s t + \frac{2\pi}{3} + \alpha)
\end{aligned}
$$

- Applying the d-q axis transformation, we obtain:

$$
\begin{aligned}
e_d =& e_T \cos(\omega_s t + \alpha - \theta_r) \\
e_q =& e_T \sin(\omega_s t + \alpha - \theta_r)
\end{aligned}
$$

where $\theta_r = \omega_r t + \theta_o \Rightarrow$ angle by which the d axis leads the q axis

Phasor Representation (II)

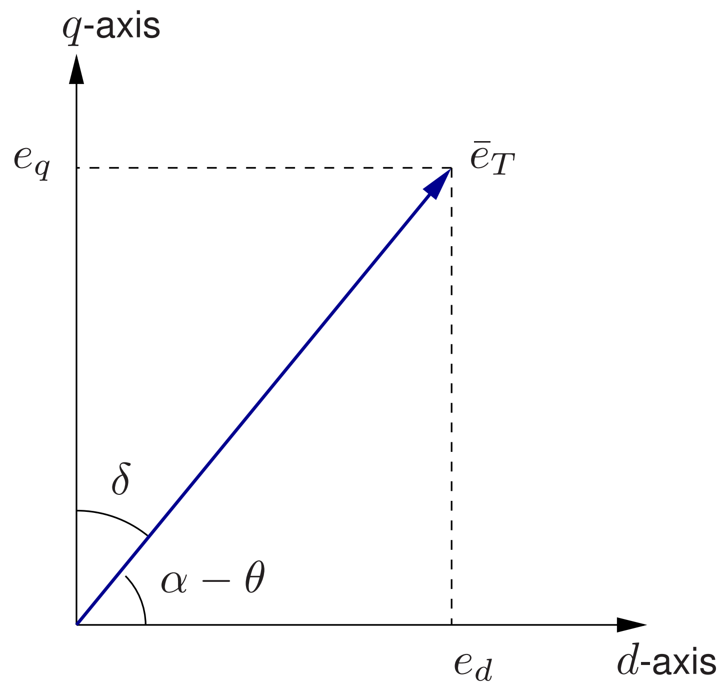
- We can define the terminal bus voltage phasor as:

$$\bar{e}_T = e_d + je_q$$

where:

$$e_d = e_T \sin\delta$$

$$e_q = e_T \cos\delta$$



where δ is the angle by which the q axis leads the phasor $\bar{e}_T$

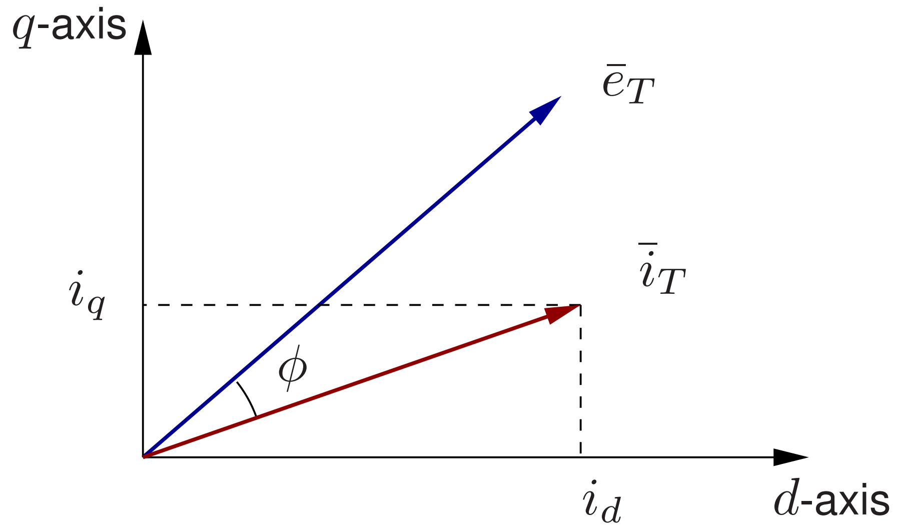
Phasor Representation (III)

- The terminal current $\bar{i}_T$ is defined as:

$$\bar{i}_T = i_d + j\, i_q$$

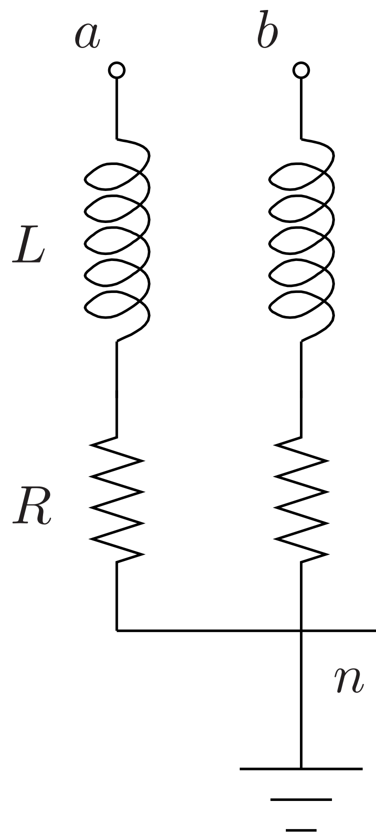
where:

$$\begin{cases} i_d = i_T\ \sin(\delta + \phi) \\ i_q = i_T\ \cos(\delta + \phi) \end{cases}$$



Example - Park's Vectors (I)

- Let consider a 3-phase symmetrical, time-invariant linear load:



Example - Park's Vectors (II)

- Let assume to apply three-phase voltages at the terminals a, b, c.
- The circuit equations, in time domain, are:

$$
\begin{aligned}
V_a(t) &= L\frac{dI_a(t)}{dt} + RI_a(t) \\
V_b(t) &= L\frac{dI_b(t)}{dt} + RI_b(t) \\
V_c(t) &= L\frac{dI_c(t)}{dt} + RI_c(t) \\
I_n(t) &= I_a(t) + I_b(t) + I_c(t)
\end{aligned}
$$

- The equations above are always valid.

Example - Park's Vectors (III)

- We can rewrite circuit equations using Park's vectors.
- Let define:

$$\begin{aligned} \bar{V}_p(t) &= V_d(t) + jV_q(t) \\ \bar{I}_p(t) &= I_d(t) + jI_q(t) \end{aligned}$$

- Then:

$$\bar{V}_p(t) = (\frac{d}{dt} + j\omega(t))L\bar{I}_p + R\bar{I}_p$$

- Where $\omega(t)$ is the speed of the Park's transformation.
- Note that $\omega(t)$ can be any function of time (i.e., it is not necessarily constant).

Example - Park's Vectors (IV)

- The vector representation can be rewritten as:

$$\begin{aligned} V_d(t) &= L\frac{dI_d(t)}{dt} - L\omega(t)I_q(t) + RI_d(t) \\ V_q(t) &= L\frac{dI_q(t)}{dt} + L\omega(t)I_d(t) + RI_q(t) \end{aligned}$$

- The equations above are always valid, but incomplete.

Example - Park's Vectors (V)

- To complete the set of equations, we write the zero component equation:

$$V_o(t) = L\frac{dI_o(t)}{dt} + RI_o(t)$$

- and the current balance:

$$I_n(t) = 3I_o(t) \quad \text{if we use } [P]$$

- and

$$I_n(t) = \sqrt{3}I_o(t) \quad \text{if we use } [\hat{P}]$$

Example - Park's Vectors (VI)

- Hence, in Park's coordinates we have:

$$
\begin{aligned}
V_d(t) &= L\frac{dI_d(t)}{dt} - L\omega(t)I_q(t) + RI_d(t) \\
V_q(t) &= L\frac{dI_q(t)}{dt} + L\omega(t)I_d(t) + RI_q(t) \\
V_o(t) &= L\frac{dI_o(t)}{dt} + RI_o(t) \\
I_n(t) &= 3I_o(t)
\end{aligned}
$$

- The equations above are always valid.

Example - Park's Vectors (VII)

- If we assume balanced and symmetrical conditions ($I_n = I_o = 0$), then the Park's vector equation fully describes the three-phase system for any transient condition.
- If we assume also steady-state conditions and that V_a, V_b and V_c are symmetric and sinusoidal, then, the Park's vector coincides, except possibly for a factor $\sqrt{2}$, with the well-known phasor representation if the reference speed ω is constant and equal to the pulsation of the voltages V_a, V_b and V_c.
- Hence, in steady-state, absolute values and for ω constant:

$$\bar{V}_p = j\omega L\bar{I}_p + R\bar{I}_p$$

Example - Park's Vectors (VIII)

- Let rewrite the Park's vector equations in per unit.
- We use the bases V_n, I_n and ω_n, then:

$$Z_n = \frac{V_n}{I_n} \qquad \text{and} \qquad L_n = \frac{Z_n}{\omega_n}$$

- If $\omega = \omega_n$, we have:

$$\bar{v}_p(t) = \frac{1}{\omega_n}\frac{d\bar{i}_p}{dt} + (r + jx)\bar{i}_p$$

- Using d- and q-axis quantities:

$$\begin{aligned} v_d(t) &= \frac{1}{\omega_n}\frac{di_d(t)}{dt} - x i_q(t) + r i_d(t) \\ v_q(t) &= \frac{1}{\omega_n}\frac{di_q(t)}{dt} + x i_d(t) + r i_q(t) \end{aligned}$$

No-load or open circuit conditions

- Let's go back to the synchronous machine model.
- In open circuit conditions $i_d = i_q = 0$
- Substituting in steady-state equations:

$$\begin{aligned} \psi_d &= x_{ad}^s i_f^s \\ \psi_q &= 0 \\ e_d &= 0 \\ e_q &= x_{ad}^s i_f^s \end{aligned}$$

- Therefore the terminal voltage is:

$$\bar{e}_T = e_d + j e_q = j x_{ad}^s i_f^s$$

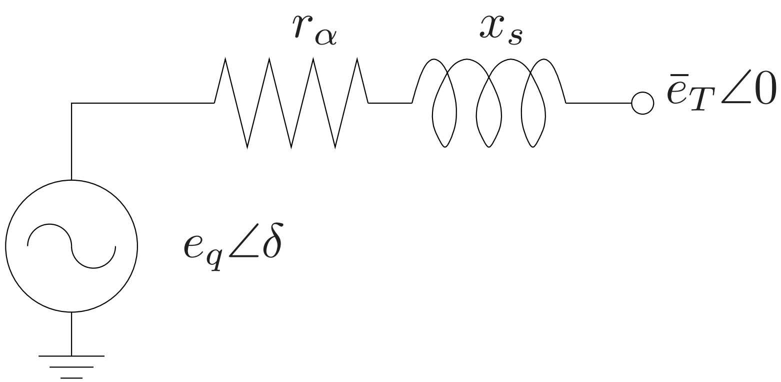
Steady-State Equivalent Circuit

- If saliency is neglected: $x_d = x_q = x_s$
  where x_s is defined as *synchronous reactance*.

- We have:

$$\bar{e}_q = \bar{e}_T + (r_a + j\, x_s)\bar{i}_T$$

where $e_q = x^s_{ad} i^s_\alpha$



Operational Impedances

- Most rotor circuits are short-circuited
- So, voltages are zero ($e_{q1} = e_{d1} = 0$) and currents of short circuited circuits can be eliminated from the system
- This leads to a formulation of stator equations in an operational form, i.e., **transfer functions**
- Hence:

$$\begin{aligned}\psi_d(s) &= -x_d(s)i_d(s) + G_f(s)v_f^s(s)\\ \psi_q(s) &= -x_q(s)i_q(s)\end{aligned}$$

Operational Impedances (II)

- Hence:

$$
\begin{aligned}
x_d(s) &= x_d \frac{(1+sT_d')(1+sT_d'')}{(1+sT_{do}')(1+sT_{do}'')} \\
x_q(s) &= x_q \frac{(1+sT_q')(1+sT_q'')}{(1+sT_{qo}')(1+sT_{qo}'')} \\
G_f(s) &= G^s \frac{(1+sT_f)}{(1+pT_{do}')(1+pT_{do}'')}
\end{aligned}
$$

Operational Impedances (III)

- The order of the transfer functions depends on the number of circuits in the rotor
- Definition of time constants:

$T'_{do}, T''_{do}, T'_{qo}, T''_{qo}$	Open circuit time constants
$T'_{d}, T''_{d}, T'_{q}, T''_{q}$	Shortcircuit time constants
$T'_{do}, T'_{qo}, T'_{d}, T'_{q}$	Transient time constants
$T''_{do}, T''_{qo}, T''_{d}, T''_{q}$	Sub-transient time constants

Definition of Time Constants

- For example:

$$
\begin{aligned}
T'_{qo} &= \frac{1}{\omega_n r^s_{q1}}(x^s_{aq} + x_{mq}) \\
T'_{do} &= \frac{1}{\omega_n r^s_{f}}(x^s_{af} + x_{md}) \\
T''_{do} &= \frac{1}{\omega_n r^s_{d1}}(x^s_{ad} + \frac{x_{md}x^s_{af}}{x_{md} + x^s_{af}})
\end{aligned}
$$

etc.

Definition of Machine Parameters

- In steady-state:

$$x_d(0) = x_d$$

x_d is the d-axis synchronous reactance

- During rapid transients, $s \to \infty$, hence:

$$x_d'' = x_d(\infty) = x_d \frac{T_d' T_d''}{T_{do}' T_{do}''}$$

x_d'' is the sub-transient d-axis reactance

Definition of Machine Parameters (II)

- If we neglect the damper winding:

$$x'_d \cong x_d(s) \cong x_d \frac{T'_d}{T'_{do}}$$

- From the definitions of the machine time constants:

$$x'_d = x_\ell + \frac{x_{md} x^s_{af}}{x_{md} + x^s_{af}}$$

x'_d is the d-axis transient reactance

- Finally:

$$x''_d = x_\ell + \frac{x_{md} x^s_{af} x^s_{ad1}}{x_{md} x^s_{af} + x_{md} x^s_{ad1} + x^s_{af} x^s_{ad1}}$$

Definition of Machine Parameters (III)

- Similarly we define x_q, x_q' (and x_q'' if we have a second q-axis amortisseur)
  - Synchronous q-axis reactance $x_q = x_q(0)$
  - Transient q-axis reactance $x_q' = x_q \frac{T_q'}{T_{qo}'}$
  - Sub-transient q-axis reactance $x_q'' = x_q \frac{T_q' T_q''}{T_{qo}' T_{qo}''}$
- Then: $x_q' = x_\ell + \frac{x_{mq} x_{aq1}^s}{x_{mq} + x_{aq1}^s}$, etc.
- The following inequalities hold:
  - $x_d \geqslant x_q > x_q' \geqslant x_d' > x_q'' \geqslant x_d''$
  - $T_{do}' > T_d' > T_{do}'' > T_d'' > T_f$
  - $T_{qo}' > T_q' > T_{qo}'' > T_q''$

Typical Values of Standard Parameters

Parameter	Hydro	Thermal
x_d	0.6-1.5	1.0-2.3
x_q	0.4-1.0	1.0-2.3
x'_d	0.2-0.5	0.15-0.4
x'_q	-	0.3-1.0
x''_d	0.15-0.35	0.12-0.25
x''_q	0.2-0.45	0.12-0.25
T'_{do}	1.5-9.0 s	3.0-10.0 s
T'_{qo}	-	0.5-2.0 s
T''_{do}	0.01-0.05 s	0.02-0.05 s
T''_{qo}	0.01-0.09 s	0.02-0.05 s
x_ℓ	0.1-0.2	0.1-0.2
r_a	0.002-0.02	0.0015-0.005

dq-axis models of Synchronous Machines

- We assume the machine is connected to bus h with voltage $\bar{v}_h = v_h \angle \theta_h$
- Common equations:
  - Power injections:

$$\begin{aligned} p_h &= v_d i_d + v_q i_q \\ q_h &= v_q i_d - v_d i_q \end{aligned}$$

  - AC-grid interface:

$$\begin{aligned} v_d &= v_h \sin(\delta - \theta_h) \\ v_q &= v_h \cos(\delta - \theta_h) \end{aligned}$$

  - Electromagnetic Torque:

$$\tau_e = \psi_d i_q - \psi_q i_d$$

Dq-axis models of Synchronous Machines

- Mechanical Equations:

$$\begin{aligned}\dot{\delta} &= \omega_n(\omega - \omega_s) \\ \dot{\omega} &= \frac{1}{2H}(\tau_m - \tau_e - D(\omega - \omega_s))\end{aligned}$$

where:

- D is a damping coefficient.
- ω_n is the base synchronous frequency in rad/s, e.g. 314.16 rad/s at 50 Hz.
- τ_m is the mechanical torque provided by the turbine.

Stator Electrical Equations

- Full dynamic equations:

$$\begin{aligned}
\dot{\psi}_d &= \omega_n(r_a i_d + \omega\psi_q + v_d) \\
\dot{\psi}_q &= \omega_n(r_a i_q - \omega\psi_d + v_q) \qquad (*)
\end{aligned}$$

- If we consider flux dynamics “fast”: $\dot{\psi}_d = 0$ and $\dot{\psi}_q = 0$, hence:

$$\begin{aligned}
0 &= r_a i_d + \omega\psi_q + v_d \\
0 &= r_a i_q - \omega\psi_d + v_q \qquad (**)
\end{aligned}$$

- If we assume that speed deviations are small then $\omega \approx 1$ pu:

$$\begin{aligned}
0 &= r_a i_d + \psi_q + v_d \\
0 &= r_a i_q - \psi_d + v_q \qquad (***)
\end{aligned}$$

Sauer-Pai's Model (I)

- Common model typically used for simulating US grids:

$$\begin{aligned}
\dot{e}'_q &= (-e'_q - (x_d - x'_d)(i_d + \gamma_{d2}\dot{\psi}''_d) + v_f)/T'_{do} \\
\dot{e}'_d &= (-e'_d + (x_q - x'_q)(i_q + \gamma_{q2}\dot{\psi}''_q))/T'_{qo} \\
\dot{\psi}''_d &= (-\psi''_d + e'_q - (x'_d - x_\ell)i_d)/T''_{do} \\
\dot{\psi}''_q &= (-\psi''_q - e'_d - (x'_q - x_\ell)i_q)/T''_{qo}
\end{aligned}$$

- Substituting the expressions of $\dot{\psi}''_d$ and $\dot{\psi}''_q$:

$$\begin{aligned}
\dot{e}'_q &= (-e'_q - (x_d - x'_d)(i_d - \gamma_{d2}\psi''_d - (1 - \gamma_{d1})i_d + \gamma_{d2}e'_q) + v_f)/T'_{do} \\
\dot{e}'_d &= (-e'_d + (x_q - x'_q)(i_q - \gamma_{q2}\psi''_q - (1 - \gamma_{q1})i_q - \gamma_{d2}e'_d))/T'_{qo} \\
\dot{\psi}''_d &= (-\psi''_d + e'_q - (x'_d - x_\ell)i_d)/T''_{do} \\
\dot{\psi}''_q &= (-\psi''_q - e'_d - (x'_q - x_\ell)i_q)/T''_{qo}
\end{aligned}$$

Sauer-Pai's Model (II)

where:

$$\gamma_{d1} = \frac{x''_d - x_\ell}{x'_d - x_\ell}$$

$$\gamma_{q1} = \frac{x''_q - x_\ell}{x'_q - x_\ell}$$

$$\gamma_{d2} = \frac{x'_d - x''_d}{(x'_d - x_\ell)^2} = \frac{1 - \gamma_{d1}}{x'_d - x_\ell}$$

$$\gamma_{q2} = \frac{x'_q - x''_q}{(x'_q - x_\ell)^2} = \frac{1 - \gamma_{q1}}{x'_q - x_\ell}$$

and with the algebraic constraints:

$$0 = \psi_d + x''_d i_d - \gamma_{d1} e'_q - (1 - \gamma_{d1})\psi''_d$$

$$0 = \psi_q + x''_q i_q - \gamma_{q1} e'_d - (1 - \gamma_{q1})\psi''_q$$

Marconato's Model(I)

- An alternative model is the following (used by ENEL, Italian ISO):

$$
\begin{aligned}
\dot{e}'_q &= \frac{(-e'_q - (x_d - x'_d - \gamma_d)i_d + (1 - \frac{T_{AA}}{T'_{do}})v_f)}{T'_{do}} \\
\dot{e}'_d &= \frac{(-e'_d + (x_q - x'_q - \gamma_q)i_q)}{T'_{qo}} \\
\dot{e}''_q &= \frac{(-e''_q + e'_q - (x'_d - x''_d + \gamma_d)i_d + \frac{T_{AA}}{T'_{do}}v_f)}{T''_{do}} \\
\dot{e}''_d &= \frac{(-e''_d + e'_d + (x'_q - x''_q + \gamma_q)i_q)}{T''_{qo}}
\end{aligned}
$$

Marconato's Model(II)

where:

$$\gamma_d = \frac{T''_{do}}{T'_{do}} \frac{x''_d}{x'_d}(x_d - x'_d)$$

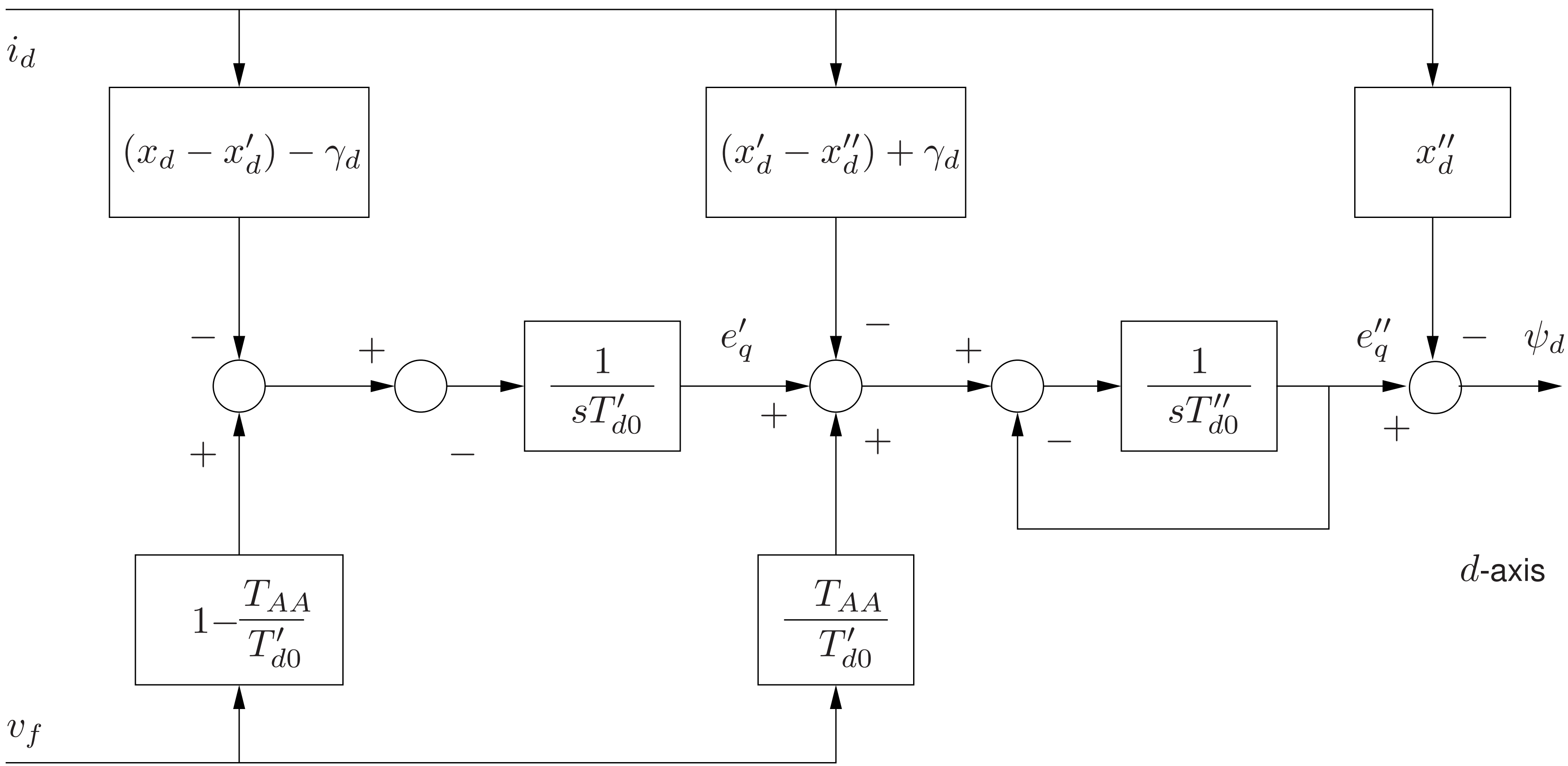
$$\gamma_q = \frac{T''_{qo}}{T'_{qo}} \frac{x''_q}{x'_q}(x_q - x'_q)$$

- and the additional algebraic constraints:

$$0 = \psi_d + x''_d i_d - e''_q$$

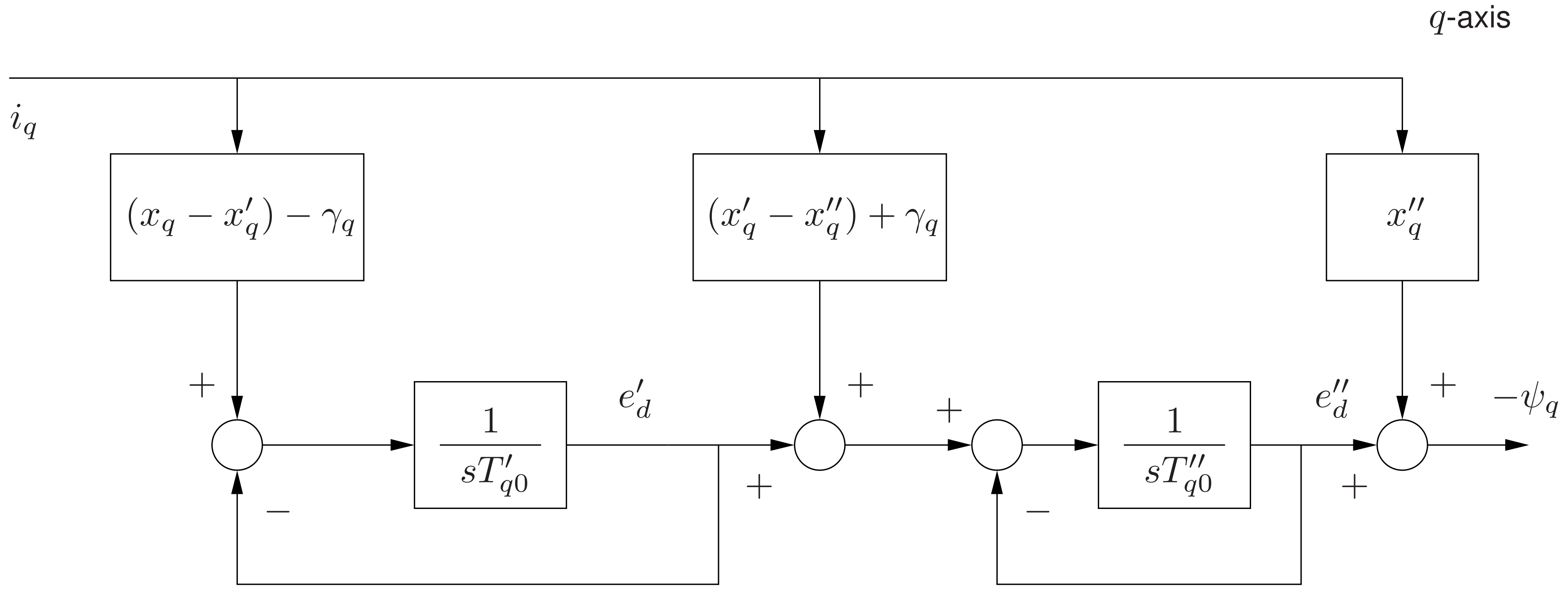
$$0 = \psi_q + x''_q i_q + e''_d$$

Marconato's Model - d-axis equivalent circuit



d-axis

Marconato's Model - q-axis equivalent circuit



Anderson-Fouad's Model

- Most common model that can be found in text books:

$$
\begin{aligned}
\dot{e}_q' &= (-e_q' - (x_d - x_d')i_d + v_f)/T_{do}' \\
\dot{e}_d' &= (-e_d' + (x_q - x_q')i_q)/T_{qo}' \\
\dot{e}_q'' &= (-e_q'' + e_q' - (x_d' - x_d'')i_d)/T_{do}'' \\
\dot{e}_d'' &= (-e_d'' + e_d' + (x_q' - x_q'')i_q)/T_{qo}''
\end{aligned}
$$

- This model can be considered a simplification of the Sauer Pai's model with:

$$e_q'' = \psi_d', \quad e_d'' = -\psi_d'', \quad \gamma_{d1} \approx \gamma_{q1} \approx 0, \quad \gamma_{d2}\dot{\psi}_d'' \approx 0, \quad \gamma_{q2}\dot{\psi}_q'' \approx 0,$$

- This model can also be viewed as a simplification of the Marconato's model with:

$$\gamma_d = \gamma_q = T_{AA} \approx 0$$

Simplified Magnetic Equations (I)

- Two d- and one q-axis model
- One d- and two q-axis model
- One d- and one q-axis model
- One d-axis model
- Electromechanical model
- “Classical” model

Two d- and one q-axis model (I)

- Let's assume $T'_{qo} \approx 0$ and $x'_q \approx x_q \rightarrow e'_d \approx 0$. Hence:

$$
\begin{aligned}
\dot{e}'_q &= (-e'_q - (x_d - x'_d)(i_d - \gamma_{d2}\psi''_d - (1-\gamma_{d1})i_d + \gamma_{d2}e'_q) + v_f)/T'_{do} \\
\dot{\psi}''_d &= (-\psi''_d + e'_q - (x'_d - x_\ell)i_d)/T''_{do} \\
\dot{\psi}''_q &= (-\psi''_q - (x'_q - x_\ell)i_q)/T''_{qo}
\end{aligned}
$$

- Plus the algebraic equations:

$$
\begin{aligned}
0 &= \psi_d + x''_d i_d - \gamma_{d1}e'_q - (1-\gamma_{d1})\psi''_d \\
0 &= \psi_q + x''_q i_q - (1-\gamma_{q1})\psi''_q
\end{aligned}
$$

Alternative Two d- and q-axis model

- Using Marconato's model and imposing $T'_{qo} \approx 0$, we obtain:

$$\begin{aligned}
\dot{e}'_q &= (-e'_q - (x_d - x'_d - \gamma_d) i_d + (1 - \frac{T_{AA}}{T'_{do}}) v_f)/T'_{do} \\
\dot{e}''_q &= (-e''_q + e'_q - (x'_d - x''_d + \gamma_d) i_d + \frac{T_{AA}}{T'_{do}} v_f)/T''_{do} \\
\dot{e}''_d &= (-e''_d + (x_q - x''_q) i_q)/T''_{qo}
\end{aligned}$$

- Plus the algebraic equations:

$$\begin{aligned}
0 &= v_q + r_a i_q - e''_q + x''_d i_d \\
0 &= v_d + r_a i_d - e''_d - x''_q i_q
\end{aligned}$$

One d- and Two q-axis model

- Let's assume that: $x'_d \approx x''_d \approx x''_q$, hence we have:

$$
\begin{aligned}
\dot{e}'_q &= (-e'_q - (x_d - x'_d)i_d + v_f)/T'_{do} \\
\dot{e}'_d &= (-e'_d + (x_q - x'_q - \gamma_q)i_q)/T'_{qo} \\
\dot{e}''_d &= (-e''_d + e'_d + (x'_q - x'_d + \gamma_q)i_q)/T''_{qo}
\end{aligned}
$$

- Plus the algebraic equations:

$$
\begin{aligned}
0 &= v_q + r_a i_q - e'_q + x'_d i_d \\
0 &= v_d + r_a i_d - e''_d - x'_q i_q
\end{aligned}
$$

One d- and one q-axis model

- If we assume that $T''_{do} \approx T''_{qo} \approx 0$, then we obtain the so-called two axis model. This is the model used in most stability studies. We have:

$$\begin{aligned} \dot{e}'_q &= (-e'_q - (x_d - x'_d)i_d + v_f)/T'_{do} \\ \dot{e}'_d &= (-e'_d + (x_q - x'_q)i_q)/T'_{qo} \end{aligned}$$

- Plus the algebraic equations:

$$\begin{aligned} 0 &= v_q + r_a i_q - e'_q + x'_d i_d \\ 0 &= v_d + r_a i_d - e'_d - x'_q i_q \end{aligned}$$

One d-axis model

- Further simplifying the machine magnetical equations, we can neglect the dynamic on the q-axis ($T'_{qo} \approx 0$). We obtain:

$$\dot{e}'_q = (-e'_q - (x_d - x'_d)i_d + v_f)/T'_{do}$$

- Plus the algebraic equations:

$$\begin{aligned} 0 &= v_q + r_a i_q - e'_q + x'_d i_d \\ 0 &= v_d + r_a i_d - x_q i_q \end{aligned}$$

Electromechanical model

- A pure electromechanical model neglects all dynamics of magnetical equations. As a consequence, the field voltage is substituted by a constant e'_q. Another assumption is that $\omega \approx 1$, hence $p_e \approx \omega T_e \approx T_e$. We have:

$$p_e = (v_q + r_a i_q) i_q + (v_d + r_a i_d) i_d$$

- Moreover if $r_a \approx 0$, $p_e = p_h$ (power injected into the grid at bus h).

- Finally let assume that $x_q = x'_d$. These assumptions lead to: e'_q = constant emf behind the transient reactance e'_d. We have:

$$\begin{aligned} 0 &= v_q + r_a i_q - e'_q + x'_d i_d \\ 0 &= v_d + r_a i_d - x'_d i_q \end{aligned}$$

Electromechanical (classical) model

- In summary the most simplified dynamic model is the following:

$$
\begin{aligned}
\dot{\delta} &= \Omega_b(\omega - 1) \\
\dot{\omega} &= (p_m - p_e - D(\omega - 1))/2H \\
0 &= (v_q + r_a i_q) i_q + (v_d + r_a i_d) i_d - p_e \\
0 &= v_q + r_a i_q - e'_q + x'_d i_d \\
0 &= v_d + r_a i_d - x'_d i_q \\
0 &= v_h \sin(\delta - \theta_h) - v_d \\
0 &= v_h \cos(\delta - \theta_h) - v_q \\
p_h &= v_d i_d + v_q i_q \\
q_h &= v_q i_d - v_d i_q
\end{aligned}
$$

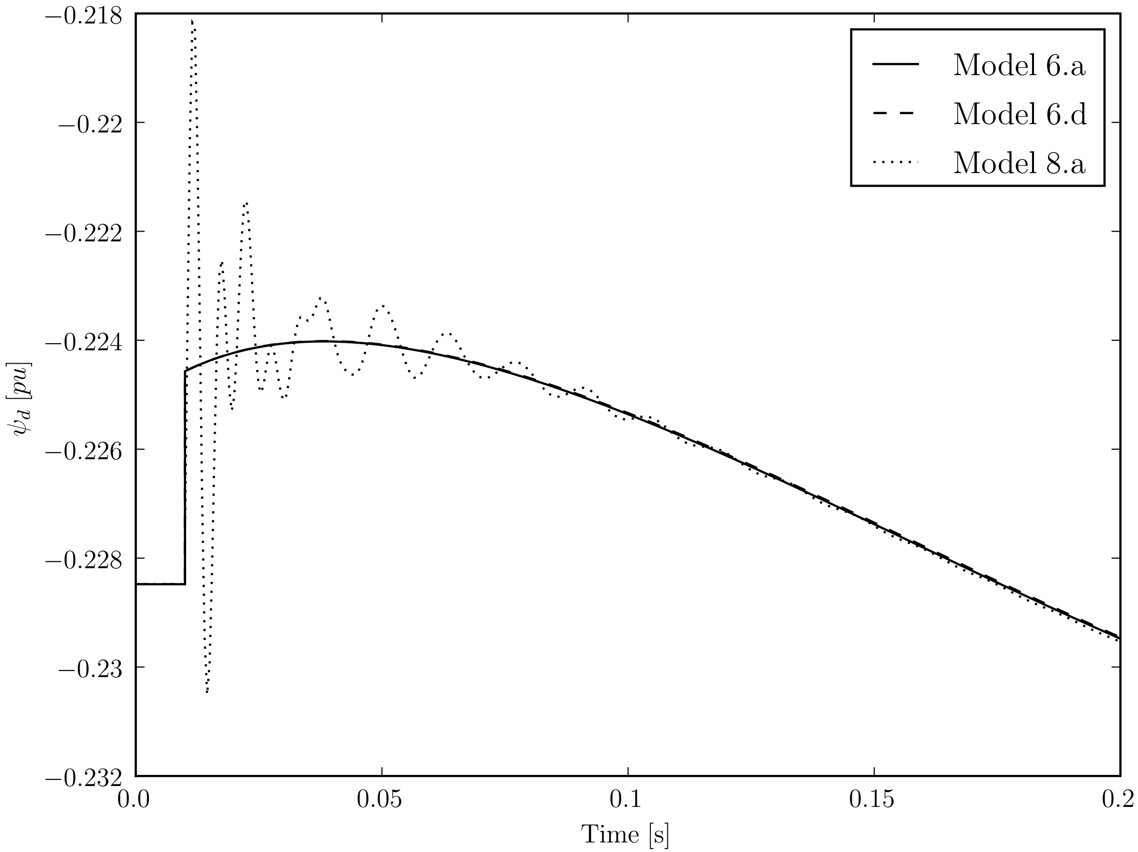
Sub-transient Electromechanical Model

- For very fast transients, it may be convenient to assume constant e''_q and e''_d.
- Hence:

$$\begin{aligned} v_d &= e''_d - r_a i_d + x''_q i_q \\ v_q &= e''_q - r_a i_q - x''_d i_d \end{aligned}$$

- This is an alternative electromechanical model where we define a "constant emf" behind the sub-transient reactance
- Observe that the so-called **classical machine model** also assumes that $r_a \approx 0$ and $D \approx 0$. $\Rightarrow$ **Lossless Model**

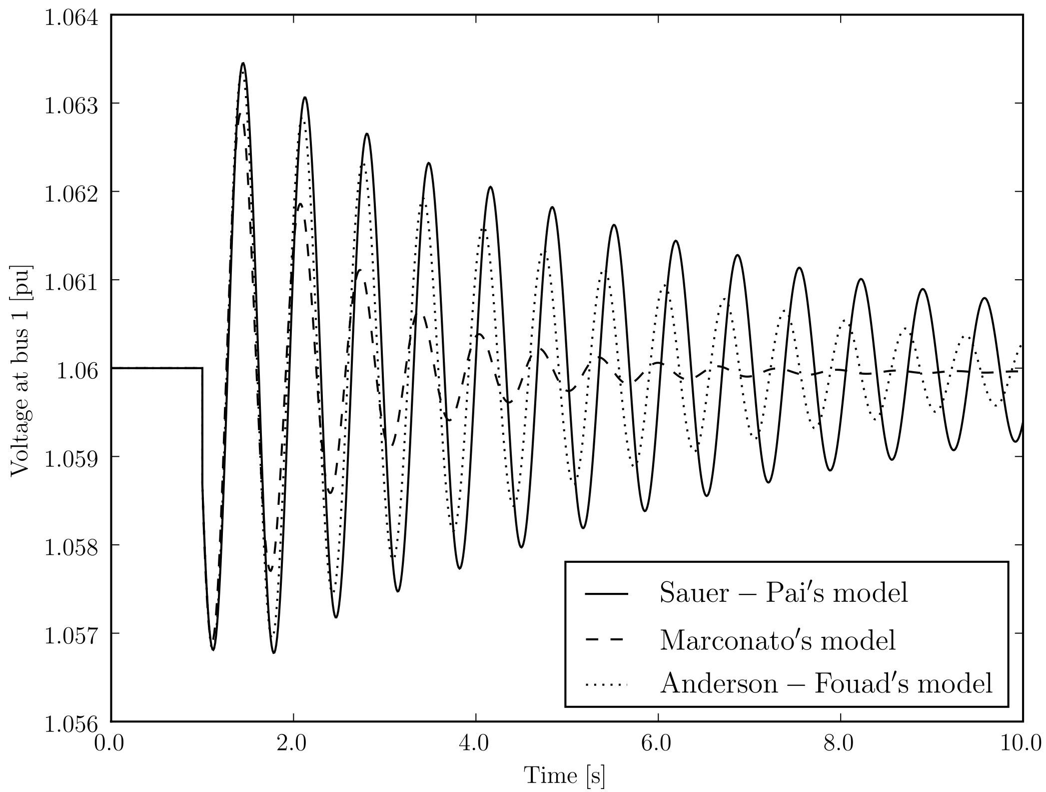
Comparison of Machine Models of Different Orders



$$6.a \Rightarrow (***)$$
$$6.d \Rightarrow (**)$$
$$8.a \Rightarrow (*)$$

All models are based on the Sauer-Pai's model for magnetical equations.

Comparison of Models of Different Types

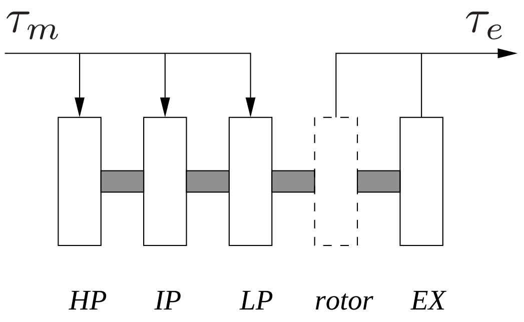


All simulations are solved using same integration step.

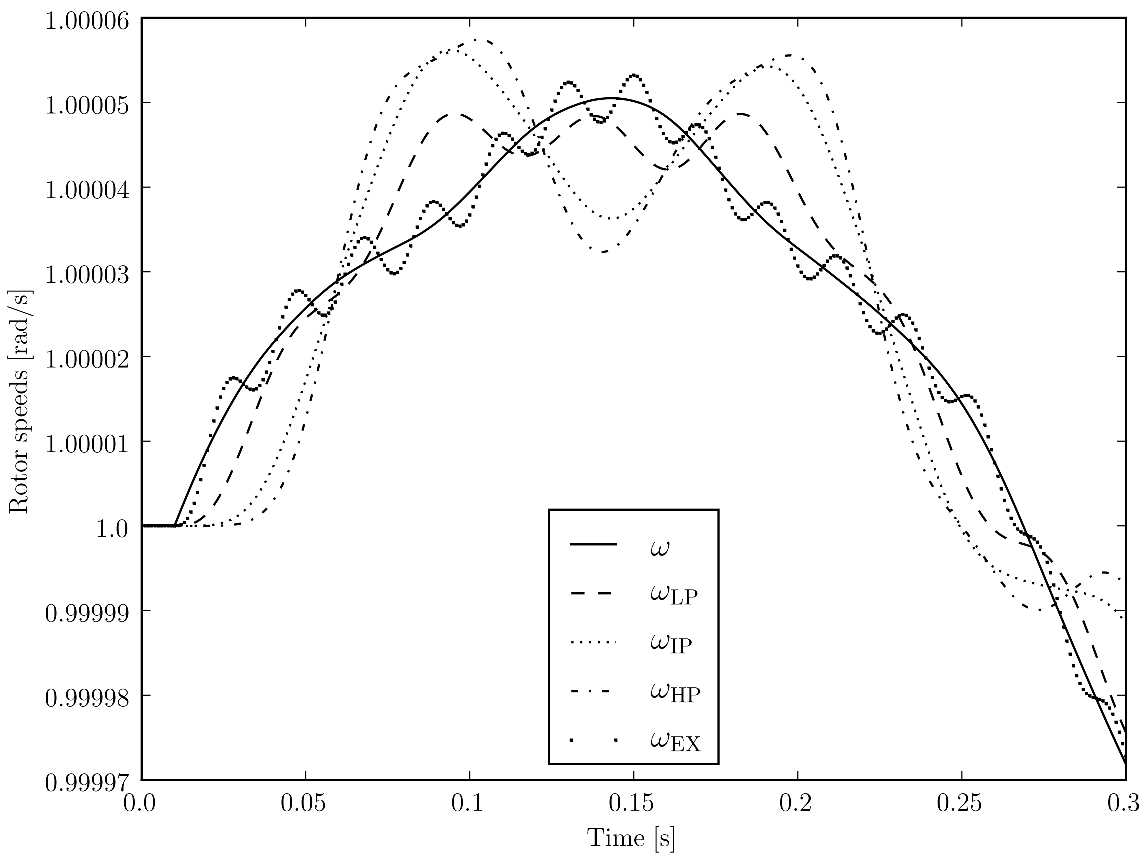
Dynamic Shaft Model (I)

$$
\begin{aligned}
\dot{\delta}_{HP} &= \omega_n(\omega_{HP}-\omega_s) \\
\dot{\omega}_{HP} &= (T_m - D_{HP}(\omega_{HP}-\omega_s) - D_{12}(\omega_{HP}-\omega_{IP}) \\
&\quad +K_{HP}(\delta_{IP}-\delta_{HP}))/2H_{HP} \\
\dot{\delta}_{IP} &= \omega_n(\omega_{IP}-\omega_s) \\
\dot{\omega}_{IP} &= (-D_{IP}(\omega_{IP}-\omega_s) - D_{12}(\omega_{IP}-\omega_{HP}) - D_{23}(\omega_{IP}-\omega_{LP}) \\
&\quad +K_{HP}(\delta_{HP}-\delta_{IP}) + K_{IP}(\delta_{LP}-\delta_{IP}))/2H_{IP} \\
\dot{\delta}_{LP} &= \omega_n(\omega_{LP}-\omega_s) \\
\dot{\omega}_{LP} &= (-D_{LP}(\omega_{LP}-\omega_s) - D_{23}(\omega_{LP}-\omega_{IP}) - D_{34}(\omega_{LP}-\omega) \\
&\quad +K_{IP}(\delta_{IP}-\delta_{LP}) + K_{LP}(\delta-\delta_{LP}))/2H_{LP} \\
\dot{\delta} &= \omega_n(\omega-1) \\
\dot{\omega} &= (-T_e - D(\omega-\omega_s) - D_{34}(\omega-\omega_{LP}) - D_{45}(\omega-\omega_{EX}) \\
&\quad K_{LP}(\delta_{LP}-\delta) + K_{EX}(\delta_{EX}-\delta))/2H \\
\dot{\delta}_{EX} &= \omega_n(\omega_{EX}-\omega_s) \\
\dot{\omega}_{EX} &= (-D_{EX}(\omega_{EX}-\omega_s) - D_{45}(\omega_{EX}-\omega) \\
&\quad +K_{EX}(\delta-\delta_{EX}))/2H_{EX}
\end{aligned}
$$

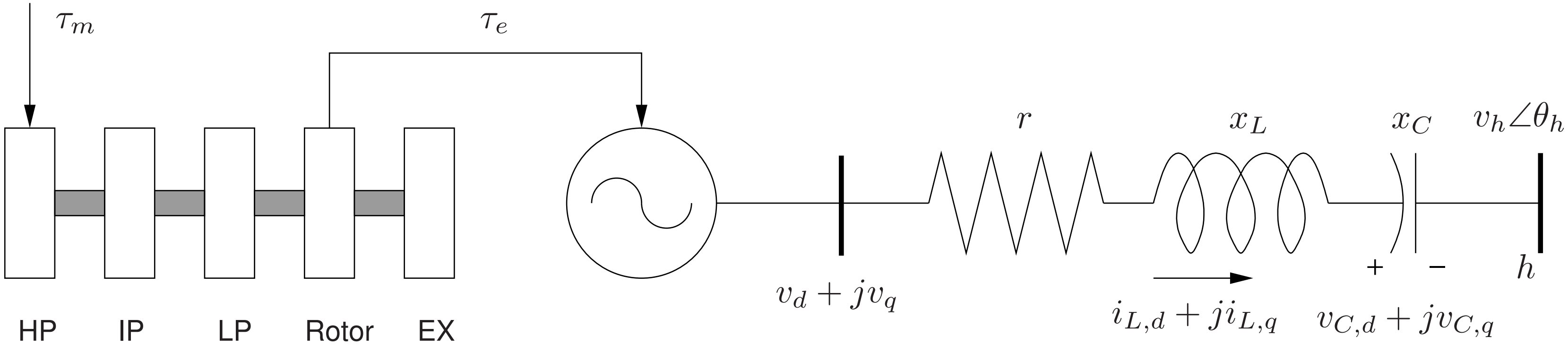
We model the shaft as a mass-spring system



Time Domain Simulation using a Dynamic Shaft



Subsynchronous Resonance Model



Line Model $\rightarrow$

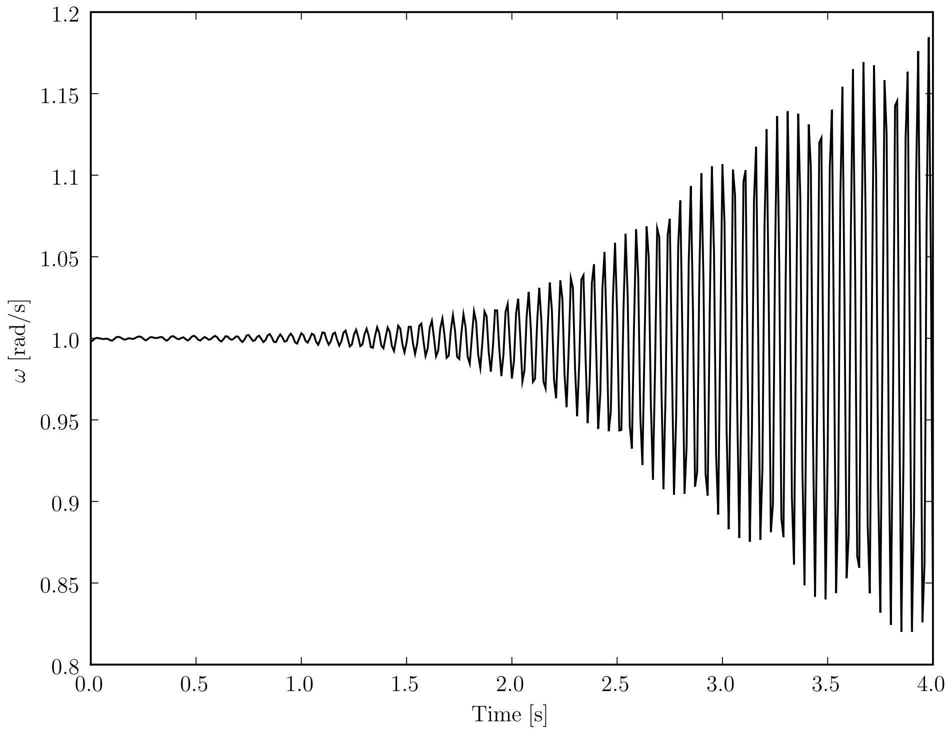
$$\begin{cases} \dot{i}_{L,d} = & \omega_n(i_{L,q} + (v_d - ri_{L,d} - v_{C,d} - v_h \sin(\delta - \theta_h))/x_L) \\ \dot{i}_{L,q} = & \omega_n(-i_{L,d} + (v_q - ri_{L,q} - v_{C,q} - v_h \cos(\delta - \theta_h))/x_L) \\ \dot{v}_{C,d} = & \omega_n(x_C i_{L,d} + v_{C,q}) \\ \dot{v}_{C,q} = & \omega_n(x_C i_{L,q} - v_{C,d}) \end{cases}$$

Generator Model $\rightarrow$

$$\begin{cases} \dot{\psi}_f = & \omega_n(v_f - x_f i_f) \\ \psi_f = & x_f i_f - (x_d - x_\ell) i_{L,d} \\ \psi_d = & (x_d - x_\ell) i_f - x_d i_{L,d} \\ \psi_q = & -x_q i_{L,q} \end{cases}$$

and flux eqs. (*)

Time domain simulation with subsynchronous resonance



The resonance model has a frequency $\approx \omega_n(1 + \sqrt{\frac{x_C}{x_L}})$

Infinite Bus Model

- Let’s consider the simplified electromechanical equations of the synchronous machine:

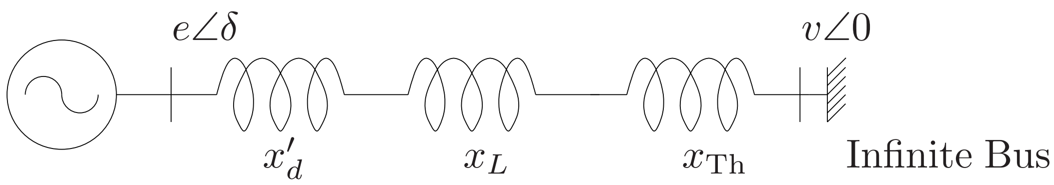
$$\dot{\delta} = \omega_n(\omega - \omega_s)$$

$$\dot{\omega} = \frac{1}{2H}(p_m - p_e(\delta))$$

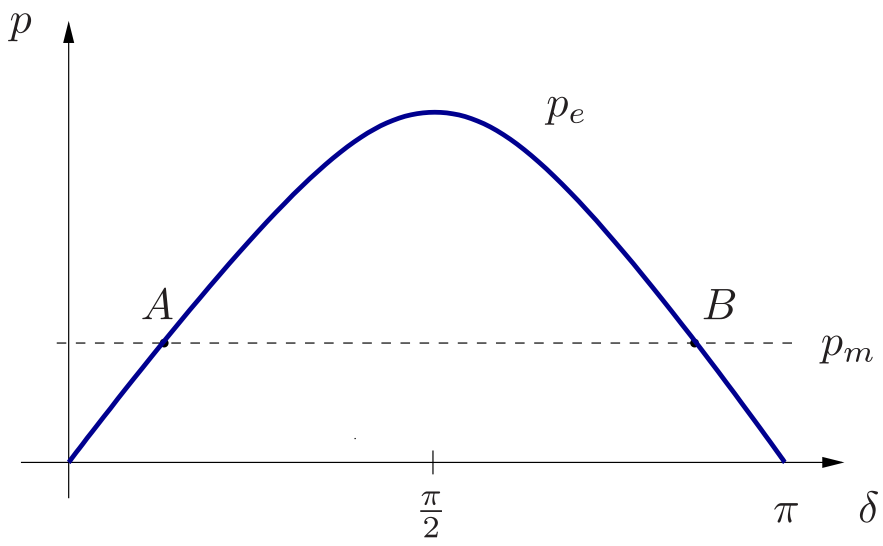
- A common approximation of network equivalents with “high” energy/power is to consider the network as a machine with $H \to \infty$ and $e'_q = \mathrm{constant}$.
- Observe that if $H \to \infty$ then $\dot{\omega} \to 0$, then $\omega = \mathrm{constant}$ (typically $\omega = \omega_s$ is assumed).
- Then if $\omega = \omega_s \Rightarrow \dot{\delta} = 0 \Rightarrow \delta = \mathrm{constant} \Rightarrow$ Phase Reference.

Electromechanical Oscillations

- Let's consider the One-Machine Infinite-Bus (OMIB) system:



- From machine equations we obtain:



$$\dot\delta = \omega_n(\omega - \omega_s)$$
$$\dot\omega = \frac{1}{2H}(p_m - p_e(\delta))$$

with: $p_e = \frac{ev}{x_{\rm eq}} \sin\delta$

$$x_{\rm eq} = x'_d + x_L + x_{\rm Th}$$

Equilibrium Points of the OMIB

- The OMIB has two equilibrium points:
- Let $\boldsymbol{x} = [\delta, \omega]^T$, then:
- Assume:

$$e = v = p_m = 1.0 \text{ pu}$$
$$x_{\text{eq}} = 0.5 \text{ pu}$$
$$\boldsymbol{x}_A = [0.5236, 1]^T$$
$$\boldsymbol{x}_B = [2.6180, 1]^T$$

Stability of the equilibrium points of the OMIB

- Point $\boldsymbol{x}_A$:

  Let's assume a small perturbation $\partial\delta > 0$

  Then $p_e(\delta_A + \partial\delta) > p_m \Rightarrow \dot{\omega} < 0 \Rightarrow \omega$ decreases

  $\Rightarrow \omega < \omega_A = 1 \Rightarrow \dot{\delta} < 0 \Rightarrow \delta$ decreases

  A similar conclusion can be drawn if $\partial\delta < 0$

  Point $\boldsymbol{x}_A$ is a “sink” $\Rightarrow$ Stable equilibrium point

- Point $\boldsymbol{x}_B$:

  Let's assume a small perturbation $\partial\delta > 0$

  Then $p_e(\delta_B + \partial\delta) < p_m \Rightarrow \dot{\omega} > 0 \Rightarrow \omega$ increases

  $\Rightarrow \omega > \omega_B = 1 \Rightarrow \dot{\delta} > 0 \Rightarrow \delta$ increases

  A similar conclusion can be drawn if $\partial\delta < 0$

  Point $\boldsymbol{x}_B$ is a “source” $\Rightarrow$ unstable equilibrium point

General Approach to define the stability of E.P.s

- Let's consider an ODE system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n$$

- Be $\boldsymbol{x}_o$ an E.P. of $\boldsymbol{f}$ such that $\mathbf{0} = \boldsymbol{f}(\boldsymbol{x}_o)$
- Then the solution $\boldsymbol{\lambda}$ of

$$\det(\boldsymbol{F}_{\boldsymbol{x}|\boldsymbol{x}_o} - \boldsymbol{\lambda}\boldsymbol{I}_n) = 0$$

  are the eigenvalues of the system (or characteristic roots)

- $\boldsymbol{F}_{\boldsymbol{x}|\boldsymbol{x}_o} \equiv \boldsymbol{A}_s$ is the system **STATE MATRIX**.
- $\boldsymbol{I}_n$ the identity matrix of order n.

First Lyapunov's Stability Criteria

- If all $\Re\{\lambda_j\} < 0$ for $j = 1, \ldots, n$ then the E.P. is stable.
- If exists at least one $\Re\{\lambda_j\} > 0$ for $j = 1, \ldots, n$ then the E.P. is unstable.
- If exists at least one $\Re\{\lambda_j\} = 0$ for $j = 1, \ldots, n$ then the stability of the E.P. cannot be defined $\Rightarrow$ the E.P. is a bifurcation point.

OMIB Example

- Let's compute the eigenvalues of the OMIB system with $e = v = p_m = 1$ pu and $x_{\rm eq} = 0.5$ pu, $H = 8$ MWs/MVA and $\omega_n = 314.16$ rad/s
- We have:

$$\det(\boldsymbol{A}_s - \boldsymbol{\lambda} \boldsymbol{I}_2) = 0$$

where

$$\boldsymbol{A}_s = \begin{bmatrix} 0 & \omega_n \\ -\frac{1}{2H}\frac{ev}{x_{\rm eq}}\cos\delta & 0 \end{bmatrix}$$

- For $\boldsymbol{x}_A \Rightarrow \lambda_{1,2} = \pm\sqrt{-\omega_n \frac{1}{2H}\frac{ev}{x_{\rm eq}}\cos\delta_A}$ = $\pm j5.8317 \rightarrow$ stable?
- For $\boldsymbol{x}_B \Rightarrow \lambda_{1,2} = \pm 5.8317 \Rightarrow$ U.E.P.

Effect of Damping

- Point $\boldsymbol{x}_A$ leads to $\Re\{\lambda_{1,2}\}$=0, hence we do not know if the point is stable or not.
- However, let consider the following modification:

$$\dot{\omega} = \frac{1}{2H}(p_m - p_e(\delta) - D(\omega - \omega_s))$$

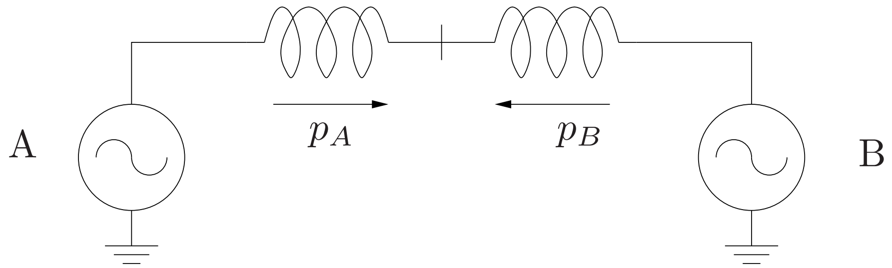
- Hence:

$$\boldsymbol{A}_s = \begin{bmatrix} 0 & \omega_n \\ -\frac{1}{2H}\frac{ev}{x_{eq}}\cos\delta & -\frac{D}{2H} \end{bmatrix}$$

For $D > 0 \Rightarrow \lambda_{1,2} = -\alpha \pm j\beta$ with $\alpha > 0 \Rightarrow$ hence $\boldsymbol{x}_A$ is a *weakly* stable equilibrium point.

Synchronisation of Synchronous Machines

- Let's assume the two-machine system:



- $p_{mA} - p_A = M_A \frac{d\omega_A}{dt}$ $\quad\quad$ $p_{mB} - p_B = M_B \frac{d\omega_B}{dt}$

  then

$$p_A = A \sin \delta_{AB} + B \cos \delta_{AB} + C$$

$$p_B = -A \sin \delta_{AB} + B \cos \delta_{AB} + D$$

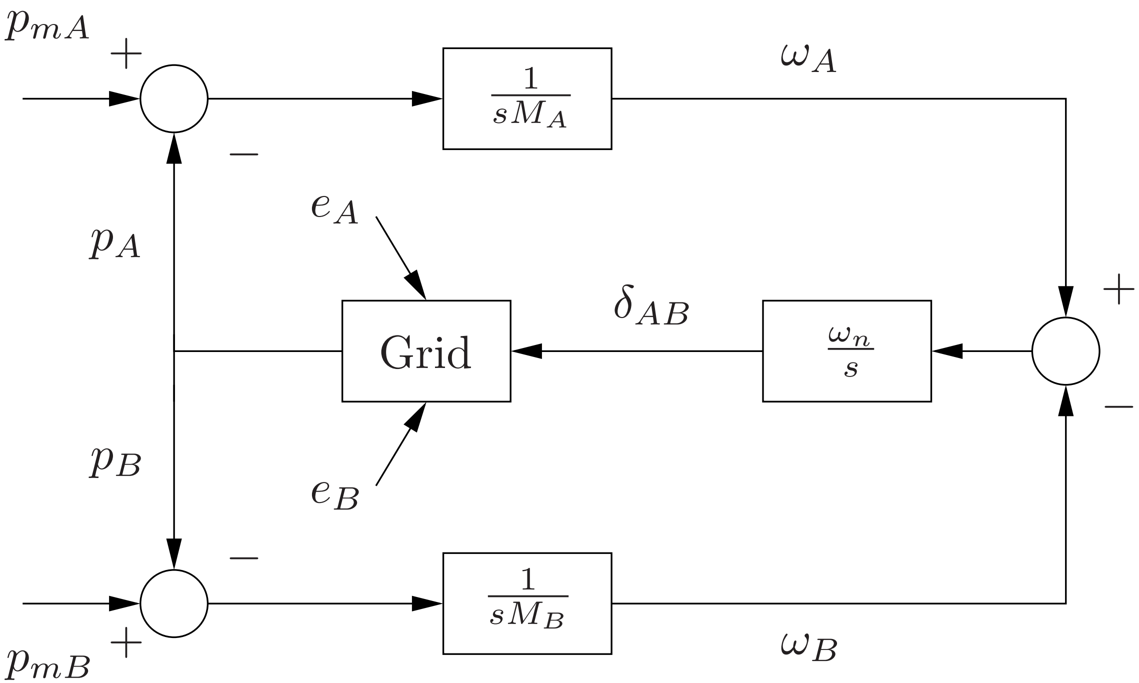
  where A, B, C and D depend on machine and system parameters:

$$e_A, e_B, x'_{dA}, x'_{dB}, x_{\text{Line}}, \text{etc.}$$

- Finally: $\frac{d\delta_A}{dt} = \omega_n(\omega_A - \omega_s)$, $\frac{d\delta_B}{dt} = \omega_n(\omega_B - \omega_s)$

Synchronisation of Synchronous Machines (II)

- If the two machines are synchronous $\Rightarrow \omega_A = \omega_B$ and $\delta_{AB} = \delta_A - \delta_B$ is constant $\Rightarrow \delta_{AB} =$ const
- Scheme of the whole system:



- In steady-state the input signals to the integrators must be 0.

Relative motion between machine rotors

- Let's consider again the two machine system and define:

$$\delta_{AB} = \delta_A - \delta_B$$
$$\omega_{AB} = \omega_A - \omega_B$$

- Then we can combine the differential equations of the two machines:

$$\frac{d\omega_{AB}}{dt} = \frac{p_{mA} - p_A}{M_A} - \frac{p_{mB} - p_B}{M_B}$$
$$\frac{d\delta_{AB}}{dt} = \omega_n(\omega_A - \omega_B)$$

Relative Motion between Machine Rotors (II)

⇒ Expanding the $\frac{d\omega_{AB}}{dt}$ equation:

$$\frac{d\omega_{AB}}{dt} = \frac{M_B p_{mA} - M_B p_A - M_A p_{mB} + M_A p_B}{M_A M_B}$$

$$= \frac{M_B p_{mA} - M_A p_{mB}}{M_A M_B} - \frac{M_B p_A - M_A p_B}{M_A M_B}$$

$$\Rightarrow \frac{d\omega_{AB}}{dt} = \frac{M_B p_{mA} - M_A p_{mB} + ((M_A + M_B)p_{mA} - (M_A + M_B)p_{mA})}{M_A M_B}$$

$$+ \frac{-M_B p_A - M_A p_B + ((M_A + M_B)p_A - (M_A + M_B)p_B)}{M_A M_B}$$

Relative Motion between Machine Rotors (III)

- Let's define:

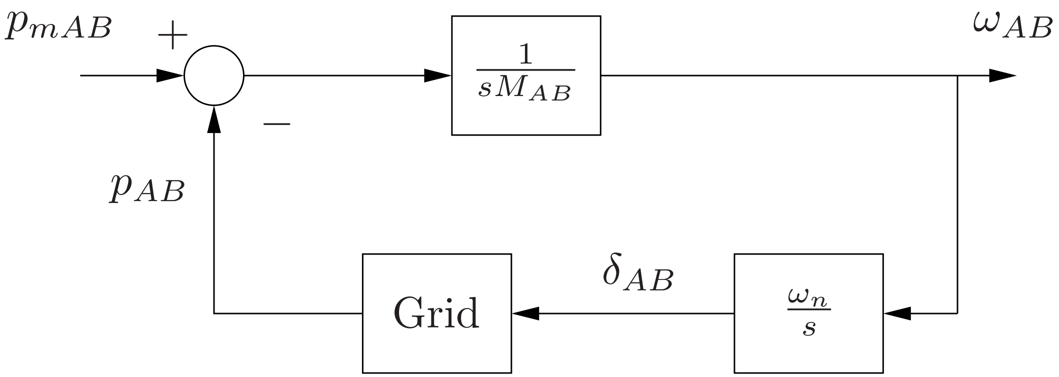
$$
\begin{aligned}
M_{AB} &= \frac{M_A M_B}{M_A + M_B} \\
p_{mAB} &= \frac{p_{mA} - M_A(p_{mA} + p_{mB})/(M_A + M_B)}{M_{AB}} \\
p_{AB} &= \frac{p_A - M_A(p_A + p_B)/(M_A + M_B)}{M_{AB}}
\end{aligned}
$$

- Hence we obtain:

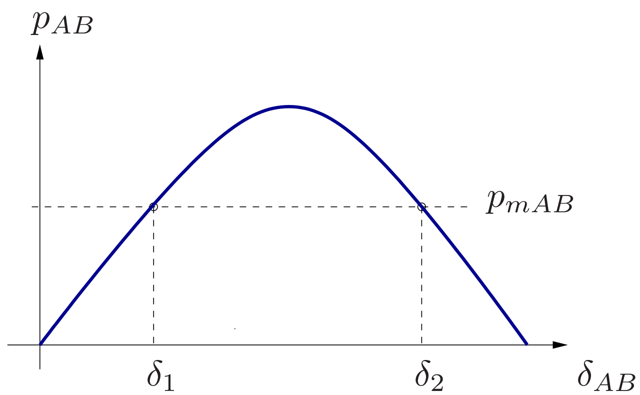
$$
\begin{aligned}
\frac{d\omega_{AB}}{dt} &= \frac{p_{mAB} - p_{AB}}{M_{AB}} \\
\frac{d\delta_{AB}}{dt} &= \omega_n \omega_{AB}
\end{aligned}
$$

Relative Motion between Machine Rotors (IV)

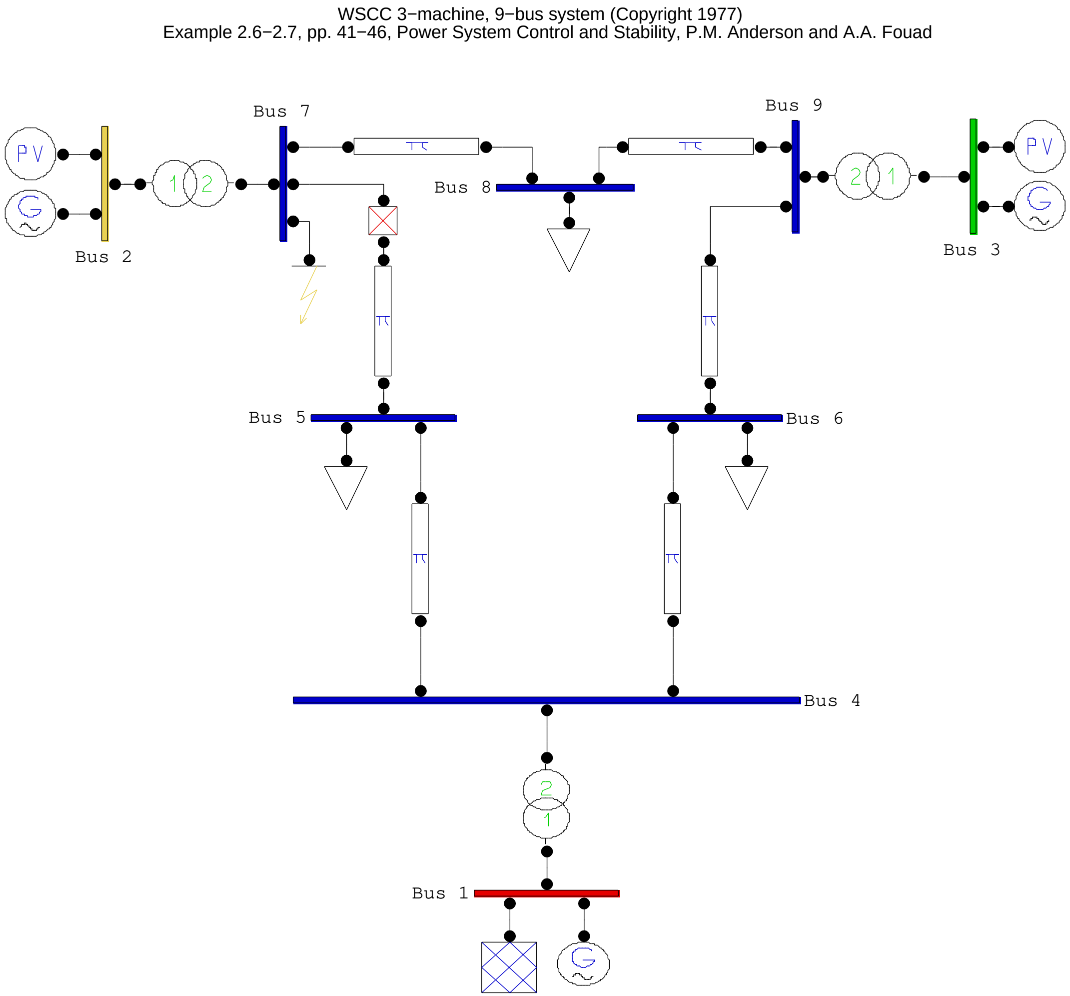
- The resulting scheme is as follows:



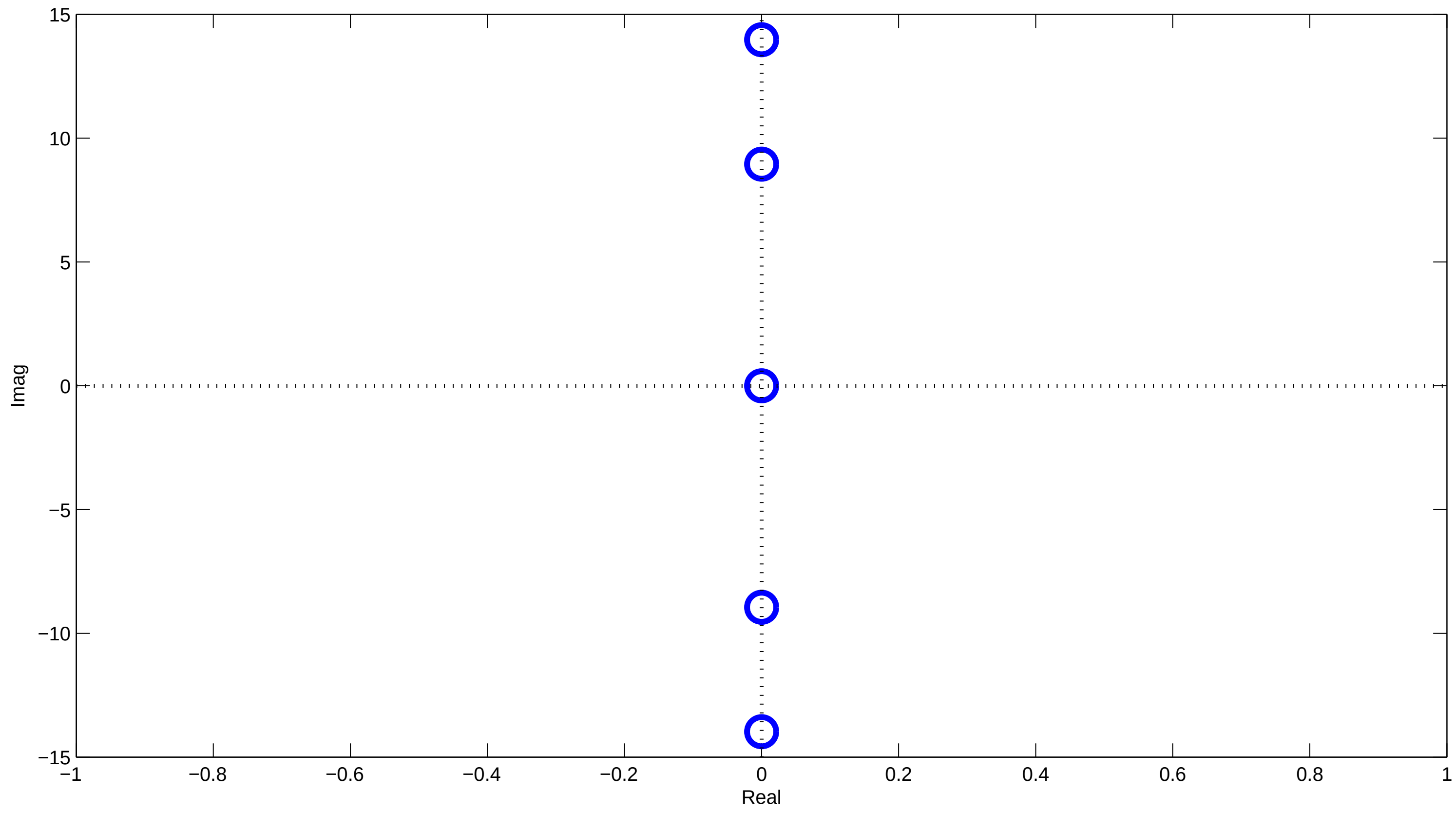
- We have obtained a system similar to the OMIB
  If $p_{mAB} = \text{constant}$, e_A and e_B are constant, then the stability can be defined based on δ_{AB}.
  (The OMIB can be obtained by imposing $M_B \rightarrow \infty$)



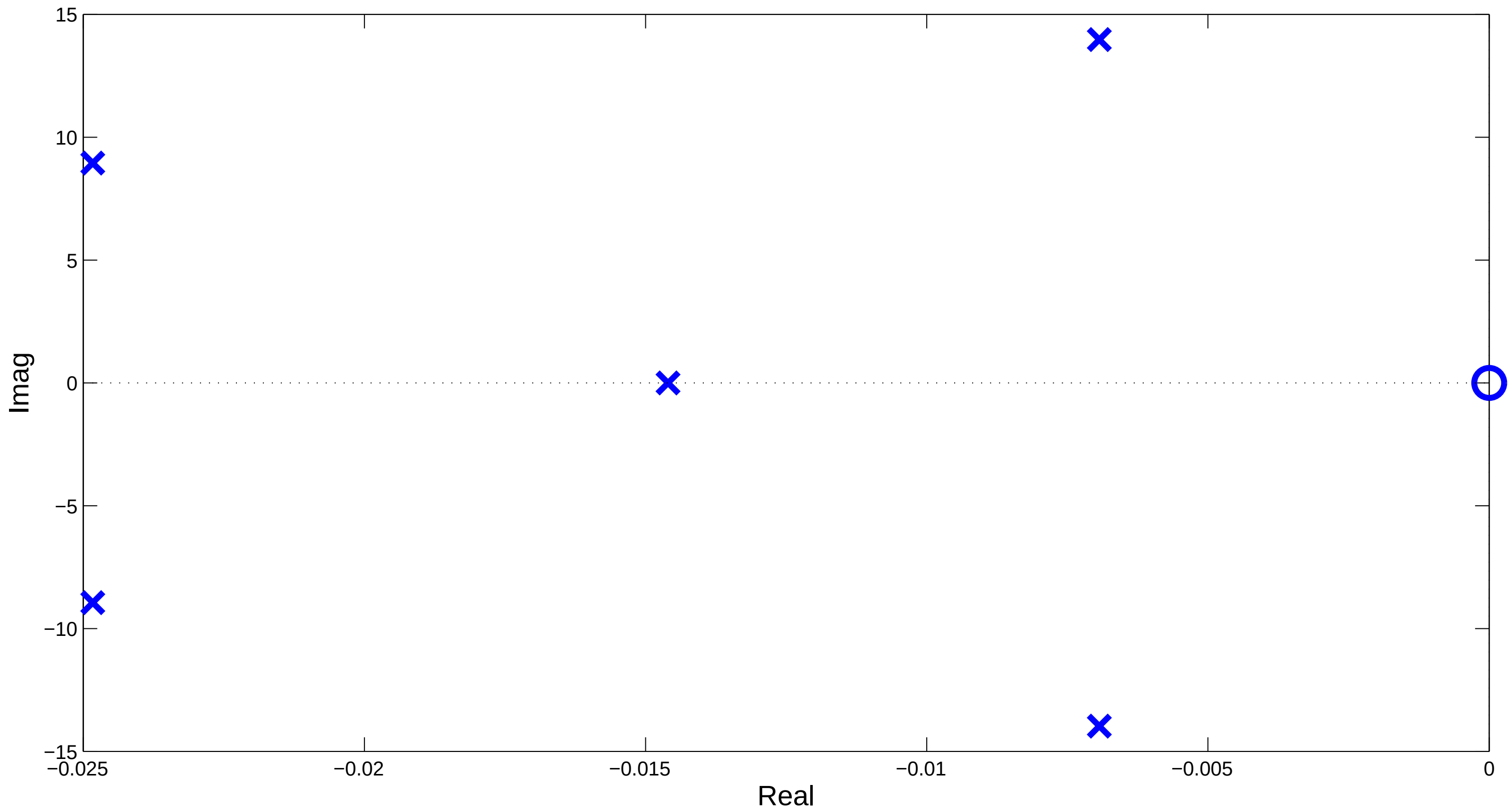
WSCC 9-bus 3-machine system



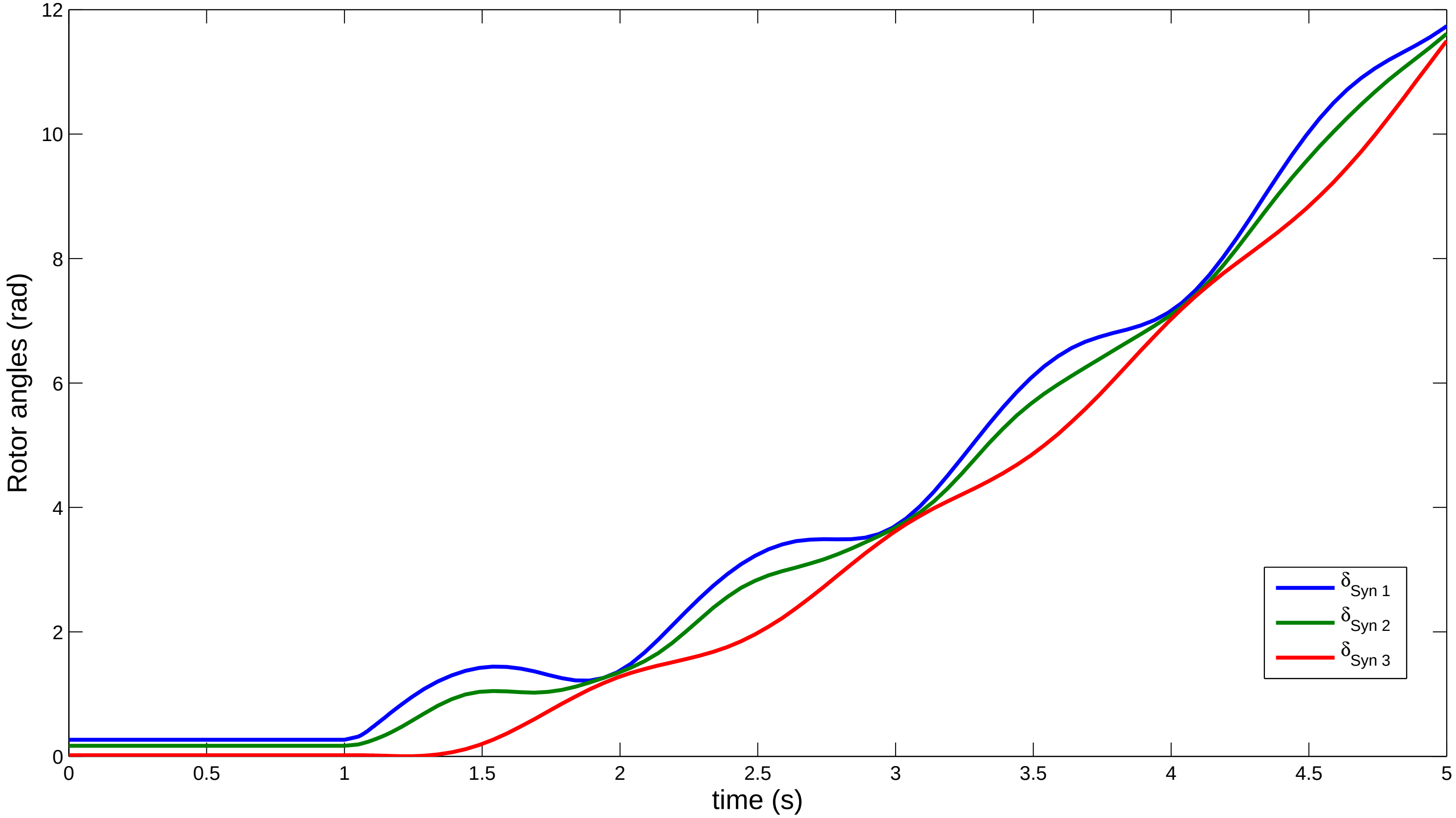
Eigenvalues at the E.P. (no damping)



Eigenvalues at the E.P. (with damping)



Transient following a fault



A fault occurs at bus 7 at $t = 1$ and is cleared at $t = 1.083$ s

Center of Inertia (COI)

- It is useful to refer machine angles and speeds to the *center of inertia* (COI), which is a weighted sum of all machine angles and speeds:

$$\delta_{\rm COI} = \frac{\sum_{j\in\mathcal{G}} H_j \delta_j}{\sum_{j\in\mathcal{G}} H_j}$$

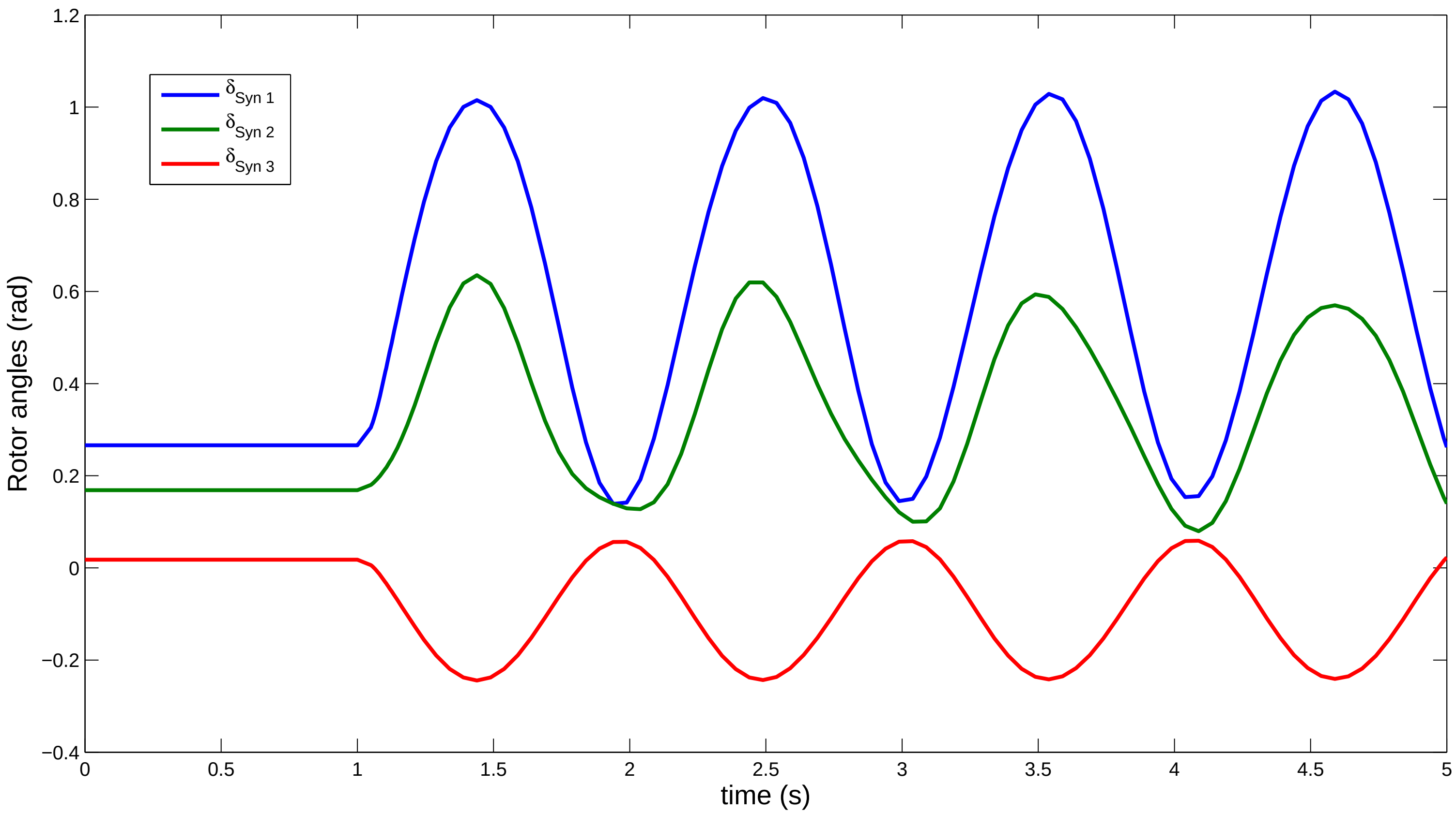
$$\omega_{\rm COI} = \frac{\sum_{j\in\mathcal{G}} H_j \omega_j}{\sum_{j\in\mathcal{G}} H_j}$$

  where $\mathcal{G}$ is the set of generators.

- Machine rotor angle equations are modified as follows:

$$\dot{\delta} = \omega_n(\omega - \omega_{\rm COI})$$

Transient following a fault (with COI)



Small Signal Stability of a General System

- In general we have several machines (with their controllers) connected to the grid (power flow equations).
- The resulting model is:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \end{aligned}$$

  where
  - $\boldsymbol{x} \in \mathbb{R}^n$,
  - $\boldsymbol{y} \in \mathbb{R}^m$,
  - $\boldsymbol{f} : \mathbb{R}^{(n+m)} \rightarrow \mathbb{R}^n$, and
  - $\boldsymbol{g} : \mathbb{R}^{(n+m)} \rightarrow \mathbb{R}^m$.
- The equilibrium point is $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ s.t. $\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{y}_0), \quad \boldsymbol{0} = \boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{y}_0)$.

State Matrix of a General System

- Let's determine the state matrix $\boldsymbol{A}_s$:

$$\begin{bmatrix} \Delta\dot{\boldsymbol{x}} \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{\boldsymbol{x}} & \boldsymbol{F}_{\boldsymbol{y}} \\ \boldsymbol{G}_{\boldsymbol{x}} & \boldsymbol{G}_{\boldsymbol{y}} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{y} \end{bmatrix} = \boldsymbol{A}_c \begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{y} \end{bmatrix}$$

$$\boldsymbol{A}_s = \boldsymbol{F}_{\boldsymbol{x}} - \boldsymbol{F}_{\boldsymbol{y}}\boldsymbol{G}_{\boldsymbol{y}}^{-1}\boldsymbol{G}_{\boldsymbol{x}}, \qquad \text{with } \boldsymbol{G}_{\boldsymbol{y}} \text{ non-singular!}$$

- Let

$$\boldsymbol{D} = \boldsymbol{F}_{\boldsymbol{y}}\boldsymbol{G}_{\boldsymbol{y}}^{-1}\boldsymbol{G}_{\boldsymbol{x}}$$

  which is often called degradation matrix

- $\boldsymbol{D}$ provides a "measure" of the effect of the grid on the stability of the dynamic system, i.e., $\boldsymbol{F}_{\boldsymbol{x}}$ (which is generally stable!)

  $\rightarrow$ **ONLY** the eigenvalues of $\boldsymbol{A}_s$ provide information on the stability of the system.