Basic Stability Concepts of Nonlinear Systems

POWER SYSTEM STABILITY ANALYSIS (EEEN40340)

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Basic Stability Concepts Outlines

- Nonlinear systems:
  - Ordinary differential equations (ODE)
  - Differential algebraic equations (DAE)
- Equilibrium points:
  - Definitions
  - Stability
- Stability regions
- Relevant bifurcation points

Some Questions . . .

- How many solutions does a *linear* systems have?
- How many solutions does a *nonlinear* systems have?
- Do low-order dynamic systems have *simpler* behaviours than high order-ones?
- Is the dynamic response of a linear system perfectly known?
- Is the stability of an equilibrium point fully defined by the state matrix of the system?
- Are continuous deterministic dynamic systems perfectly predictable (e.g., observable and controllable)?

Ordinary Differential Equations (ODEs)

ODE Systems

- Nonlinear systems are represented by a set of nonlinear differential equations:

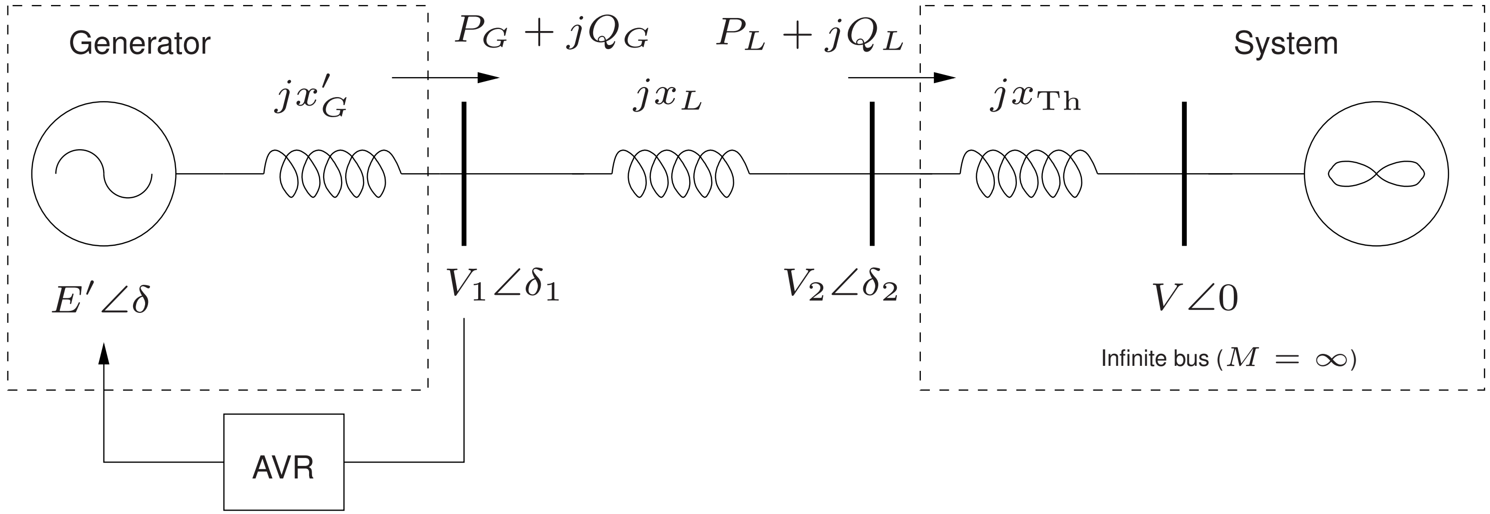
$$\dot{\boldsymbol{x}} = \boldsymbol{s}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{\mu})$$

where

  - $\boldsymbol{x} \rightarrow n$ state variables (e.g. generator angles)
  - $\boldsymbol{p} \rightarrow k$ controllable parameters (e.g. compensation)
  - $\boldsymbol{\mu} \rightarrow \ell$ uncontrollable parameters (e.g. loads)
  - $\boldsymbol{s}(\cdot) \rightarrow n$ nonlinear functions (e.g. generator equations)

ODE Systems

- For example, for a simple generator-system model:



- The generator is modeled as a simple d axis transient voltage behind transient reactance.

ODE Systems

- The generator has only the mechanical dynamics:

$$
\begin{aligned}
\dot{\delta} &= \omega = \omega_r - \omega_0 \\
\dot{\omega} &= \frac{1}{M}(P_L - P_G - D\omega)
\end{aligned}
$$

where

$$
\begin{aligned}
P_G &= \frac{E'V}{x'_G + x_L + x_{\mathrm{Th}}} \sin\delta \\
&= \frac{V_1 V}{x_L + x_{\mathrm{Th}}} \sin\delta_1
\end{aligned}
$$

ODE Systems

- If the AVR is modeled, V_1 may be assumed to be kept constant by varying E', with the generator's reactive power within limits:

$$Q_G = \frac{V_1^2}{x_L + x_{\mathrm{Th}}} - \frac{V_1 V}{x_L + x_{\mathrm{Th}}} \cos \delta_1$$

$$\Rightarrow \quad Q_{G\,\min} \le Q_G \le Q_{G\,\max}$$

ODE Systems

- If the AVR is not modeled:
  - $\boldsymbol{x} = [\delta, \omega]^T \rightarrow$ state variables
  - $\boldsymbol{p} = [E', V]^T \rightarrow$ controlled parameters
  - $\mu = P_L \rightarrow$ uncontrolled parameters
  - Hence, assuming $\boldsymbol{p} = [1.5, 1]^T$, $M = D = 0.1$, and $x'_G + x_L + x_{\mathrm{Th}} = 0.75$:

$$\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{p}, \mu) = \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 10\mu - 20 \sin x_1 - x_2 \end{cases}$$

- These are basically the pendulum equations.

Differential Algebraic Equations (DAEs)

DAE Systems

- Differential Algebraic Equation (DAE) models are defined as:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{\mu}) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{\mu})\end{aligned}$$

  where:

  - $\boldsymbol{y} \to m$ algebraic variables (e.g. load voltages)
  - $\boldsymbol{f}(\cdot) \to n$ nonlinear differential equations (e.g. generator equations)
  - $\boldsymbol{g}(\cdot) \to m$ nonlinear algebraic equations (e.g. reactive power equations)

DAE Systems

- For example, for the generator-infinite bus example with AVR, for $Q_{G\,\min} \leq Q_G \leq Q_{G\,\max}$:

$$
\begin{aligned}
\dot{\delta} &= \omega \\
\dot{\omega} &= \frac{1}{M}\left(P_L - \frac{E'V}{x'_G + x_L + x_{\mathrm{Th}}}\sin\delta - D\omega\right) \\
0 &= \frac{E'V}{x}\sin\delta - \frac{V_1 V}{x_L + x_{\mathrm{Th}}}\sin\delta_1 \\
0 &= -Q_G - \frac{V_1^2}{x'_G} + \frac{V_1 E'}{x'_G}\cos(\delta_1 - \delta) \\
0 &= Q_G - \frac{V_1^2}{x_L + x_{\mathrm{Th}}} + \frac{V_1 V}{x_L + x_{\mathrm{Th}}}\cos\delta_1
\end{aligned}
$$

DAE Systems

- Thus, for $Q_{G\,\min} \leq Q_G \leq Q_{G\,\max}$:
  - $\boldsymbol{x} = [\delta, \omega]^T$
  - $\boldsymbol{y} = [E', \delta_1, Q_G]^T$
  - $\boldsymbol{p} = [V_1, V]^T = [1, 1]^T$
  - $\mu = P_L$
  - $M = D = 0.1, x'_G = 0.25, x_L + x_{\mathrm{Th}} = 0.5$

$$
\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \mu) = \begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = 10\mu - 13.33 y_1 \sin x_1 - x_2 \end{cases}
$$

$$
\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \mu) = \begin{cases} 0 & = 1.333 y_1 \sin x_1 - 2 \sin y_2 \\ 0 & = -y_3 - 4 + 4 y_1 \cos(y_2 - x_1) \\ 0 & = y_3 - 2 + 2 \cos y_2 \end{cases}
$$

DAE Systems

- If $Q_G = Q_{G\,\max}$ or $Q_G = Q_{G\,\min}$:
  - $\boldsymbol{x} = [\delta, \omega]^T$
  - $\boldsymbol{y} = [E', \delta_1, V_1]^T$
  - $\boldsymbol{p} = [Q_G, V]^T = [\pm 0.5, 1]^T$
  - $\mu = P_L$
  - $M = D = 0.1, x'_G = 0.25, x_L + x_{\mathrm{Th}} = 0.5$

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \mu) = \begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = 10\mu - 13.33 y_1 y_3 \sin x_1 - x_2 \end{cases}$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \mu) = \begin{cases} 0 & = 1.333 y_1 \sin x_1 - 2 y_3 \sin y_2 \\ 0 & = \mp 0.5 - 4 y_3^2 + 4 y_1 y_3 \cos(y_2 - x_1) \\ 0 & = \pm 0.5 - 2 y_3^2 + 2 y_3 \cos y_2 \end{cases}$$

DAE Systems

- If the Jacobian matrix $D_y \boldsymbol{g} = [\partial g_i / \partial y_j]_{m \times m}$ is nonsingular, i.e. invertible, along the trajectory solutions, the system can be transformed into an ODE system (Implicit Function Theorem):

$$
\begin{aligned}
& \boldsymbol{y} &=& \quad \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{\mu}) \\
\Rightarrow \quad & \dot{\boldsymbol{x}} &=& \quad \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{\mu}), \boldsymbol{p}, \boldsymbol{\mu}) \\
& &=& \quad \boldsymbol{s}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{\mu})
\end{aligned}
$$

- In practice, this is a purely “theoretical” exercise that is not carry out due to its complexity.
- The system model should be revised when $D_y \boldsymbol{g}$ is singular.

Equilibria

Equilibria of ODEs

- For the ODE system, equilibria are defined as the solution $\boldsymbol{x}_0$ for given parameter values $\boldsymbol{p}_0$ and $\boldsymbol{\mu}_0$ of the set of equations

$$\boldsymbol{s}(\boldsymbol{x}_0, \boldsymbol{p}_0, \boldsymbol{\mu}_0) = \boldsymbol{0}$$

- There can be multiple solutions to this problem, i.e., multiple equilibrium points.
- Due to the nonlinearity of $\boldsymbol{s}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{\mu})$, to find one equilibrium can be a non-trivial task (see, for example, the power flow problem).

Stability of Equilibria

- The "stability" of these equilibria is defined by linearizing the nonlinear system around $\boldsymbol{x}_0$, i.e.

$$\begin{aligned}\Delta\dot{\boldsymbol{x}} &= \left[\frac{\partial s_i}{\partial x_j}(\boldsymbol{x}_0, \boldsymbol{p}_0, \boldsymbol{\mu}_0)\right]_{n\times n} \underbrace{\boldsymbol{x} - \boldsymbol{x}_0}_{\Delta\boldsymbol{x}} \\ &= D_x\boldsymbol{s}|_0 \Delta\boldsymbol{x}\end{aligned}$$

- where $D_x\boldsymbol{s}|_0 = D_x\boldsymbol{s}(\boldsymbol{x}_0, \boldsymbol{p}_0, \boldsymbol{\mu}_0) = \partial\boldsymbol{s}/\partial\boldsymbol{x}|_0$ is the system Jacobian matrix evaluated at the equilibrium point.

- The following notations are used to indicate the state matrix:

$$D_x\boldsymbol{s}|_0 = \boldsymbol{A}_S = \nabla^T\boldsymbol{s}|_0 = \boldsymbol{s}_{\boldsymbol{x}}(\boldsymbol{x}_0, \boldsymbol{p}_0, \boldsymbol{\mu}_0)$$

- In the remainder of this notes, $\boldsymbol{s}_{\boldsymbol{x}}$ will be used, omitting the formal dependence on $(\boldsymbol{x}_0, \boldsymbol{p}_0, \boldsymbol{\mu}_0)$ if there is no ambiguity.

Definitions (for Continuous-time Systems)

- Let consider an autonomous nonlinear dynamical system:

$$\dot{\boldsymbol{x}} = \boldsymbol{s}(\boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

where $\boldsymbol{x}(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ denotes the system state vector, $\mathcal{X}$ an open set containing the origin, and $\boldsymbol{s} : \mathcal{X} \to \mathbb{R}^n$ continuous on $\mathcal{X}$. Suppose $\boldsymbol{s}$ has an equilibrium $\boldsymbol{x}_e$.

  - The equilibrium of the above system is said to be Lyapunov stable, if, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, if $\|\boldsymbol{x}(0) - \boldsymbol{x}_e\| < \delta$, then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| < \epsilon$, for every $t \geq 0$.
  - The equilibrium of the above system is said to be asymptotically stable if it is Lyapunov stable and if there exists $\delta > 0$ such that if $\|\boldsymbol{x}(0) - \boldsymbol{x}_e\| < \delta$, then $\lim_{t\to\infty} \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| = 0$.
  - The equilibrium of the above system is said to be exponentially stable if it is asymptotically stable and if there exist $\alpha, \beta, \delta > 0$ such that if $\|\boldsymbol{x}(0) - \boldsymbol{x}_e\| < \delta$, then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \leq \alpha\|\boldsymbol{x}(0) - \boldsymbol{x}_e\|e^{-\beta t}$, for $t \geq 0$.

Definitions (conceptual meaning)

- Conceptually, the meanings of the above terms are the following:

  1. Lyapunov stability of an equilibrium means that solutions starting “close enough” to the equilibrium (within a distance δ from it) remain “close enough” forever (within a distance ϵ from it). Note that this must be true for any ϵ that one may want to choose.
  2. Asymptotic stability means that solutions that start close enough not only remain close enough but also eventually converge to the equilibrium.
  3. Exponential stability means that solutions not only converge, but in fact converge faster than or at least as fast as a particular known rate $\alpha\|\boldsymbol{x}(0) - \boldsymbol{x}_e\|e^{-\beta t}$.

Attractive Trajectories

- The trajectory $\tilde{\boldsymbol{x}}(t)$ is (locally) attractive if

$$\|\boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)\| \to 0$$

  for $t \to \infty$ for all trajectories that start close enough to $\boldsymbol{x}(t)$, and globally attractive if this property holds for all trajectories.

- Hence, if $\tilde{\boldsymbol{x}}(t)$ belongs to the interior of its stable manifold, $\tilde{\boldsymbol{x}}(t)$ is asymptotically stable if it is both attractive and stable.

- Note: there are counterexamples showing that attractivity does not imply asymptotic stability.

Lyapunov's First Method (I)

- From linear system theory, the linear system stability is defined by the eigenvalues ρ_i of the state matrix $\boldsymbol{s_x}$, which are defined as the solutions of the equation:

$$\begin{aligned}
\boldsymbol{s_x}\boldsymbol{v} &= \rho\boldsymbol{v} \quad \rightarrow \quad \text{right e-vector} \\
\boldsymbol{s_x}^T\boldsymbol{w} &= \rho\boldsymbol{w} \quad \rightarrow \quad \text{left e-vector}
\end{aligned}$$

$$\Rightarrow \qquad \det(\boldsymbol{s_x} - \rho\boldsymbol{I}_n) = 0$$

$$\Rightarrow \qquad a_n\rho^n + a_{n-1}\rho^{n-1} + \ldots + a_1\rho + a_0 = 0$$

Eigenvalue Analysis

- There are n complex eigenvalues, left and right eigenvectors associated with the system Jacobian $\boldsymbol{s_x}$.
- In practice, these eigenvalues are not computed using characteristic polynomial but other more efficient numerical techniques, as the costs associated with computing these values is rather large in realistic power systems.

Lyapunov's First Method (II)

- These eigenvalues define the small perturbation stability of the ODE system, i.e. the "local" stability of the nonlinear system near the equilibrium points:
  - *Stable equilibrium point* (s.e.p.): The system is locally stable about $\boldsymbol{x}_0$ if all the eigenvalues $\rho_i(\boldsymbol{s}_{\boldsymbol{x}})$ are on the left-half (LH) of the complex plane.
  - *Unstable equilibrium point* (u.e.p.): The system is locally unstable about $\boldsymbol{x}_0$ if at least one eigenvalue $\rho_i(\boldsymbol{s}_{\boldsymbol{x}})$ is on the right-half (RH) of the complex plane.

Lyapunov's First Method (III)

- If at least one eigenvalue $\rho_i(\boldsymbol{s_x})$ is on the imaginary axis of the complex plane, then the state matrix is not informative on the stability of the equilibrium point $\boldsymbol{x}_0$.
- *Bifurcation points* are characterized by a state matrix that shows at least one eigenvalue $\rho_i(\boldsymbol{s_x})$ is on the imaginary axis of the complex plane at the equilibrium.
- However, note that not all equilibria with eigenvalues on the imaginary axis are necessarily bifurcation points.

Equilibria - Example I

- Let consider the following scalar differential equation:

$$\dot{x} = s(x) = \mu - x^2$$

- For $\mu > 0$, there are two equilibrium points:

$$x_1 = \sqrt{\mu}, \quad \text{and} \quad x_2 = -\sqrt{\mu}$$

- According to the state matrix $s_x(x, \mu) = -2x$, x_1 is stable and x_2 is unstable.
- The point $(x_0, \mu_0) = (0, 0)$, is a bifurcation point. In fact $s_x(0, 0) = 0$.
- Finally, there is no equilibrium for $\mu < 0$.

Equilibria - Example II

- Let consider again the simple one machine-infinite bus (OMIB) example with no AVR:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= 10\mu - 20\sin x_1 - x_2\end{aligned}$$

- The equilibrium points can be found from the steady-state equations:

$$\begin{aligned}0 &= x_{20} \\ 0 &= 10\mu - 20\sin x_{10} - x_{20}\end{aligned}$$

Equilibria - Example II

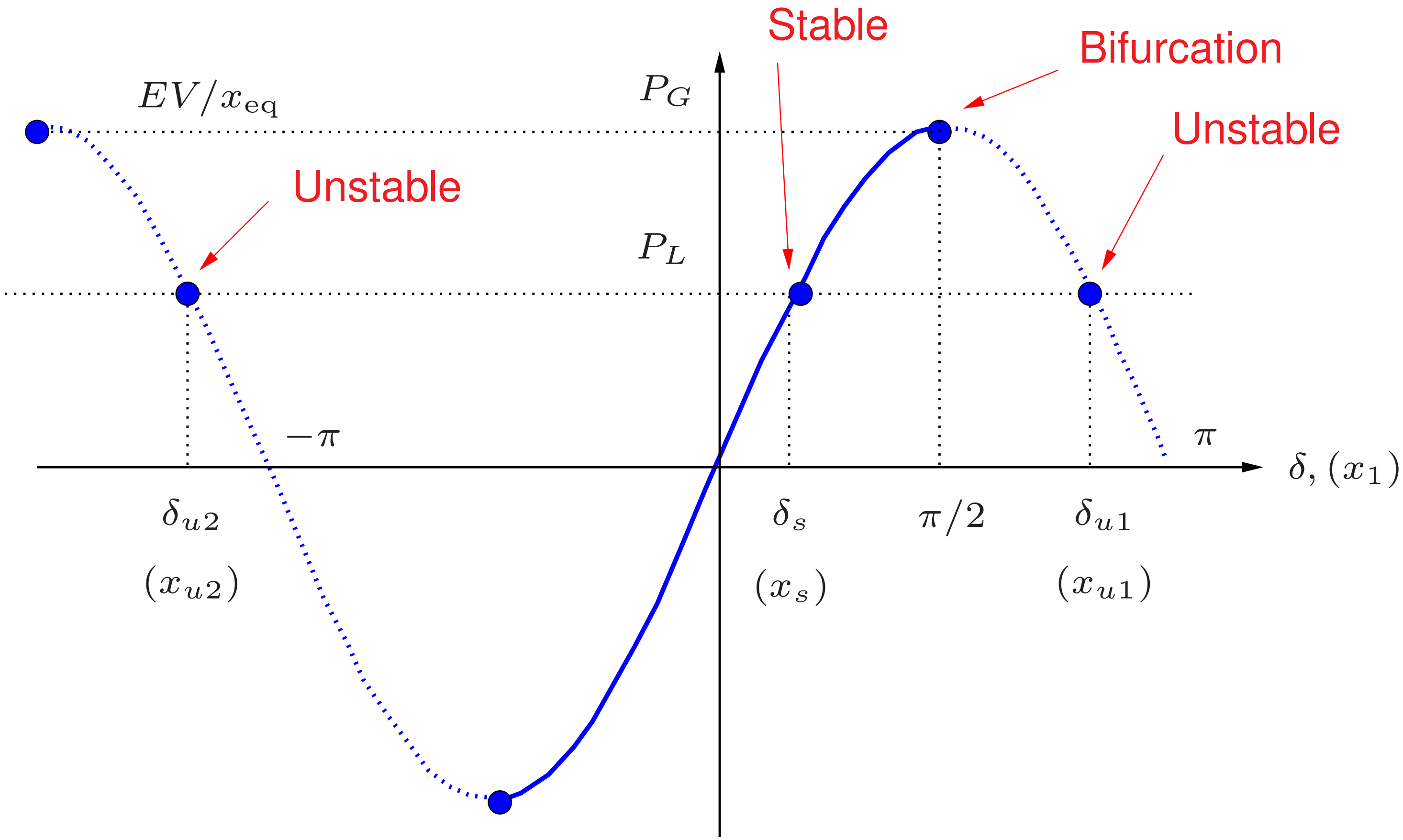
- which leads to the solutions:

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} \sin^{-1}(\mu/2) \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \delta_0 \\ \omega_0 \end{bmatrix} = \begin{bmatrix} \sin^{-1}(P_L/2) \\ 0 \end{bmatrix}$$

- This yields basically three equilibrium points (other solutions are just “repetitions” of these three):
  - s.e.p $\rightarrow -\pi/2 < x_{1s} < \pi/2$
  - u.e.p.$_1 \rightarrow x_{1u1} = x_{1s} + \pi$
  - u.e.p.$_2 \rightarrow x_{1u2} = x_{1s} - \pi$

Equilibria - Example II

- This corresponds to the intersections of $P_G = E'V/x_{\text{eq}} \sin\delta$ with P_L:



Equilibria - Example II

- The stability of these equilibria is determined using the system Jacobian:

$$\begin{aligned}
\boldsymbol{s_x} &= \begin{bmatrix} \partial s_1/\partial x_1|_0 & \partial s_1/\partial x_2|_0 \\ \partial s_2/\partial x_1|_0 & \partial s_2/\partial x_2|_0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -20\cos x_{10} & -1 \end{bmatrix}
\end{aligned}$$

Equilibria - Example II

$$\Rightarrow \quad \det(\boldsymbol{s}_{\boldsymbol{x}} - \rho \boldsymbol{I}_2) = \det \begin{bmatrix} -\rho & 1 \\ -20\cos x_{10} & -1-\rho \end{bmatrix}$$

$$= \rho^2 + \rho + 20\cos x_{10}$$

$$= 0$$

$$\Rightarrow \quad \rho_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 80\cos x_{10}}$$

$$P_L = 1 \quad \Rightarrow \quad \rho_{1,2}(\boldsymbol{s}_{\boldsymbol{x}}(x_s)) = -0.5 \pm j4.132$$

$$\rho_{1,2}(\boldsymbol{s}_{\boldsymbol{x}}(x_{u_1,u_2})) = \begin{cases} 3.192 \\ -5.192 \end{cases}$$

Equilibria - Example II

- For this system, the equilibria are:
  - stable if
$$\frac{\partial P_G}{\partial \delta} > 0$$
  - unstable if
$$\frac{\partial P_G}{\partial \delta} < 0$$
  - bifurcation point for
$$\frac{\partial P_G}{\partial \delta} = 0 \quad \Rightarrow \quad \delta = \pi/2$$

Equilibria of DAEs

- For DAE systems, the equilibria $\boldsymbol{z}_0 = (\boldsymbol{x}_0, \boldsymbol{y}_0)$ for parameter values $\boldsymbol{p}_0$ and $\boldsymbol{\mu}_0$ are defined as the solution to the nonlinear, steady state problem:

$$\boldsymbol{F}(\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{p}_0, \boldsymbol{\mu}_0) = \begin{cases} \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{p}_0, \boldsymbol{\mu}_0) & = \boldsymbol{0} \\ \boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{p}_0, \boldsymbol{\mu}_0) & = \boldsymbol{0} \end{cases}$$

- In this case, the linearization about $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ yields:

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= D_x\boldsymbol{f}|_0\Delta\boldsymbol{x} + D_y\boldsymbol{f}|_0\Delta\boldsymbol{y} \\ \boldsymbol{0} &= D_x\boldsymbol{g}|_0\Delta\boldsymbol{x} + D_y\boldsymbol{g}|_0\Delta\boldsymbol{y} \end{aligned}$$

or, using the compact notation:

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= \boldsymbol{f}_{\boldsymbol{x}}\Delta\boldsymbol{x} + \boldsymbol{f}_{\boldsymbol{y}}\Delta\boldsymbol{y} \\ \boldsymbol{0} &= \boldsymbol{g}_{\boldsymbol{x}}\Delta\boldsymbol{x} + \boldsymbol{g}_{\boldsymbol{y}}\Delta\boldsymbol{y} \end{aligned}$$

Equilibria of DAEs

- Hence, by eliminating $\Delta \boldsymbol{y}$ from these equations, one obtains:

$$D_x \boldsymbol{s}|_0 = D_x \boldsymbol{f}|_0 - D_y \boldsymbol{f}|_0 D_y \boldsymbol{g}|_0^{-1} D_x \boldsymbol{g}|_0$$

or

$$\boldsymbol{s}_{\boldsymbol{x}} = \boldsymbol{f}_{\boldsymbol{x}} - \boldsymbol{f}_{\boldsymbol{y}} \boldsymbol{g}_{\boldsymbol{y}}^{-1} \boldsymbol{g}_{\boldsymbol{x}}$$

- Observe that, as mentioned before, the nonsingularity of the Jacobian $\boldsymbol{g}_{\boldsymbol{y}}$ in this case is required.

- The same local stability conditions apply in this case based on the eigenvalues of $\boldsymbol{s}_{\boldsymbol{x}}$.

Equilibria - Example III

- For the generator-infinite bus example with AVR example (within Q_G limits), the steady-state solutions are obtained from solving the steady-state or power flow equations:

$$
\begin{aligned}
0 &= x_{20} \\
0 &= 10\mu - 13.33 y_{10} \sin x_{10} - x_{20} \\
0 &= 1.333 y_{10} \sin x_{10} - 2 \sin y_{20} \\
0 &= -y_{30} - 4 + 4 y_{10} \cos(y_{20} - x_{10}) \\
0 &= y_{30} - 2 + 2 \cos y_{20}
\end{aligned}
$$

Equilibria - Example III

- Which in MATLAB format are:

```
function f = dae_eqs(z)

global mu

x10 = z(1);
x20 = z(2);
y10 = z(3);
y20 = z(4);
y30 = z(5);

f(1,1) = x20;
f(1,2) = 10*mu - 13.33 * y10 * sin(x10) - x20;
f(1,3) = 1.333 * y10 * sin (x10) - 2 * sin(y20);
f(1,4) = -y30 - 4 + 4 * y10 * cos(y20 - x10);
f(1,5) = y30 - 2 + 2 * cos(y20);
```

Equilibria - Example III

- This generates the equilibrium point for $\mu = P_L = 1$:

```
>> mu = 1;
>> z0 = fsolve(@dae_eqs,[0 0 1 0 1],optimset('Display','iter'))

                                            Norm of      First-order   Trust-region
 Iteration  Func-count     f(x)             step         optimality    radius
     0          6               102                          133              1
     1         12           1.52385               1         10.2              1
     2         18         0.0050436        0.298687        0.219            2.5
     3         24       2.72816e-05       0.0607967       0.0534            2.5
     4         30        2.0931e-13     0.000413599     4.89e-06            2.5
     5         36       4.98474e-28     5.13313e-08     2.23e-13            2.5
Optimization terminated: first-order optimality is less than options.TolFun.

z0 =

    0.7539         0    1.0959    0.5236    0.2679
```

Equilibria - Example III

- And the symbolic Jacobian matrices:

```
syms x1 x2 y1 y2 y3 real
z = [x1 x2 y1 y2 y3];
F = dae_eqs(z);
f = F(1:2);
g = F(3:5);

Dxf = jacobian(f,[x1,x2])
Dyf = jacobian(f,[y1,y2,y3])
Dxg = jacobian(g,[x1,x2])
Dyg = jacobian(g,[y1,y2,y3])
```

Equilibria - Example III

- And the symbolic Jacobian matrices:

```
Dxf =
[                    0,                    1]
[ -1333/100*y1*cos(x1),                   -1]

Dyf =
[                 0,                 0,                 0]
[ -1333/100*sin(x1),                 0,                 0]

Dxg =
[ 1333/1000*y1*cos(x1),                    0]
[    -4*y1*sin(-y2+x1),                    0]
[                    0,                    0]

Dyg =
[ 1333/1000*sin(x1),        -2*cos(y2),                 0]
[     4*cos(-y2+x1), 4*y1*sin(-y2+x1),                -1]
[                 0,        -2*sin(y2),                 1]
```

Equilibria - Example III

- These generate the following eigenvalues at the equilibrium point:

```
Dxf =
[                    0,                     1]
[ -1333/100*y1*cos(x1),                    -1]

Dyf =
[                 0,                  0,                  0]
[ -1333/100*sin(x1),                  0,                  0]

Dxg =
[ 1333/1000*y1*cos(x1),                     0]
[    -4*y1*sin(-y2+x1),                     0]
[                    0,                     0]

Dyg =
[ 1333/1000*sin(x1),         -2*cos(y2),                  0]
[     4*cos(-y2+x1),  4*y1*sin(-y2+x1),                 -1]
[                 0,         -2*sin(y2),                  1]
```

Equilibria - Example III

- These generate the following eigenvalues at the equilibrium point:

```
x1 = z0(1); x2 = z0(2); y1 = z0(3); y2 = z0(4); y3 = z0(5);

A = vpa(subs(Dxf),5);
B = vpa(subs(Dyf),5);
C = vpa(subs(Dxg),5);
D = vpa(subs(Dyg),5);
Dxs = A - B * inv(D) * C;
ev = vpa(eig(Dxs),5)

ev =

 -.50000+3.5698*i
 -.50000-3.5698*i
```

- Hence, this is a s.e.p.

Equilibria - Example III

- A u.e.p. can be computed as well from these equations (neglecting Q_G limits):

```
>> z0 = fsolve(@dae_eqs,[3 0 0.1 3 3],optimset('Display','iter'));

z0 =

    2.7465         0    1.9491    2.6180    3.7321

>> x1 = z0(1); x2 = z0(2); y1 = z0(3); y2 = z0(4); y3 = z0(5);
>> A = vpa(subs(Dxf),5);
>> B = vpa(subs(Dyf),5);
>> C = vpa(subs(Dxg),5);
>> D = vpa(subs(Dyg),5);
>> Dxs = A - B * inv(D) * C;
>> ev = vpa(eig(Dxs),5)

ev =

  4.2892
 -5.2892
```

Z-domain - I

- The eigenvalue analysis discussed so far is based on the so called S-domain.
- While the S-domain is widely used in power system analysis, there are alternative domains that can be used to solve the eigenvalue analysis.
- For example, Z-domain bilinear transformation is as follows:

$$\boldsymbol{A}_Z = (\boldsymbol{A}_S + \chi\boldsymbol{I})(\boldsymbol{A}_S - \chi\boldsymbol{I}_n)^{-1} \tag{1}$$

where $\boldsymbol{A}_S$ is the original state matrix, $\boldsymbol{I}$ the identity matrix of the same size as $\boldsymbol{A}_S$, and χ is a weighting factor that, based on heuristic considerations, can be set to $\chi = 8$.

Z-domain - II

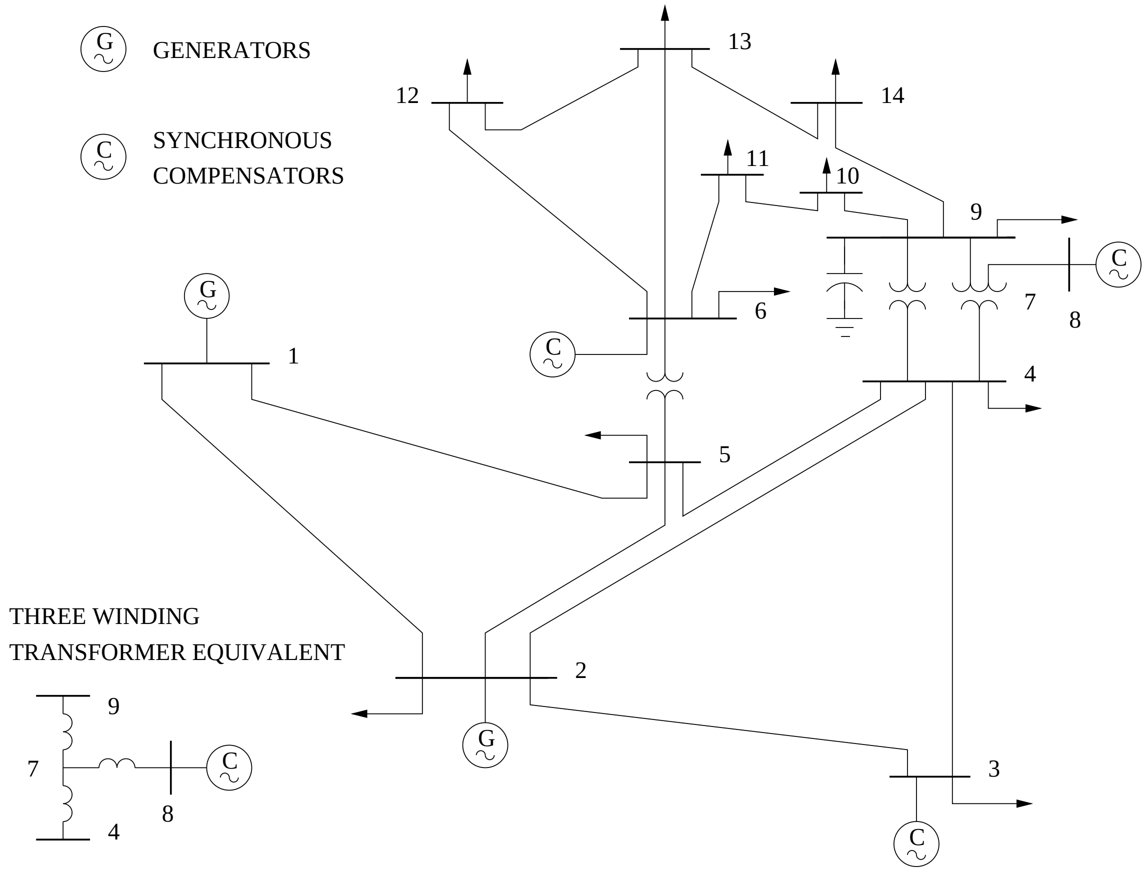
- The first Lyapunov method applied to the Z-domain, becomes:
  - If $|\rho_i| < 1 \; \forall \; i = 1, \ldots, n$, the equilibrium point is asymptotically stable.
  - If $|\rho_i| > 1$ for some i, the equilibrium point is unstable.
  - If $|\rho_i| = 1$ for some i, further investigation is needed.

Z-domain - III

- Computing $\boldsymbol{A}_Z$ is more expensive than $\boldsymbol{A}_S$ but using $\boldsymbol{A}_Z$ can be useful to:
  - Better visualize stiff systems, as the eigenvalues falls within or close to the unitary circle.
  - To speed up the determination of the maximum amplitude eigenvalue (e.g., by means of the Arnoldi iteration), especially in case of unstable equilibrium points with only few eigenvalues outside the unit circle.

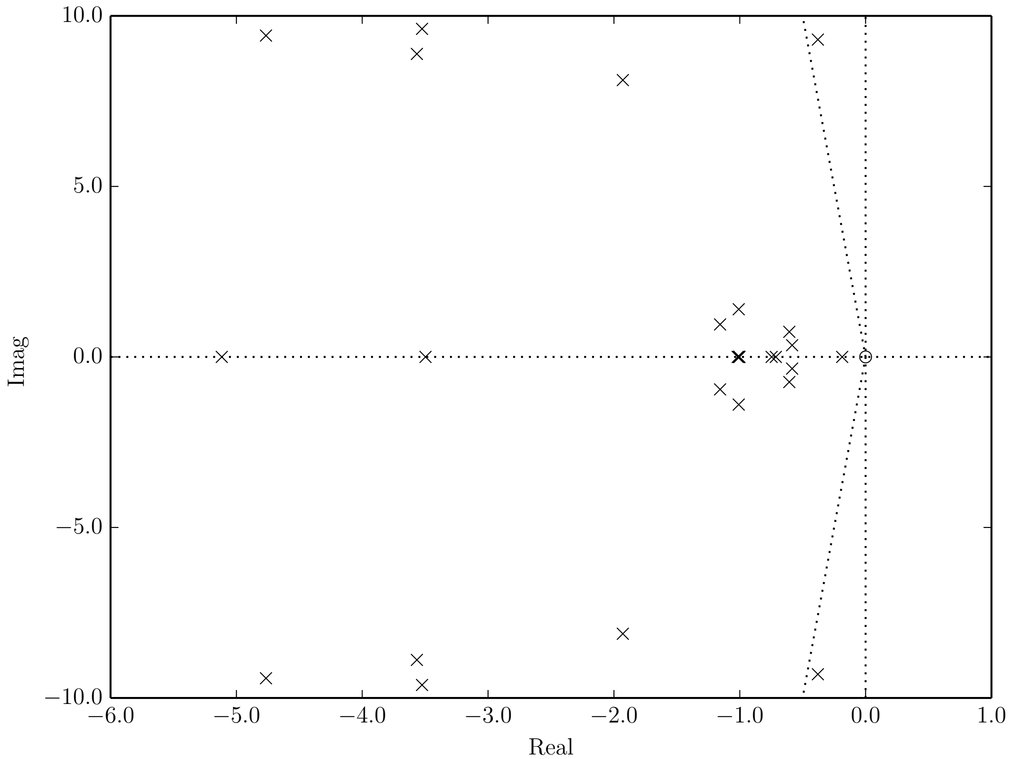
Equilibria - Example IV

- Let consider the IEEE 14-bus system and base case conditions.



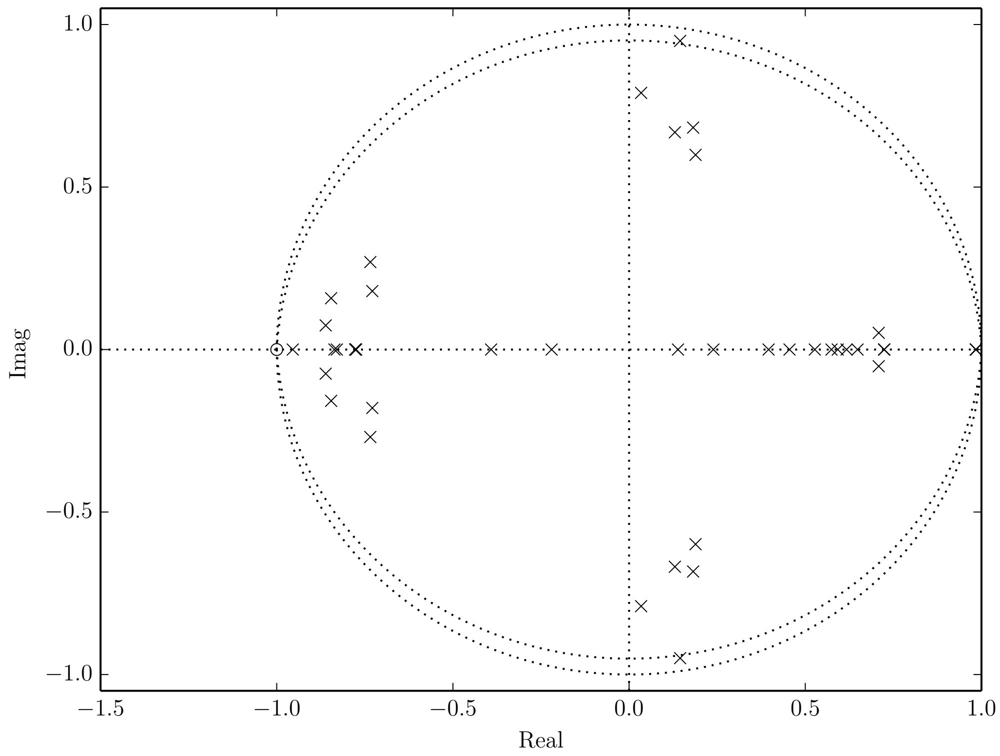
Equilibria - Example IV

- Dominant modes of the IEEE 14-bus system in the S-plane.



Equilibria - Example IV

- Eigenvalues of the IEEE 14-bus system in the Z-plane.



Algorithms to Compute Eigenvalues

- The problem can be tackled in two ways:
  - To find the roots of a polynomial (slow even for small problems!)
  - To find *all* eigenvalues and eigenvectors of the state matrix (e.g., Shur method). This can be adequate for small to medium size problems.
  - To find a subset of eigenvalues, based on some properties (e.g., largest magnitude, smallest real part, etc.). This is probably the most promising approach for stability analysis.

Example: Power Iteration - I

- The *power iteration* is arguably the simplest method to compute eigenvalues.
- The power iteration algorithm starts with a vector $\boldsymbol{b}_0$, which may be an approximation to the dominant eigenvector or a random vector.
- The method is described by the map:

$$\boldsymbol{b}_{k+1} = \frac{\boldsymbol{A}_S \boldsymbol{b}_k}{\|\boldsymbol{A}_S \boldsymbol{b}_k\|} \ . \tag{2}$$

Example: Power Iteration - II

- At every iteration, the vector $\boldsymbol{b}_k$ is multiplied by the matrix $\boldsymbol{A}_S$ and normalized.
- Under the assumptions:
  - $\boldsymbol{A}_S$ has an eigenvalue that is strictly greater in magnitude than its other eigenvalues
  - The starting vector $\boldsymbol{b}_0$ has a nonzero component in the direction of an eigenvector associated with the *dominant eigenvalue*.[a]
- Then, (2) converges to an eigenvector associated tot he dominant eigenvalue.

[a] *Dominant mode* and *Dominant eigenvalue* must not be confused!

Example: Power Iteration - III

- A simple proof of the functioning of the power method is given below.
- Let assume that the state matrix is diagonalizable and of order n.
- Let $\rho_1, \rho_2, \ldots, \rho_n$ be the n eigenvalues (counted with multiplicity) of $\boldsymbol{A}_S$ and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be the corresponding eigenvectors.
- Suppose that ρ_1 is the dominant eigenvalue, so that $|\rho_1| > |\rho_j|$ for $j > 1$.

Example: Power Iteration - IV

- Let choose an initial vector $\boldsymbol{b}_0$ can be written:

$$\boldsymbol{b}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n \ . \tag{3}$$

- If $\boldsymbol{b}_0$ is chosen randomly (with uniform probability), then $c_1 \neq 0$ with probability 1. Now,

$$\begin{aligned} \boldsymbol{A}_S^k\boldsymbol{b}_0 &= c_1\boldsymbol{A}_S^k\boldsymbol{v}_1 + c_2\boldsymbol{A}_S^k\boldsymbol{v}_2 + \cdots + c_m\boldsymbol{A}_S^k\boldsymbol{v}_n \\ &= c_1\rho_1^k\boldsymbol{v}_1 + c_2\rho_2^k\boldsymbol{v}_2 + \cdots + c_n\rho_n^k\boldsymbol{v}_n \\ &= c_1\rho_1^k\left(\boldsymbol{v}_1 + \frac{c_2}{c_1}\left(\frac{\rho_2}{\rho_1}\right)^k\boldsymbol{v}_2 + \cdots + \frac{c_n}{c_1}\left(\frac{\rho_n}{\rho_1}\right)^k\boldsymbol{v}_n\right) . \end{aligned} \tag{4}$$

- The expression within parentheses converges to $\boldsymbol{v}_1$ because $|\rho_j/\rho_1| < 1$ for $j > 1$.

Example: Power Iteration - V

- On the other hand, we have

$$\boldsymbol{b}_k = \frac{\boldsymbol{A}_S^k \boldsymbol{b}_0}{\|\boldsymbol{A}_S^k \boldsymbol{b}_0\|}. \tag{5}$$

- Therefore, $\boldsymbol{b}_k$ converges to (a multiple of) the eigenvector $\boldsymbol{v}_1$.
- The convergence is geometric, with ratio

$$\left|\frac{\rho_2}{\rho_1}\right|, \tag{6}$$

  where ρ_2 denotes the second dominant eigenvalue.
- Thus, the method converges slowly if there is an eigenvalue close in magnitude to the dominant eigenvalue.

Example: Power Iteration - VI

- Once the estimation of the eigenvector has been obtained, the correspondent eigenvalue can be computed using the *Rayleigh quotient*:

$$\rho = \frac{\boldsymbol{v}^T \boldsymbol{A}_S \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}} \tag{7}$$

- Note that, in stability analysis, we are interested in the rightmost eigenvalues, which might not be the dominant ones.
- Actually, rightmost eigenvalues are those closer to the imaginary axis and tend to have small magnitude.

Example: Inverse and Rayleigh Quotient Iteration

- The *inverse iteration* is a variant of the power method and allow to find such eigenvalues.
- The method is described by the iteration

$$\boldsymbol{b}_{k+1} = \frac{(\boldsymbol{A}_S - \tilde{\rho}\boldsymbol{I}_n)^{-1}\boldsymbol{b}_k}{C_k}, \tag{8}$$

  where C_k are some constants usually chosen as $C_k = \|(\boldsymbol{A}_S - \tilde{\rho}\boldsymbol{I}_n)^{-1}\boldsymbol{b}_k\|$ and $\tilde{\rho}$ is an estimation of the sought eigenvalue.
- Since eigenvectors are defined up to multiplication by constant, the choice of C_k can be arbitrary in theory.
- The Rayleigh quotient iteration consists in updating ρ at each iteration using (7) with the current vector $\boldsymbol{b}_k$.

Some Libraries for Eigenvalue Analysis

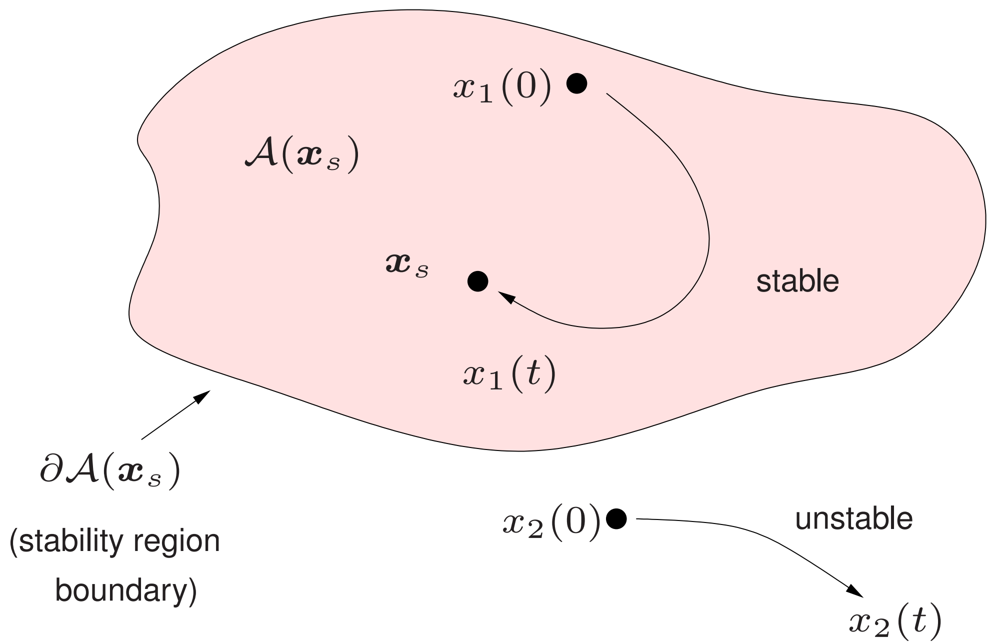
- Full specturm:
  - LAPACK: state-of-the-art of the QR factorization and Shur method.
  - EISPACK: superseded by LAPACK.
  - MAGMA: GPU-based Shur method.
  - GSL: Double-shift Francis method

- Partial spectrum:
  - ARPACK: it provides the state-of-the art for the Arnoldi iteration. It requires a good algorithm to factorize the state matrix (e.g., KLU)
  - SLEPc: it is based on PETSc and provides several algorithms: Arnoldi iteration, Krylov-Shur method, Lanczos method, generalized Davidson method.
  - HSL: Subspace iteration as well as Arnoldi iteration with Chebychev acceleration.

Stability Region

Region of Attraction of S.E.P.

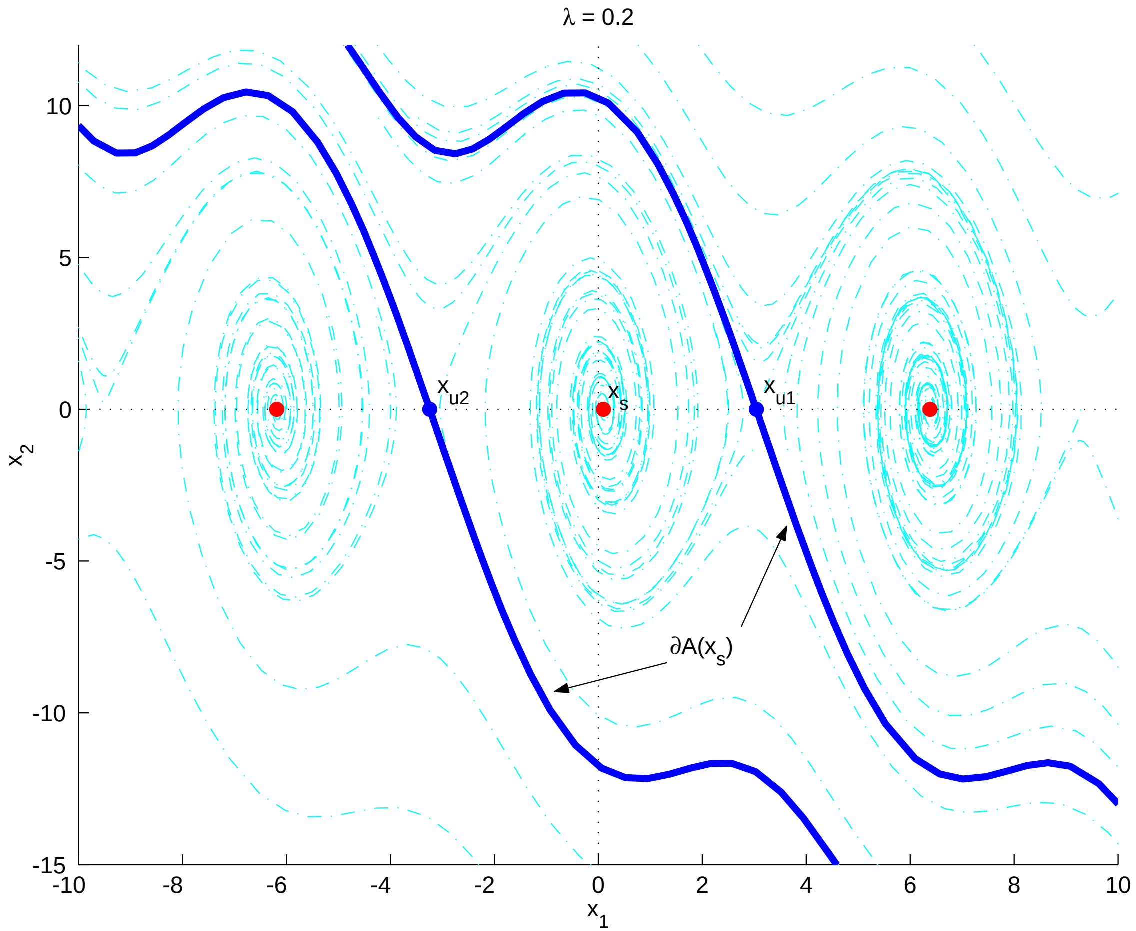
- Associated with every s.e.p. $\boldsymbol{x}_s$ there is a stability region $\mathcal{A}(\boldsymbol{x}_s)$, which basically corresponds to the region of system variables that are all attracted to $\boldsymbol{x}_s$, i.e. $\boldsymbol{x}(t \to \infty) \to \boldsymbol{x}_s$:



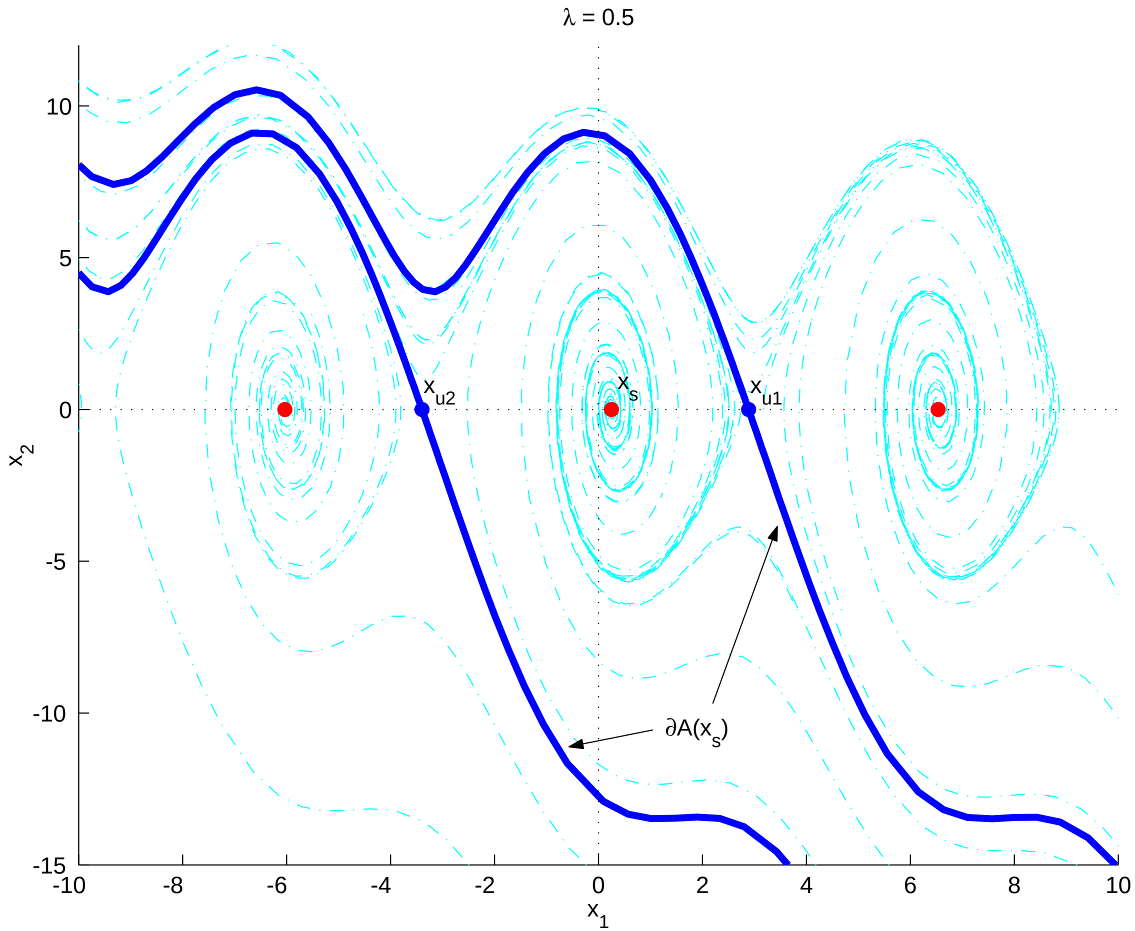
Region of Attraction of S.E.P.

- if $\mathcal{A}(\boldsymbol{x}_s)$ is known, the stability of a system for large perturbations can be readily evaluated.
- However, determining this region is a rather difficult task.
- This can realistically be accomplished only for low-dimendional systems using "sophisticated" nonlinear system analysis techniques.

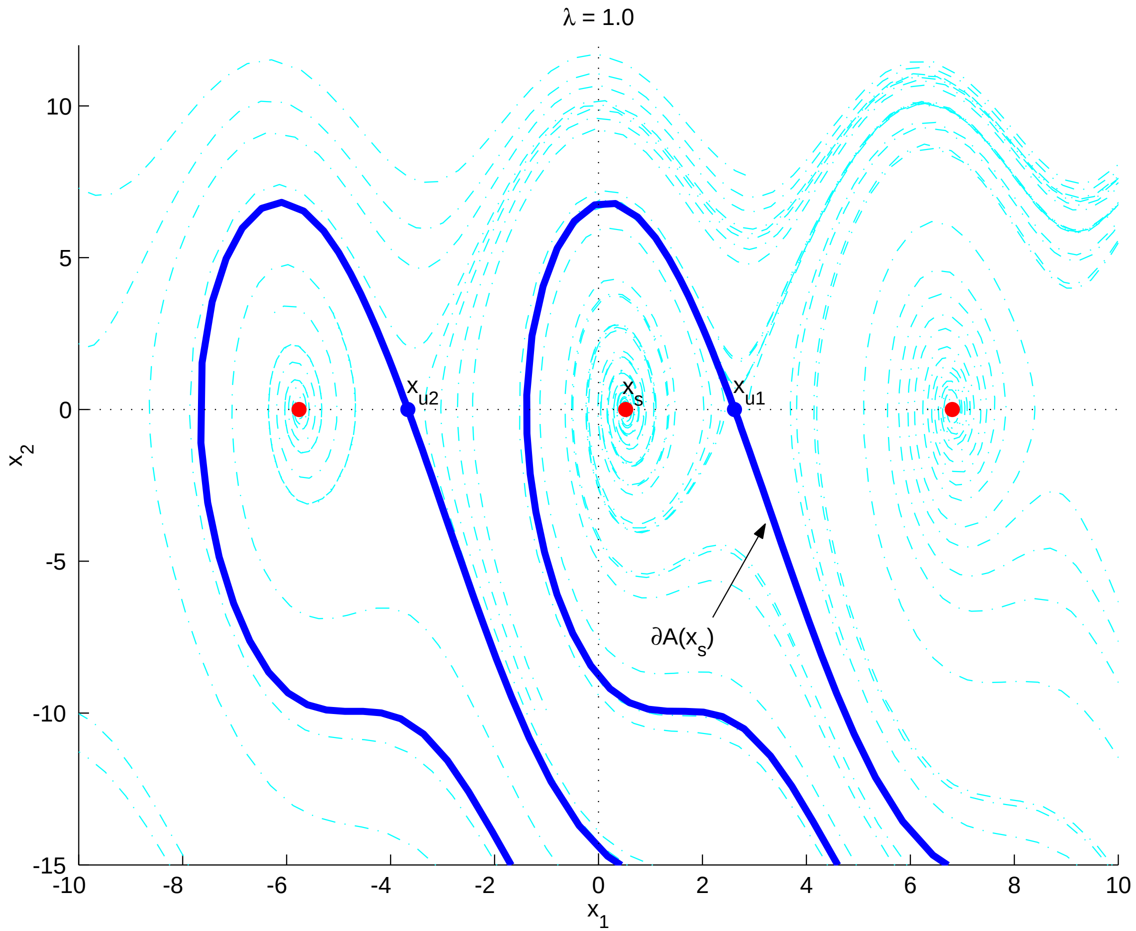
Region of Attraction of the OMIB System



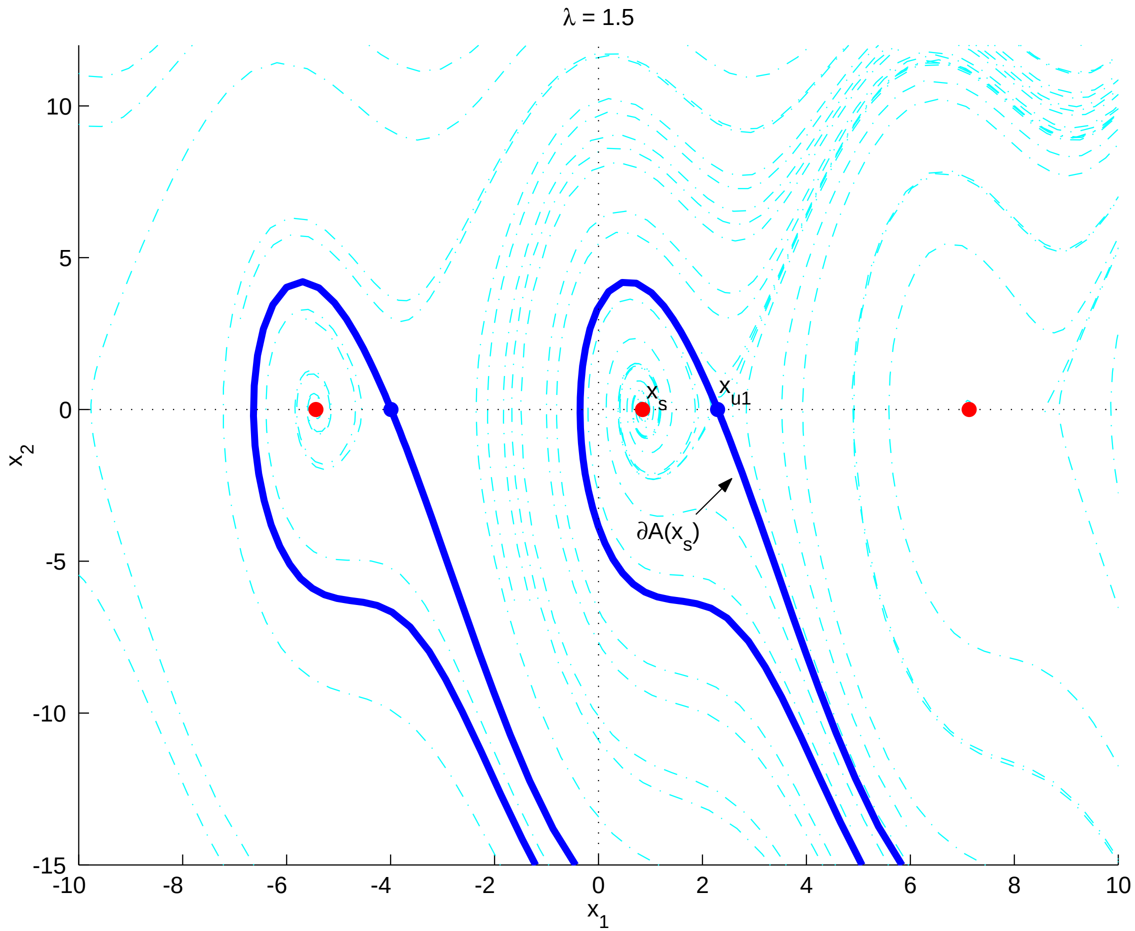
Region of Attraction of the OMIB System



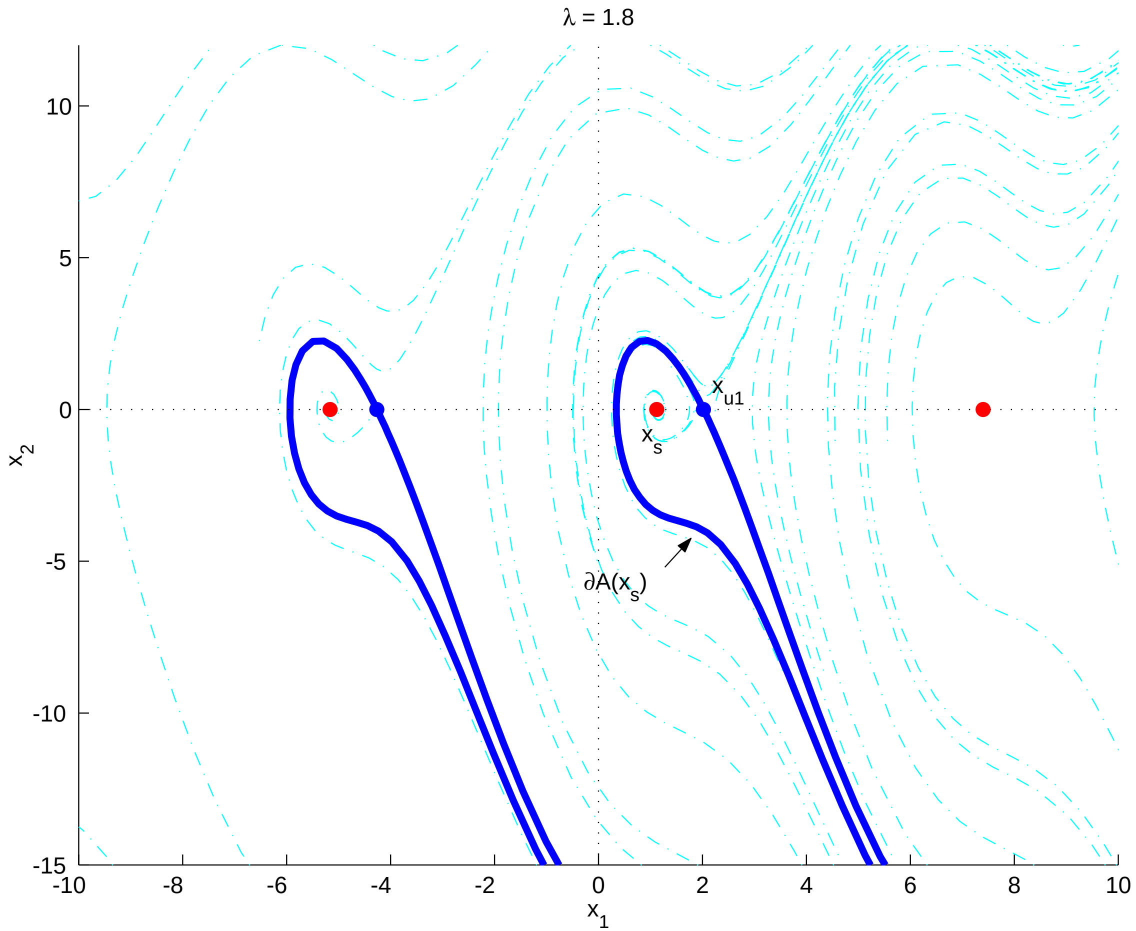
Region of Attraction of the OMIB System



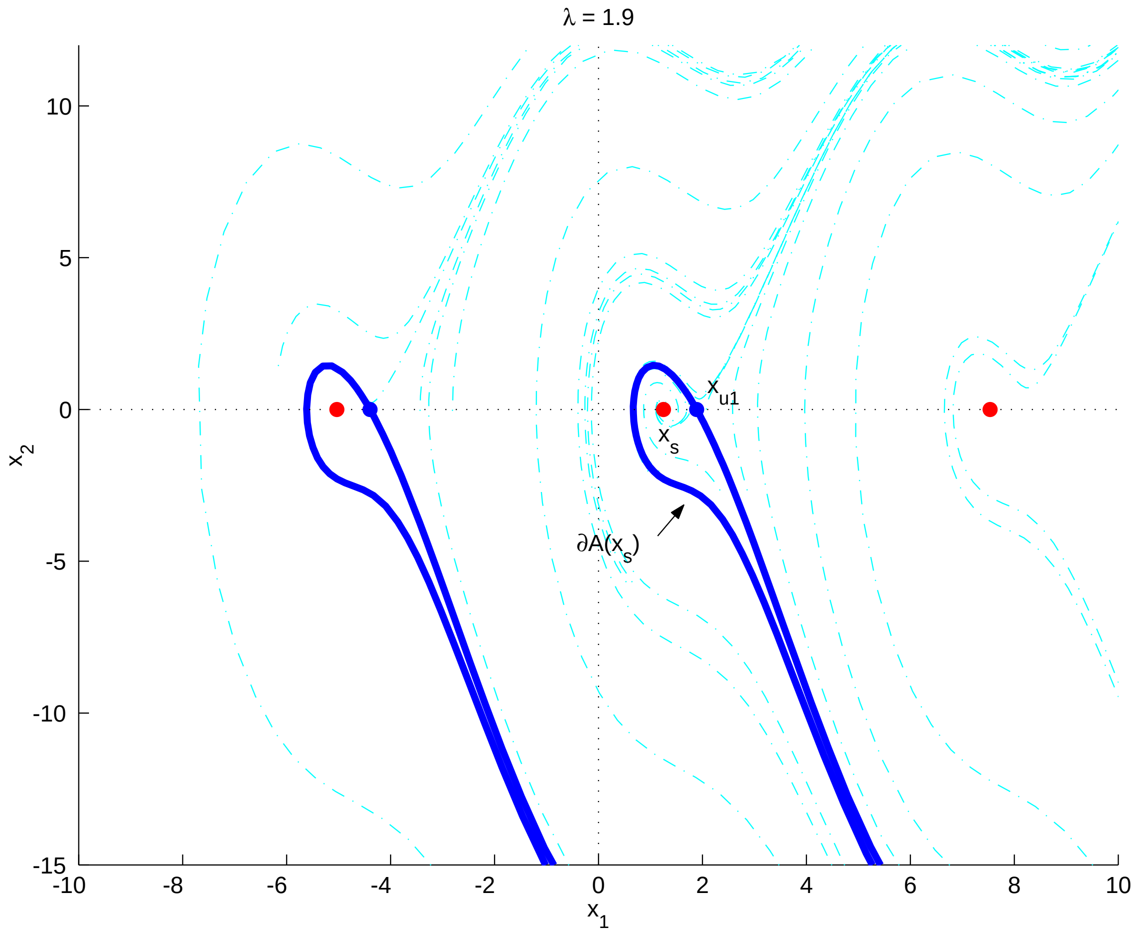
Region of Attraction of the OMIB System



Region of Attraction of the OMIB System



Region of Attraction of the OMIB System



Region of Attraction of S.E.P.

- In real systems, trial-and-error techniques are usually used:
  - A contingency yields a given initial condition $\boldsymbol{x}(0)$.
  - For the post-contingency system, the time trajectories $\boldsymbol{x}(t)$ can be computed by numerical integration.
  - If $\boldsymbol{x}(t)$ converges to the post-contingency equilibrium point $\boldsymbol{x}_s$, the system is stable, i.e. $\boldsymbol{x}(0) \in \mathcal{A}(\boldsymbol{x}_s)$.
  - If it diverges, the system is unstable.

Outlines of Bifurcation Theory

and Relevant Bifurcation Points

Beyond Linearization

- As shown above, the linearization is inconclusive for equilibrium points whose state matrix shows eigenvalues on the imaginary axis.
- In these cases, we need some other mathematical tool able to define the (local) stability of the nonlinear system.
- Bifurcation theory helps define the properties of certain equilibrium points whenever linearization is inconclusive.

A Qualitative Definition of Bifurcation Theory

- Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations.
- Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden "qualitative" or topological change in its behaviour.
- The bifurcation theory attempts to classify *stationary points*, e.g., equilibrium points.
- The classification of bifurcations is far from being complete!

Bifurcation Types

- It is useful to divide bifurcations into two principal classes:
  - Local bifurcations, which can be analysed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds; and
  - Global bifurcations, which often occur when larger invariant sets of the system "collide" with each other, or with equilibria of the system. They cannot be detected purely by a stability analysis of the equilibria (fixed points).

- We will consider *only* local bifurcations.

Examples of Local Bifurcations

- Examples of local bifurcations include:
  - Saddle-node (fold) bifurcation
  - Transcritical bifurcation
  - Limit-induced bifurcation
  - Period-doubling (flip) bifurcation
  - Hopf bifurcation

- We will see examples of power system models showing most bifurcations above.

Examples of Global Bifurcations

- For the sake of example, let cite some global bifurcations:
  - Homoclinic bifurcation in which a limit cycle collides with a saddle point.
  - Heteroclinic bifurcation in which a limit cycle collides with two or more saddle points.
  - Infinite-period bifurcation in which a stable node and saddle point simultaneously occur on a limit cycle.
  - Blue sky catastrophe in which a limit cycle collides with a nonhyperbolic cycle.

- Global bifurcations can also involve more complicated sets such as chaotic attractors (e.g. crises).

Codimension of a Bifurcation

- *The **codimension** of a bifurcation is the number of parameters which must be varied for the bifurcation to occur.*
- This corresponds to the codimension of the parameter set for which the bifurcation occurs within the full space of parameters.
- Saddle-node bifurcations and Hopf bifurcations are the only generic local bifurcations which are really codimension-one (the others all having higher codimension).
- However, transcritical and pitchfork bifurcations are also often thought of as codimension-one, because the normal forms can be written with only one parameter.
- We will consider *only* bifurcations of codimension 1.

Robustness of Bifurcation Points

- An important feature of a bifurcation point is its ***robustness*** with respect to the parameters and approximations of the systems.
- If the bifurcation disappears when the model of the system is modified (e.g., losses are taken into account), then the bifurcation is not robust.
- Clearly, only robust bifurcations are relevant, as they are an intrinsical property of the system.
- In power systems, saddle-node, limit-induced and Hopf bifurcations are robust, i.e., they can be expected to appear *always* for certain values of the exogenous (uncontrollable) variables.

Example - Saddle-Node Bifurcation (I)

- Let consider again the following scalar differential equation:

$$\dot{x} = \mu - x^2$$

- The behavior of the system and the properties of the equilibrium points change *structurally* as μ varies:
  - Two equilibrium points (one stable) for $\mu > 0$.
  - No equilibrium points for $\mu < 0$.
  - One equilibrium point for $\mu = 0$.

- One can intuitively see that the point $(x, \mu) = (0, 0)$ defines a structural change of the system.

- We will see that this is the simplest system that shows a saddle-node bifurcation.

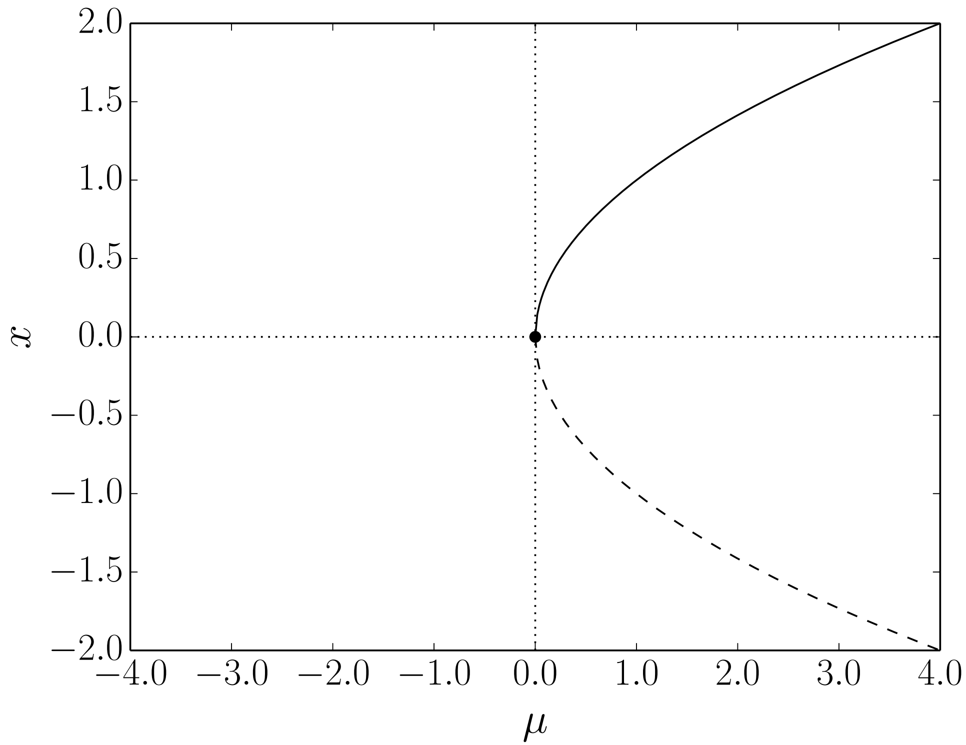
Bifurcation Diagram

- A bifurcation diagram shows the possible long-term values (equilibria/fixed points or periodic orbits) of a system as a function of a bifurcation parameter in the system.
- It is usual to represent stable solutions with a solid line and unstable solutions with a dotted or dashed line.

Example - Saddle-Node Bifurcation (II)

- The bifurcation diagram of the system $\dot{x} = \mu - x^2$ is:

$$x = \pm\sqrt{\mu}$$



ODE System for Bifurcation Analysis

- Consider the following ODE system:

$$\dot{\boldsymbol{x}} = \boldsymbol{s}(\boldsymbol{x}, \mu) \tag{9}$$

where $\boldsymbol{s}$ is smooth ($\boldsymbol{s} : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$), $\boldsymbol{x} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$.

- Let $(\boldsymbol{x}_0, \mu_0)$ be an equilibrium for (9), hence:

$$\boldsymbol{0} = \boldsymbol{s}(\boldsymbol{x}_0, \mu) \tag{10}$$

- The DAE case can be always reduced to the ODE case, provided that the hypothesis of the implicit function theorem apply.
- So we will focus only on the ODE system (9).

Limit Points

- Given the state matrix $\boldsymbol{s_x}$ at the equilibrium (10), let $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{w} \in \mathbb{R}^n$ be the normalized right and left eigenvectors associated to a certain eigenvalue ρ of $\boldsymbol{s_x}$, then:

$$\boldsymbol{s_x v} = \rho \boldsymbol{v} \tag{11}$$

$$\boldsymbol{s_x^T w} = \rho \boldsymbol{w}$$

- A *limit point* of $\boldsymbol{s}$ is a point for which $\boldsymbol{s_x}$ is singular.

- At the limit point, there exist a left eigenvector $\boldsymbol{w}$ and a right eigenvector $\boldsymbol{v}$, such that $\boldsymbol{s_x^T w} = \boldsymbol{0}$ and $\boldsymbol{s_x v} = \boldsymbol{0}$.

Transversality Conditions (I)

- To introduce formally the transversality conditions of bifurcation points, consider the identity:

$$\boldsymbol{s}(\boldsymbol{x}(p), \mu(p)) \equiv \boldsymbol{0} \tag{12}$$

where p is a diffeomorphic parametrization of a branch through the ordinary point $(\boldsymbol{x}, \mu)$ so that:

$$\boldsymbol{x}_p = \frac{d\boldsymbol{x}}{dp}, \quad \mu_p = \frac{d\mu}{dp} \tag{13}$$

and $(\boldsymbol{x}_p^T, \mu_p)^T \neq \boldsymbol{0}$ and $\boldsymbol{x}_p \equiv \boldsymbol{v}$ is a nonzero right eigenvector of $\boldsymbol{s}_{\boldsymbol{x}}$.

Transversality Conditions (II)

- Differentiating (12) allows defining higher order identities, as follows:

  First order differentiation:

$$\boldsymbol{s}_{\boldsymbol{x}}\boldsymbol{x}_p + \boldsymbol{s}_{\mu}\mu_p \equiv \mathbf{0} \tag{14}$$

  Second order differentiation:

$$\boldsymbol{s}_{\boldsymbol{x}\boldsymbol{x}}\boldsymbol{x}_p\boldsymbol{x}_p + \boldsymbol{s}_{\mu\mu}\mu_p\mu_p + 2\boldsymbol{s}_{\boldsymbol{x}\mu}\boldsymbol{x}_p\mu_p + \boldsymbol{s}_{\boldsymbol{x}}\boldsymbol{x}_{pp} + \boldsymbol{s}_{\mu}\mu_{pp} \equiv \mathbf{0} \tag{15}$$

  Third order differentiation:

$$\begin{aligned} & \boldsymbol{s}_{\boldsymbol{x}\boldsymbol{x}\boldsymbol{x}}\boldsymbol{x}_p\boldsymbol{x}_p\boldsymbol{x}_p + \boldsymbol{s}_{\mu\mu\mu}\mu_p\mu_p\mu_p + \\ & 2\boldsymbol{s}_{\boldsymbol{x}\boldsymbol{x}\mu}\boldsymbol{x}_p\boldsymbol{x}_p\mu_p + 2\boldsymbol{s}_{\boldsymbol{x}\mu\mu}\boldsymbol{x}_p\mu_p\mu_p + \\ & 3\boldsymbol{s}_{\boldsymbol{x}\boldsymbol{x}}\boldsymbol{x}_p\boldsymbol{x}_{pp} + 3\boldsymbol{s}_{\mu\mu}\mu_p\mu_{pp} + \\ & 2\boldsymbol{s}_{\boldsymbol{x}\mu}\boldsymbol{x}_{pp}\mu_p + 2\boldsymbol{s}_{\boldsymbol{x}\mu}\boldsymbol{x}_p\mu_{pp} + \\ & \boldsymbol{s}_{\boldsymbol{x}}\boldsymbol{x}_{ppp} + \boldsymbol{s}_{\mu}\mu_{ppp} \equiv \mathbf{0} \end{aligned} \tag{16}$$

Saddle-Node Bifurcation (I)

- The saddle-node bifurcation is a limit point that occurs when two equilibria, typically one stable and one unstable, merge and disappear as the parameter μ changes.
- The saddle-node bifurcation is also called *fold* bifurcation or *tangent* bifurcation.
- The transversality conditions computed at the equilibrium $(\boldsymbol{x}_0, \mu_0)$ are:

$$
\begin{aligned}
(\boldsymbol{s_x v})^T = \boldsymbol{w}^T \boldsymbol{s_x} &= \boldsymbol{0} \\
\boldsymbol{w}^T \boldsymbol{s}_\mu &\neq 0 \\
\boldsymbol{w}^T \boldsymbol{s_{xx} v v} &\neq 0
\end{aligned}
\tag{17}
$$

where $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{w} \in \mathbb{R}^n$ are the normalized right and left eigenvectors of the Jacobian matrix $\boldsymbol{s_x}$.

Saddle-Node Bifurcation (II)

- The first condition implies that the Jacobian matrix has a simple zero eigenvalue and, in turns, that the Jacobian matrix is singular.
- The second and third conditions impose the *nondegeneracy* of the equilibrium.
- Note: a **degenerate case** is a limiting case in which an element of a class of objects is qualitatively different from the rest of the class and hence belongs to another, usually simpler, class.

Saddle-Node Bifurcation (III)

- Multiplying (14) by $\boldsymbol{w}^T$:

$$(\boldsymbol{w}^T \boldsymbol{s}_{\boldsymbol{x}})\boldsymbol{x}_p + (\boldsymbol{w}^T \boldsymbol{s}_\mu)\mu_p = \boldsymbol{0} \tag{18}$$

- Hence, from the first and the second of (17), one obtains that $\mu_p = 0$.
- Moreover, from the second and the third of (17), from multiplying (15) by $\boldsymbol{w}^T$, and using the condition $\mu_p = 0$, one has:

$$\boldsymbol{w}^T \boldsymbol{s}_{\boldsymbol{x}\boldsymbol{x}} \boldsymbol{x}_p \boldsymbol{x}_p + (\boldsymbol{w}^T \boldsymbol{s}_\mu)\mu_{pp} = \boldsymbol{0} \tag{19}$$

  that implies $\mu_{pp} \neq 0$.
- The conditions $\mu_p = 0$ and $\mu_{pp} \neq 0$ are equivalent to (17).

Saddle-Node Bifurcation (IV)

- The *normal form* of the saddle-node bifurcation is:

$$\dot{x} = \epsilon_1 \mu + \epsilon_2 x^2 \tag{20}$$

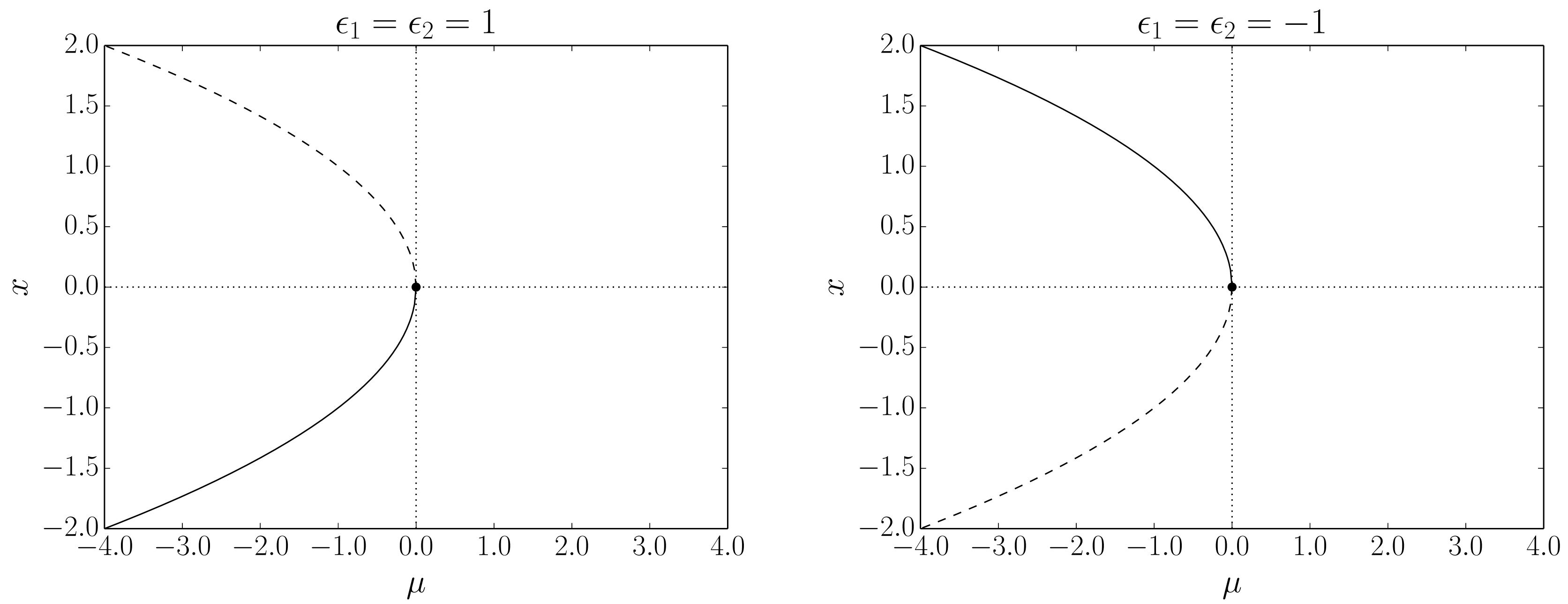
where $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$ and with bifurcation point $(x_0, \mu_0) = (0, 0)$.

- The conditions (17) applied to (20) become:

$$\begin{aligned} s_x &= 2\epsilon_2 x_0 = 0 \\ s_\mu &= \epsilon_1 \neq 0 \\ s_{xx} &= 2\epsilon_2 \neq 0 \end{aligned} \tag{21}$$

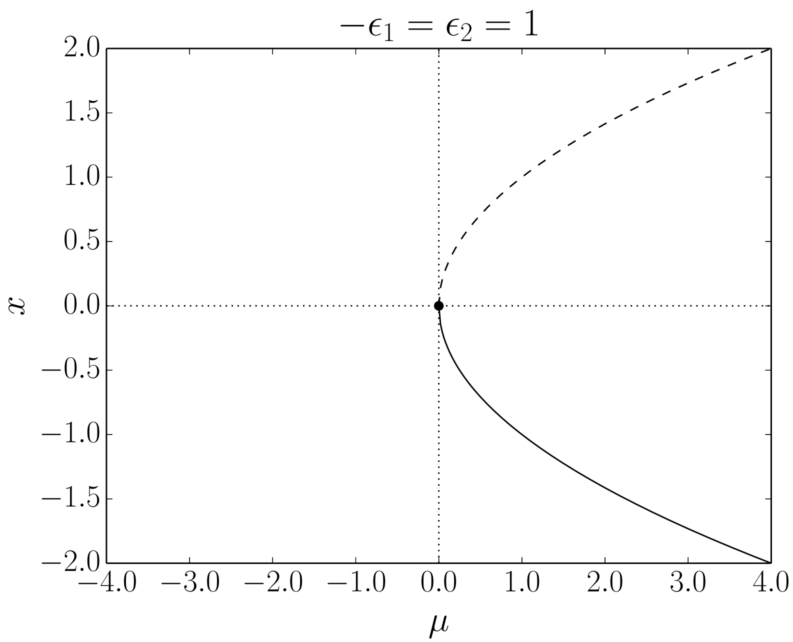
Saddle-Node Bifurcation (V)

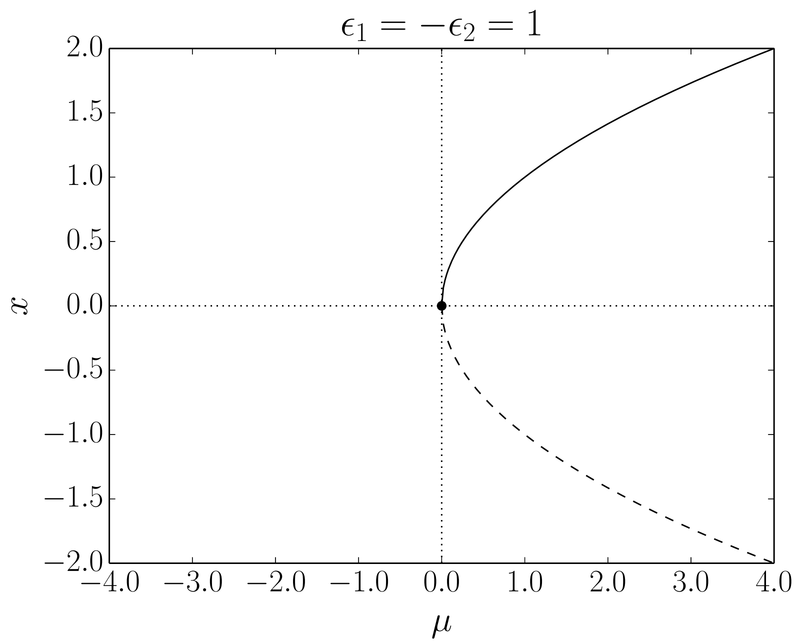
- Illustration of the saddle-node bifurcation through the normal form (20). Solid branches are stable; dashed branches are unstable.



Saddle-Node Bifurcation (VI)

- Illustration of the saddle-node bifurcation through the normal form (20). Solid branches are stable; dashed branches are unstable.





Transcritical Bifurcation (I)

- A transcritical bifurcation is one in which a fixed point exists for all values of a parameter and is never destroyed.
- However, such a fixed point interchanges its stability with another fixed point as the parameter is varied.
- Both before and after the bifurcation, there is one unstable and one stable fixed point.
- Their stability is exchanged when they collide.
- So the unstable fixed point becomes stable and *vice versa*.

Transcritical Bifurcation (II)

- The transversality conditions for the transcritical bifurcations are:

$$
\begin{aligned}
(\boldsymbol{s_x v})^T = \boldsymbol{w}^T \boldsymbol{s_x} &= \boldsymbol{0} \\
\boldsymbol{w}^T \boldsymbol{s}_\mu &= 0 \\
\boldsymbol{w}^T \boldsymbol{s_{xx} v v} &\neq 0 \\
\boldsymbol{w}^T \boldsymbol{s}_{\boldsymbol{x}\mu} \boldsymbol{v} &\neq 0
\end{aligned} \tag{22}
$$

- Equation (14) and the first and the second of (22) do not lead to any condition on μ_s.

- However, from (15) and the third and the fourth of (22), one obtains:

$$
(\boldsymbol{w}^T \boldsymbol{s_{xx}}) \boldsymbol{x}_p \boldsymbol{x}_p + 2(\boldsymbol{w}^T \boldsymbol{s}_{\boldsymbol{x}\mu}) \boldsymbol{x}_p \mu_p + (\boldsymbol{w}^T \boldsymbol{s}_{\mu\mu}) \mu_p \mu_p = 0 \tag{23}
$$

from which $\mu_p \neq 0$ follows.

Transcritical Bifurcation (III)

- The normal form of the transcritical bifurcation is:

$$\dot{x} = \epsilon_1 \mu x + \epsilon_2 x^2 \tag{24}$$

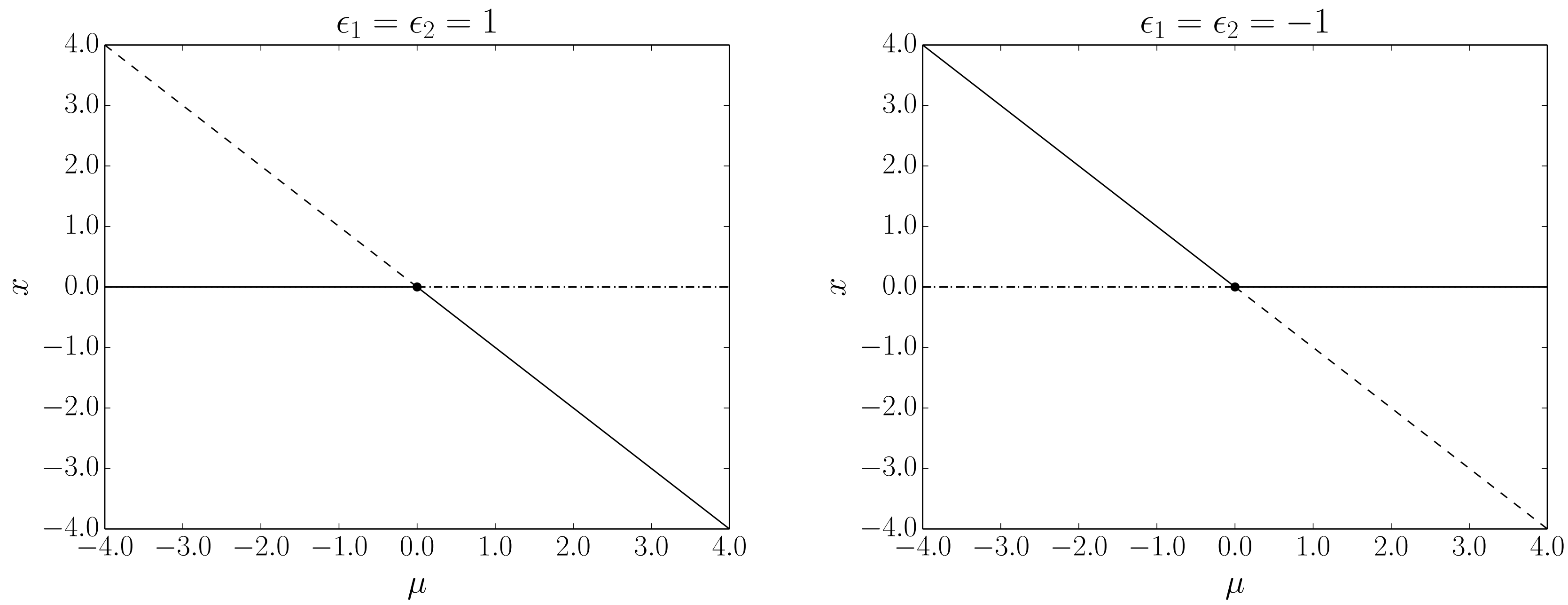
where $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$ and with bifurcation point $(x_0, \mu_0) = (0, 0)$.

- The conditions (22) applied to (24) become:

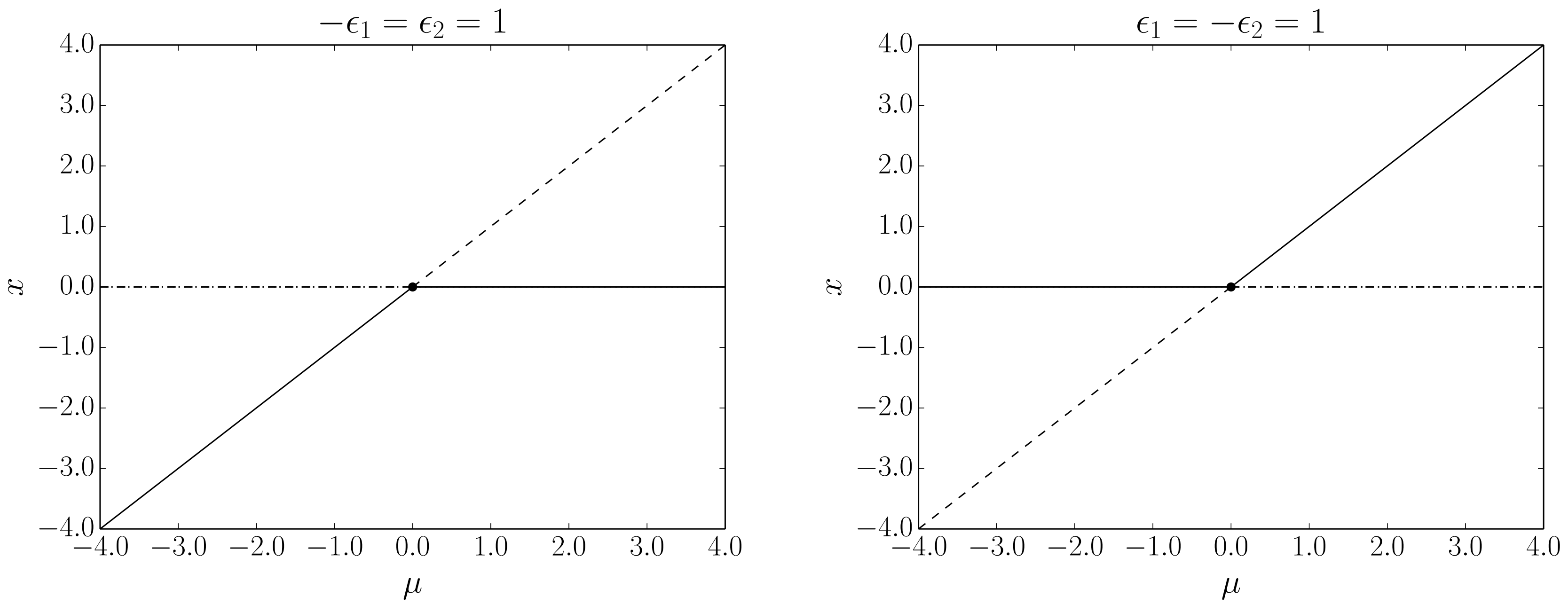
$$\begin{aligned} s_x &= \epsilon_1 \mu_0 + 2\epsilon_2 x_0 = 0 \\ s_\mu &= \epsilon_1 x_0 = 0 \\ s_{xx} &= 2\epsilon_2 \neq 0 \\ s_{x\mu} &= \epsilon_1 \neq 0 \end{aligned} \tag{25}$$

Transcritical Bifurcation (IV)

- Illustration of the transcritical bifurcation through the normal form (24). Solid branches are stable; dashed branches are unstable.



- Illustration of the transcritical bifurcation through the normal form (24). Solid branches are stable; dashed branches are unstable.



Switching Manifold

- Let assume the following vector field (the notation is general and common in circuit analysis):

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{s}_1(\boldsymbol{x}), & \text{if} \quad \boldsymbol{h}(\boldsymbol{x}) < \boldsymbol{0} \\ \boldsymbol{s}_2(\boldsymbol{x}), & \text{if} \quad \boldsymbol{h}(\boldsymbol{x}) \geq \boldsymbol{0} \end{cases} \tag{26}$$

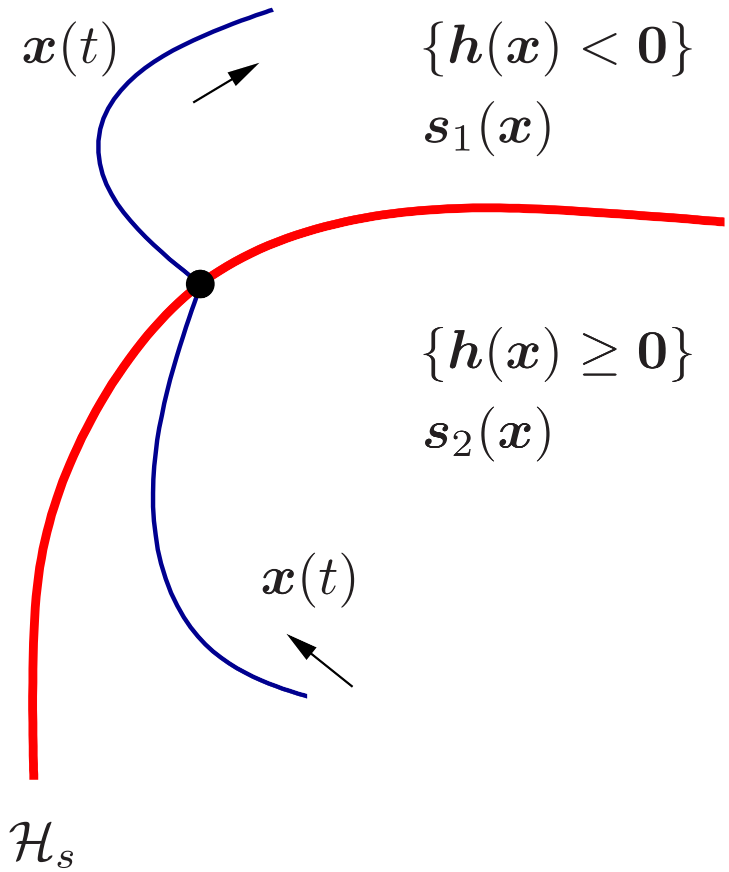
  where the boundary between the two domains is called the *switching manifold* $\mathcal{H}_s = \{\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}\}$.

- When so-called switching manifolds are reached by an evolving trajectory the systems switches from one vector field to another.

- The system trajectories are continuous but non-differentiable at the points where the aforementioned switchings take place.

Illustration of the Switching Manifold

- A trajectory $\boldsymbol{x}(t)$ that hits the switching manifold $\mathcal{H}_s$:



Filippov System

- Filippov systems are defined as follows:

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{s}_1(\boldsymbol{x}, \mu), & \text{if} \quad \boldsymbol{h}(\boldsymbol{x}, \mu) < \boldsymbol{0} \\ \boldsymbol{s}_2(\boldsymbol{x}, \mu), & \text{if} \quad \boldsymbol{h}(\boldsymbol{x}, \mu) \geq \boldsymbol{0} \end{cases} \tag{27}$$

- Dynamical systems governed by discontinuous but piecewise smooth ordinary differential equations.
- Hitting or not the switching manifold $\mathcal{H}_s$ depends on the parameter μ.

Limit-Induced Bifurcation (I)

- A limit-induced bifurcation is an equilibrium point *on* the switching manifold.
- In power systems, switching manifolds can be due to:
  - Hard limits of control systems (e.g. saturation of the field voltage of the synchronous machine). The vector field is continuous but not differentiable.
  - Switching limits are events that can be described by discrete variables (e.g., breaker operations). These limits generally cause also *jumps* of the vector field.
- In the remainder, we discuss only limit-induced bifurcations characterized by a continuous vector field.

Limit-Induced Bifurcation (II)

- At the limit-induced bifurcation point, we have:

$$\mathbf{0} = \begin{cases} \boldsymbol{s}_1(\boldsymbol{x}_0, \mu_0) \\ \boldsymbol{s}_2(\boldsymbol{x}_0, \mu_0) \\ \boldsymbol{h}(\boldsymbol{x}_0, \mu_0) \end{cases} \tag{28}$$

- We add the following transversality conditions:

$$\begin{aligned} \text{det}(\boldsymbol{s}_{1,\boldsymbol{x}}) &\neq 0 \\ \text{det}(\boldsymbol{s}_{2,\boldsymbol{x}}) &\neq 0 \end{aligned} \tag{29}$$

which imply that the equilibrium is **not** a limit point.

- Do not confuse *limit point* and *limit-induced bifurcation point*!

Limit-Induced Bifurcation (III)

- Depending on the signs of det$(\boldsymbol{s}_{1,\boldsymbol{x}})$ and det$(\boldsymbol{s}_{2,\boldsymbol{x}})$, it is possible to distinguish between non-critical (or *dynamic*) limit-induced bifurcations and critical (or *static*) ones.
- In particular, defining

$$\alpha = \frac{\det(\boldsymbol{s}_{1,\boldsymbol{x}})}{\det(\boldsymbol{s}_{2,\boldsymbol{x}})} \tag{30}$$

  if $\alpha > 0$, the bifurcation is non-critical, while if $\alpha < 0$ the bifurcation is critical.
- Only critical limit-induced bifurcations can lead to the disappearance of the equilibrium point, similarly to what happens in the case of saddle-node bifurcations.
- Again, observe that limit-induced bifurcations are **not** limit points, i.e., $\mu_p \neq 0$.

Limit-Induced Bifurcation (IV)

- In common practice, actuation limit functions typically define some algebraic variable limits.
- A typical actuation limit function is as follows:

$$r = \begin{cases} r^{\max}, & \text{if} \quad \phi(\boldsymbol{x}) > r^{\max} \\ \phi(\boldsymbol{x}), & \text{if} \quad r^{\min} \leq \phi(\boldsymbol{x}) \leq r^{\max} \\ r^{\min}, & \text{if} \quad \phi(\boldsymbol{x}) < r^{\min} \end{cases} \tag{31}$$

  which introduces two limit-induced bifurcations, one for $r = r^{\max}$ and one for $r = r^{\min}$.

- See the OMIB example with inclusion of reactive power limit constraints.

Limit-Induced Bifurcation (V)

- A possible normal form of the limit-induced bifurcation is:

$$\dot{x} = \begin{cases} s_1 = \epsilon_1 \mu + m_1 x, & \text{if} \quad x > 0 \\ s_2 = \epsilon_1 \mu + m_2 x, & \text{if} \quad x \leq 0 \end{cases} \tag{32}$$

where $\epsilon_1 = \pm 1$, $m_1 \neq 0$, and $m_2 \neq 0$.

Limit-Induced Bifurcation (VI)

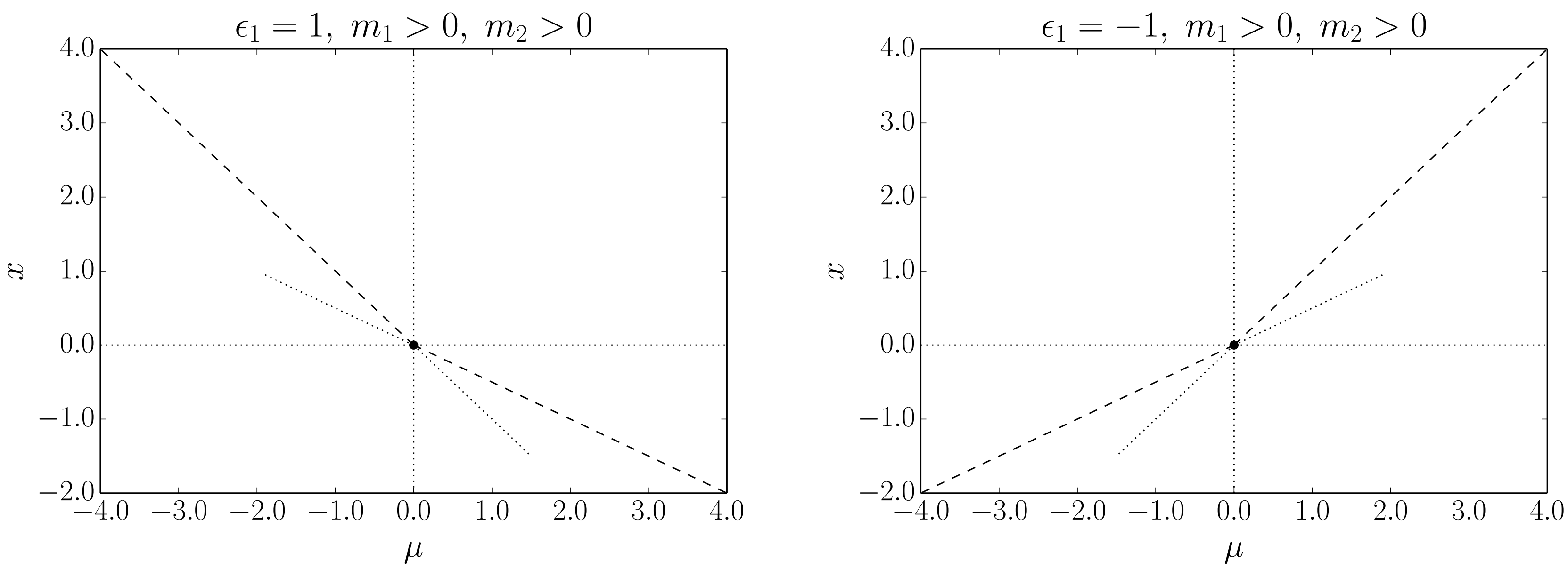
- The bifurcation point of (32) is $(x_0, \mu_0) = (0, 0)$ at which the conditions (29) and (30) become:

$$
\begin{aligned}
s_1 = s_2 &= 0 \\
s_{1,x} = m_1 &\neq 0 \\
s_{2,x} = m_2 &\neq 0 \\
\alpha = \frac{s_{1,x}}{s_{2,x}} &= \frac{m_1}{m_2}
\end{aligned}
\tag{33}
$$

- Hence, the bifurcation is critical only if sign$(m_1) \neq$ sign(m_2).

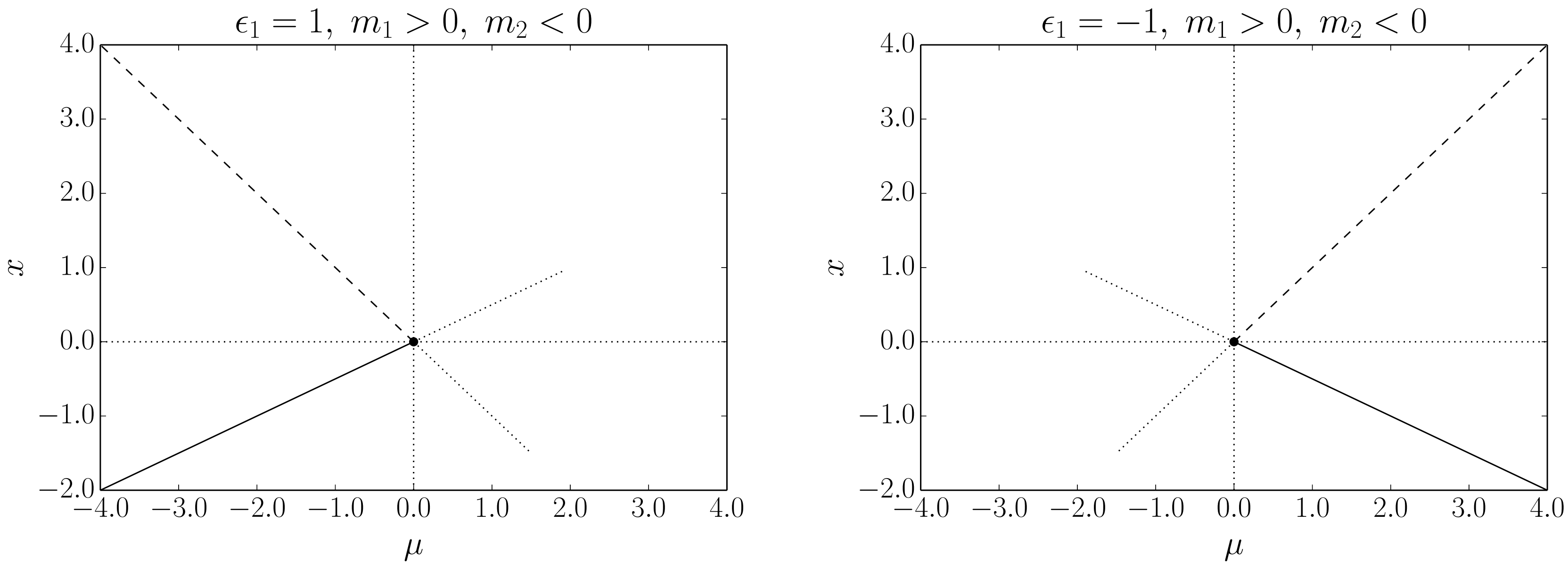
Limit-Induced Bifurcation (VII)

- Illustration of the limit-induced bifurcation through the normal form (32). Solid branches are stable; dashed branches are unstable.



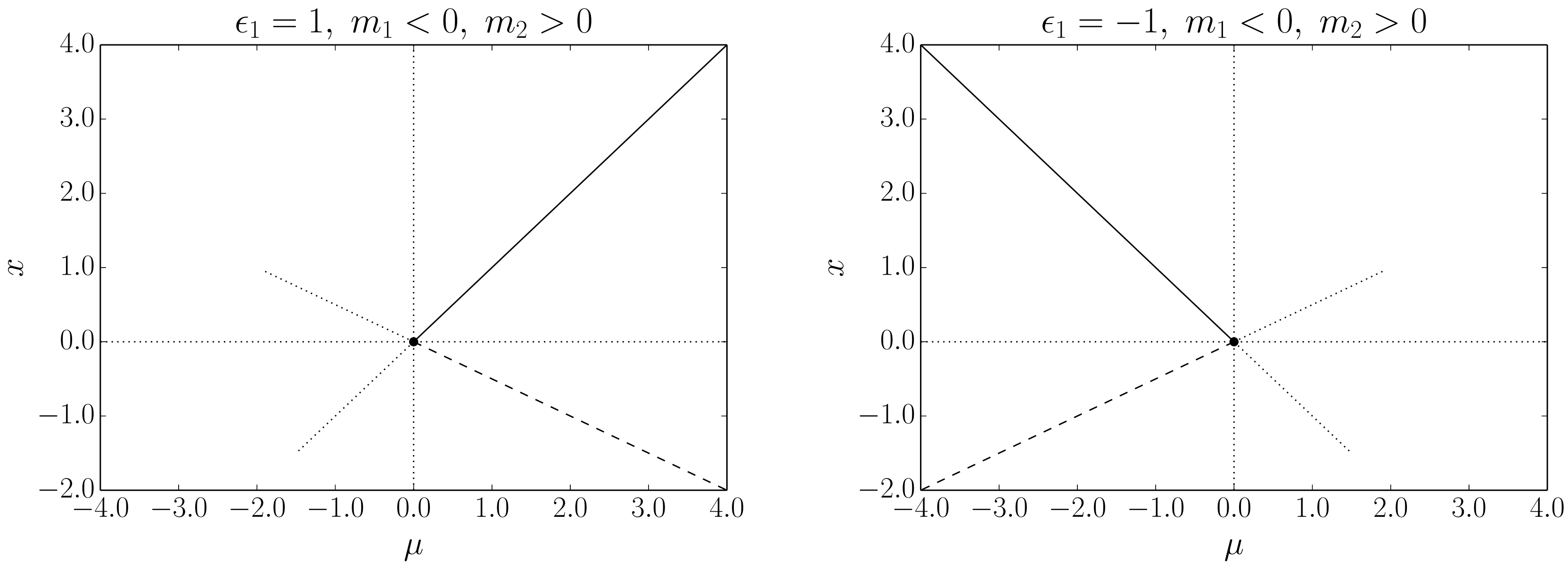
Limit-Induced Bifurcation (VIII)

- Illustration of the limit-induced bifurcation through the normal form (32). Solid branches are stable; dashed branches are unstable.



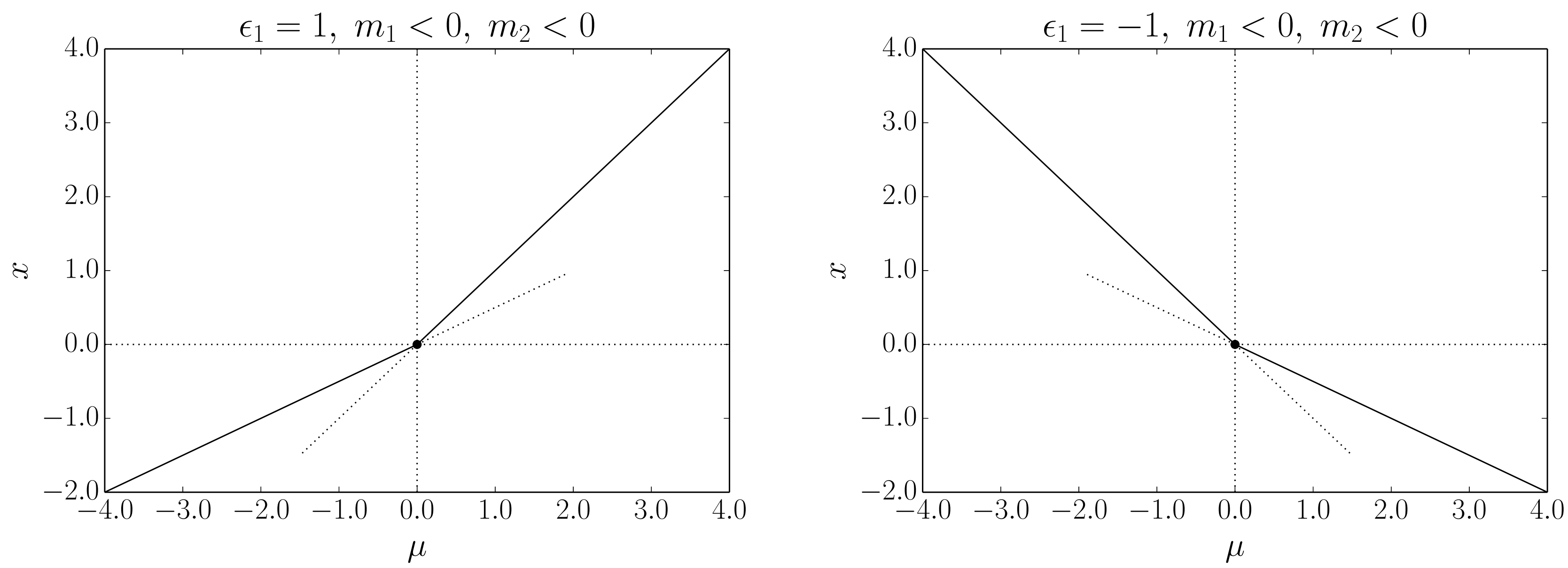
Limit-Induced Bifurcation (IX)

- Illustration of the limit-induced bifurcation through the normal form (32). Solid branches are stable; dashed branches are unstable.



Limit-Induced Bifurcation (X)

- Illustration of the limit-induced bifurcation through the normal form (32). Solid branches are stable; dashed branches are unstable.



Singularity-Induced Bifurcation (I)

- Singularity induced bifurcations only occurs in differential algebraic equations and are characterized by an equilibrium point for which an eigenvalue of the system Jacobian becomes unbounded.

- Consider a parametrized semiexplicit differential algebraic equation (DAE):

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \mu) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \mu) \end{aligned} \tag{34}$$

- Define:

$$\Delta(\boldsymbol{x}, \boldsymbol{y}, \mu) = \det(\boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}, \mu)) \tag{35}$$

- Assume that $(\boldsymbol{x}_0, \boldsymbol{y}_0, \mu_0)$ is an equilibrium of (34).

Singularity-Induced Bifurcation (II)

- Then, $(\boldsymbol{x}_0, \boldsymbol{y}_0, \mu_0)$ is a singularity-induced bifurcation point if:

$$\boldsymbol{g}_{\boldsymbol{y}} \boldsymbol{y}_p = \mathbf{0} \tag{36}$$

$$\det\left(\begin{bmatrix} \boldsymbol{f}_{\boldsymbol{x}} & \boldsymbol{f}_{\boldsymbol{y}} \\ \boldsymbol{g}_{\boldsymbol{x}} & \boldsymbol{g}_{\boldsymbol{y}} \end{bmatrix}\right) = \det(\boldsymbol{A}_C) \neq 0$$

$$\det\left(\begin{bmatrix} \boldsymbol{f}_{\boldsymbol{x}} & \boldsymbol{f}_{\boldsymbol{y}} & \boldsymbol{f}_{\mu} \\ \boldsymbol{g}_{\boldsymbol{x}} & \boldsymbol{g}_{\boldsymbol{y}} & \boldsymbol{g}_{\mu} \\ \Delta_{\boldsymbol{x}} & \Delta_{\boldsymbol{y}} & \Delta_{\mu} \end{bmatrix}\right) \neq 0$$

  where $\boldsymbol{y}_p$ is a nonzero right eigenvector of $\boldsymbol{g}_{\boldsymbol{y}}$.

- The first condition of (36) implies that $\boldsymbol{g}_{\boldsymbol{y}}$ has a simple zero eigenvalue.

Singularity-Induced Bifurcation (III)

- Let define:

$$b = -\text{trace}\,\left(\boldsymbol{f_y}\ \text{adj}(\boldsymbol{g_y})\ \boldsymbol{g_x}\right) \tag{37}$$

$$c = \Delta_\mu - \begin{bmatrix} \Delta_{\boldsymbol{x}} & \Delta_{\boldsymbol{y}} \end{bmatrix} \begin{bmatrix} \boldsymbol{f_x} & \boldsymbol{f_y} \\ \boldsymbol{g_x} & \boldsymbol{g_y} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{f}_\mu \\ \boldsymbol{g}_\mu \end{bmatrix}$$

  where adj$(\bullet)$ is the adjoint (i.e., the transpose of the cofactor matrix) of a given matrix and trace$(\bullet)$ is the sum of the diagonal elements of a given matrix.

- Then, if μ increases through μ_0, one eigenvalue of $\boldsymbol{s_x} = \boldsymbol{f_x} - \boldsymbol{f_y}\boldsymbol{g_y}^{-1}\boldsymbol{g_x}$ evaluated along the equilibrium locus, moves from $\mathbb{R}^-$ to $\mathbb{R}^+$ if $b/c > 0$ (respectively, from $\mathbb{R}^+$ to $\mathbb{R}^-$ if $b/c < 0$) along the real axis by diverging through ∞.

Singularity-Induced Bifurcation (IV)

- Since $\boldsymbol{A}_C$ is invertible, it is always possible to rewrite (34) as follows:

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \tilde{\boldsymbol{y}}, \hat{\boldsymbol{y}}, \mu) \\
\boldsymbol{T}_\epsilon \dot{\tilde{\boldsymbol{y}}} &= \tilde{\boldsymbol{g}}(\boldsymbol{x}, \tilde{\boldsymbol{y}}, \hat{\boldsymbol{y}}, \mu) \\
\boldsymbol{0} &= \hat{\boldsymbol{g}}(\boldsymbol{x}, \tilde{\boldsymbol{y}}, \hat{\boldsymbol{y}}, \mu)
\end{aligned}
\tag{38}
$$

  where $\tilde{\boldsymbol{y}}$ and $\hat{\boldsymbol{y}}$ ($\boldsymbol{y} = [\tilde{\boldsymbol{y}}^T, \hat{\boldsymbol{y}}^T]^T$) indicate *slow* and *fast* algebraic variables, respectively, and $\boldsymbol{T}_\epsilon$ is a diagonal matrix of *small* time constants, so that $\hat{\boldsymbol{g}}_{\hat{\boldsymbol{y}}}$ is nonsingular at the equilibrium.

- Thus, in practice, singularity-induced bifurcations can be eliminated by reformulating the DAE system (34).

- However, identifying $\tilde{\boldsymbol{y}}$ might not be trivial.

Alternative Way to Obtain an ODE from a DAE (I)

- Let consider again the original DAE system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \mu) \\ \boldsymbol{0} &= \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \mu) \end{aligned} \tag{39}$$

- Deriving the algebraic equations with respect of time leads to:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \mu) \\ \boldsymbol{0} &= \boldsymbol{g}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}, \mu)\dot{\boldsymbol{x}} + \boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}, \mu)\dot{\boldsymbol{y}} \end{aligned} \tag{40}$$

- And, if $\boldsymbol{g}_{\boldsymbol{y}}$ is not singular, one can define the following ODE:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \mu) \\ \dot{\boldsymbol{y}} &= -\boldsymbol{g}_{\boldsymbol{y}}^{-1}(\boldsymbol{x}, \boldsymbol{y}, \mu)\boldsymbol{g}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}, \mu)\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \mu) \end{aligned} \tag{41}$$

Alternative Way to Obtain an ODE from a DAE (II)

- If $\boldsymbol{g}_{\boldsymbol{y}}$ is singular, one can derive again the second equation of (40) until $\dot{\boldsymbol{y}}$ (and its higher order derivatives) can be defined.
- The number of derivatives needed to define the ODE is the index of the Hessenberg form of the original DAE (39).
- To know the index of the DAE is useful to define the robustness of the numerical integration scheme.
- Only index 1 DAEs can be safely integrated through stiff implicit formulas.
- However, *good* integration schemes (e.g., BDF methods) can also generally handle index 2 DAEs and DAEs whose index is greater than 2 are quite uncommon in pratice.

Alternative Way to Obtain an ODE from a DAE (III)

- Previous equations seem to solve the problem of defining an ODE based on a DAE, however . . .
- . . . in (41), $\boldsymbol{y}$ must be continuous, while in (39), the algebraic variables can *jump*.
- Hence, unfortunately, the solutions of (41) are **NOT** necessarily the same as those of (39).
- Then, typically, (41) has *more* solutions than (39).
- Hence, this approach cannot be used to avoid DAEs . . .

Singularity-Induced Bifurcation (V)

- Let consider the following example:

$$\dot{x} = \mu - x - \epsilon_1 y \tag{42}$$
$$0 = x + \mu(y + 1)$$

where $\epsilon_1 = \pm 1$.

- In this case, the algebraic variable y can be obtained explicitly as a function of x and μ, but let keep the DAE for the sake of example.

Singularity-Induced Bifurcation (VI)

- The equilibrium $(x_0, y_0, \mu_0) = (0, 0, 0)$ is a singularity-induced bifurcation point, in fact:

$$g_y = \mu_0 = 0 \tag{43}$$

$$\det\left(\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1 & -\epsilon_1 \\ 1 & \mu_0 \end{bmatrix}\right) = \epsilon_1$$

$$\det\left(\begin{bmatrix} f_x & f_y & f_\mu \\ g_x & g_y & g_\mu \\ \Delta_x & \Delta_y & \Delta_\mu \end{bmatrix}\right) = \det\left(\begin{bmatrix} -1 & -\epsilon_1 & 1 \\ 1 & \mu_0 & (1 + y_0) \\ 0 & 0 & 1 \end{bmatrix}\right) = \epsilon_1$$

  where $\Delta(x, y, \mu) = \mu$.

- Moreover, $c = 1$.

Singularity-Induced Bifurcation (VII)

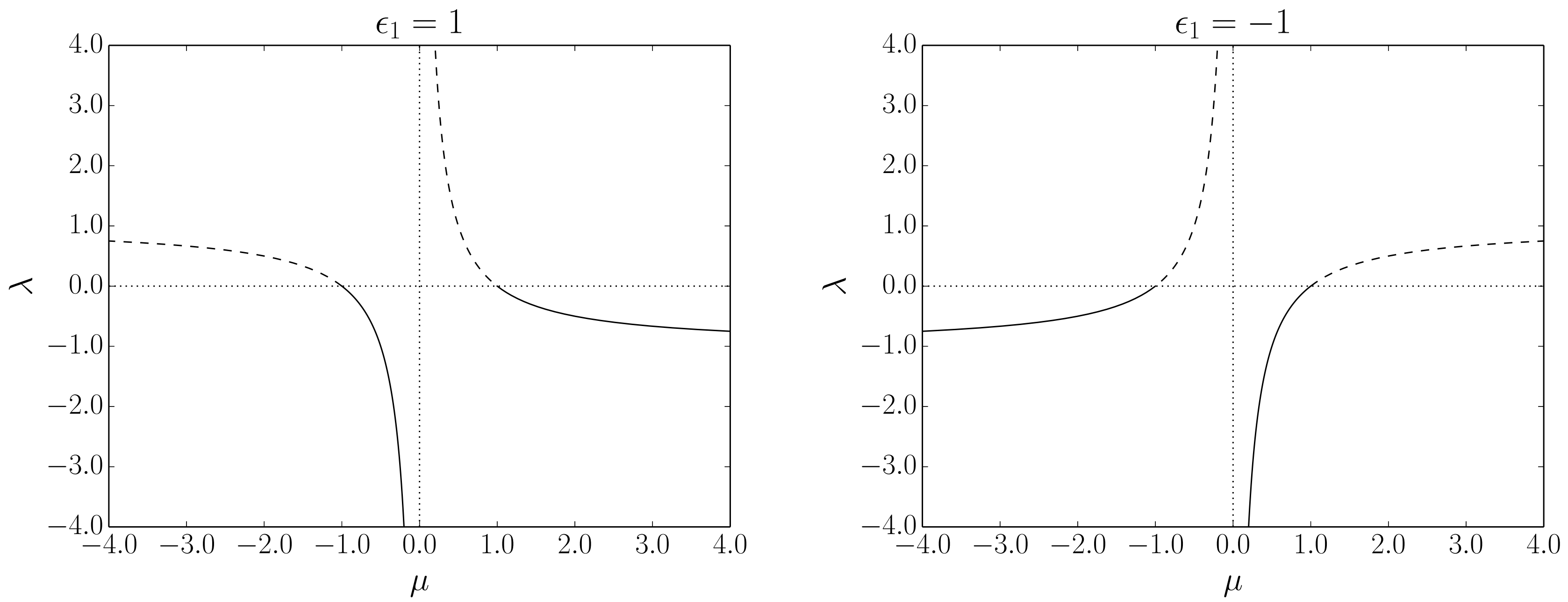
- Since g_y is scalar, the adjoint matrix of g_y (and thus b) is not defined.
- However, observe that:

$$\rho = f_x - f_y \frac{g_x}{g_y} = -1 + \frac{\epsilon_1}{\mu_0} \tag{44}$$

- Hence, as μ increases through μ_0, if $\epsilon_1 = 1$ (respectively, $\epsilon_1 = -1$), the eigenvalue ρ of (42) moves from $\mathbb{R}^-$ ($\mathbb{R}^+$) to $\mathbb{R}^+$ ($\mathbb{R}^-$) along the real axis by diverging through ∞.

Singularity-Induced Bifurcation (VIII)

- Illustration of the singularity-induced bifurcation occurring in (44). Solid branches are stable; dashed branches are unstable.



Singularity-Induced Bifurcation (IX)

- One can easily remove the singularity-induced bifurcation by rewriting (42) as follows:

$$\begin{aligned} \dot{x} &= \mu - x - \epsilon_1 \hat{y} \\ T_\epsilon \dot{\hat{y}} &= x + \mu(\hat{y} + 1) \end{aligned} \tag{45}$$

  for which the eigenvalues at the equilibrium $(x_0, \hat{y}_0, \mu_0) = (0, 0, 0)$ are:

$$\rho_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\epsilon_1}{T_\epsilon}} \tag{46}$$

- Proposed exercise: find the conditions on T_ϵ for the system (45) to be stable. Assume $T_\epsilon > 0$, and $\epsilon_1 = \pm 1$.

Pitchfork Bifurcation (I)

- The picthfork bifurcation is another bifurcation in which a fixed point exists for all values of a parameter and is never destroyed.
- Two branches intersect at the bifurcation point so that there are two possible situations:

  Super-critical case: there are intervals having a single stable fixed point and three fixed points, two of which are stable and one of which is unstable.

  Sub-critical case: there are intervals having a single unstable fixed point and three fixed points, two of which are unstable and one of which is stable.

Pitchfork Bifurcation (II)

- The conditions for $(\boldsymbol{x}_0, \mu_0)$ to be a pitchfork bifurcation point are:

  1. $\boldsymbol{s}$ is an odd function, i.e.,

  $$\boldsymbol{s}(-\boldsymbol{x}, \mu) = -\boldsymbol{s}(\boldsymbol{x}, \mu) \tag{47}$$

  2. The following transversality conditions hold:

  $$\begin{aligned} (\boldsymbol{s}_{\boldsymbol{x}}\boldsymbol{v})^T = \boldsymbol{w}^T\boldsymbol{s}_{\boldsymbol{x}} &= \boldsymbol{0} \\ \boldsymbol{w}^T\boldsymbol{s}_{\mu} &= 0 \\ \boldsymbol{w}^T\boldsymbol{s}_{\boldsymbol{x}\boldsymbol{x}}\boldsymbol{v}\boldsymbol{v} &= 0 \\ \boldsymbol{w}^T\boldsymbol{s}_{\boldsymbol{x}\mu}\boldsymbol{v} &\neq 0 \\ \boldsymbol{w}^T\boldsymbol{s}_{\boldsymbol{x}\boldsymbol{x}\boldsymbol{x}}\boldsymbol{v}\boldsymbol{v}\boldsymbol{v} &\neq 0 \end{aligned} \tag{48}$$

Pitchfork Bifurcation (III)

- The kind of the pitchfork bifurcation is given by the sign of the third derivative:

$$\boldsymbol{w}^T \boldsymbol{s}_{\boldsymbol{xxx}} \boldsymbol{vvv} = \begin{cases} < 0, & \text{super-critical} \\ > 0, & \text{sub-critical} \end{cases} \tag{49}$$

- Equation (14) and the first and the second of (48) do not lead to any condition on μ_s.

- From (15) and the third and the fourth of (48), one obtains:

$$2(\boldsymbol{w}^T \boldsymbol{s}_{\boldsymbol{x}\mu}) \boldsymbol{x}_p \mu_p + (\boldsymbol{w}^T \boldsymbol{s}_{\mu\mu}) \mu_p \mu_p = 0 \tag{50}$$

from which it follows that $\mu_s = 0$ if and only if $\boldsymbol{w}^T \boldsymbol{s}_{\mu\mu} = 0$.

Pitchfork Bifurcation (IV)

- The normal form of the pitchfork bifurcation is:

$$\dot{x} = \epsilon_1 \mu x + \epsilon_2 x^3 \tag{51}$$

where $\epsilon_1 = \pm 1$ and $\epsilon_2 \pm 1$ and with bifurcation point $(x_0, \mu_0) = (0, 0)$.

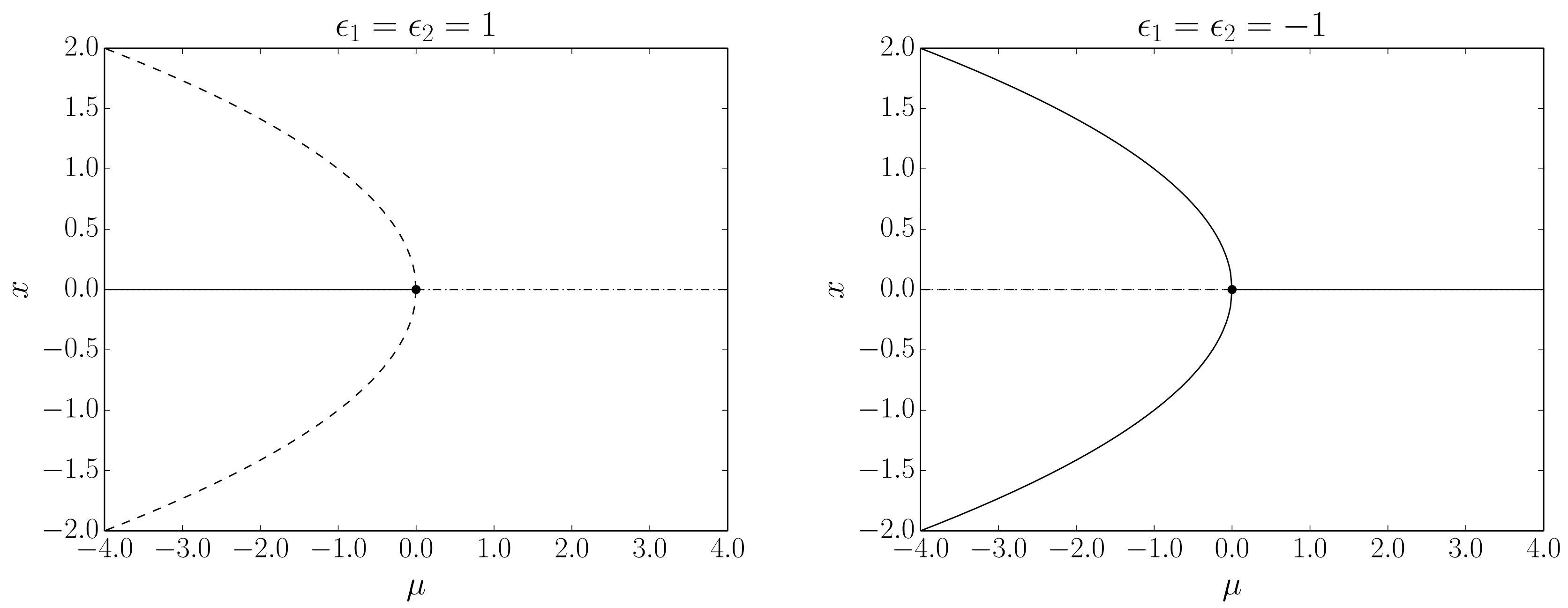
- The conditions (48) applied to (24) become:

$$\begin{aligned} s_x &= \epsilon_1 \mu_0 + 3\epsilon_2 x_0^2 = 0 \\ s_\mu &= \epsilon_1 x_0 = 0 \\ s_{xx} &= 6\epsilon_2 x_0 = 0 \\ s_{x\mu} &= \epsilon_1 \neq 0 \\ s_{xxx} &= 6\epsilon_2 \neq 0 \end{aligned} \tag{52}$$

- Hence, sign$(s_{xxx}) = \epsilon_2$, if $\epsilon_2 = -1$, the bifurcation is super-critical, sub-critical otherwise.

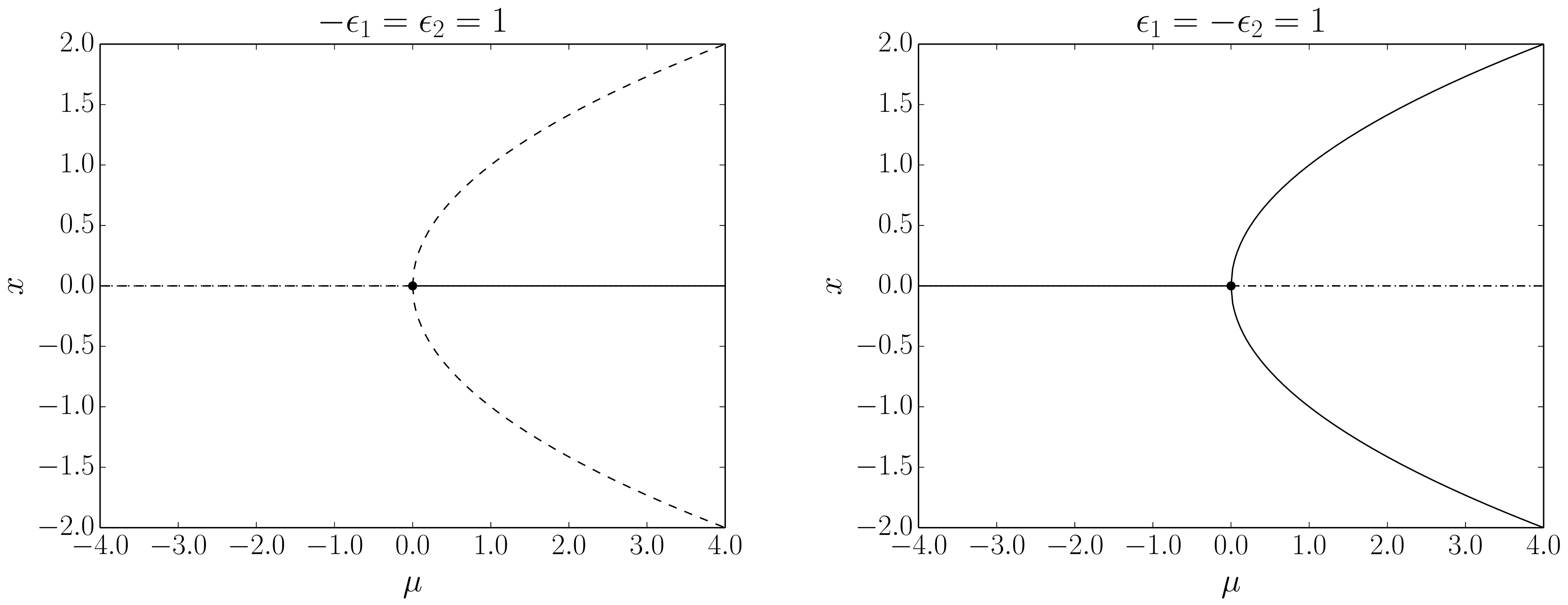
Pitchfork Bifurcation (V)

- Illustration of the pitchfork bifurcation occurring in (51). Solid branches are stable; dashed branches are unstable.



Pitchfork Bifurcation (VI)

- Illustration of the pitchfork bifurcation occurring in (51). Solid branches are stable; dashed branches are unstable.



Hopf Bifurcation (I)

- A Hopf or Andronov-Hopf bifurcation is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane.
- Under reasonably generic assumptions about the dynamical system, the Hopf bifurcation is the birth of a limit cycle from an equilibrium in dynamical systems generated by ODEs, when the equilibrium changes stability via a pair of purely imaginary eigenvalues.
- The bifurcation can be super-critical or sub-critical, resulting in stable or unstable (within an invariant two-dimensional manifold) limit cycle, respectively.

Hopf Bifurcation (II)

- The condition for the equilibrium to be a Hopf bifurcation is that at the equilibrium the Jacobian matrix $\boldsymbol{s_x}$ has one pair of complex eigenvalues:

$$\lambda(\mu) = \alpha(\mu) \pm j\omega(\mu) \tag{53}$$

  that becomes purely imaginary at μ_0, i.e., $\alpha(\mu_0) = 0$ and $\omega(\mu_0) = \omega_0 > 0$.

- Moreover, generically, as μ passes through $\mu = \mu_0$, the equilibrium changes stability and a unique limit cycle bifurcates from it.

Hopf Bifurcation (III)

- Formally, the nondegeneracy conditions of the Hopf bifurcations are:

$$l_1(\boldsymbol{x}_0, \mu_0) \neq 0 \tag{54}$$

$$\left.\frac{d\lambda}{d\mu}\right|_{\boldsymbol{x}_0, \mu_0} \neq 0$$

where $l_1(\mu_0)$ is the first Lyapunov's coefficient that is defined below.

Lyapunov Coefficient (I)

- Let $\boldsymbol{v} \in \mathbb{C}^n$ be a complex eigenvector of $\boldsymbol{s_x}$ corresponding to the eigenvalue $j\omega_0$: $\boldsymbol{s_x v} = j\omega_0 \boldsymbol{v}$.
- Introduce also the adjoint eigenvector $\boldsymbol{w} \in \mathbb{C}^n$: $\boldsymbol{s_x}^T \boldsymbol{w} = -j\omega_0 \boldsymbol{w}$, $\langle \boldsymbol{w}, \boldsymbol{v} \rangle = 1$,
- where $\langle \boldsymbol{w}, \boldsymbol{v} \rangle = \boldsymbol{w}^H \boldsymbol{v} = (\boldsymbol{w}^*)^T \boldsymbol{v}$ is the inner product in $\mathbb{C}^n$.

Lyapunov Coefficient (II)

- Then:

$$l_1(\boldsymbol{x}_0, \mu_0) = \frac{1}{2\omega_0}\Re\{\langle \boldsymbol{w}, \boldsymbol{s}_{\boldsymbol{xxx}}\boldsymbol{vvv}^*\rangle - 2\langle \boldsymbol{w}, \boldsymbol{s}_{\boldsymbol{xx}}\boldsymbol{v}(\boldsymbol{s}_{\boldsymbol{x}}^{-1}(\boldsymbol{s}_{\boldsymbol{xx}}\boldsymbol{vv}^*))\rangle + \quad (55)$$
$$\langle \boldsymbol{w}, \boldsymbol{s}_{\boldsymbol{xx}}\boldsymbol{v}^*((2j\omega_0\boldsymbol{I}_n - \boldsymbol{s}_{\boldsymbol{x}})^{-1}(\boldsymbol{s}_{\boldsymbol{xx}}\boldsymbol{vv}))\rangle\}$$

  where $\boldsymbol{I}_n$ is the unit $n \times n$ matrix.

- Note that the value but not the sign of $l_1(\mu_0)$ depends on the scaling of the eigenvector $\boldsymbol{v}$.

- The normalization $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 1$ is one of the options to remove this ambiguity.

Hopf Bifurcation (IV)

- The normal form of the Hopf bifurcation with $(\boldsymbol{x}_0, \mu_0) = ((x_{1,0}, x_{2,0})^T, \mu_0) = (\mathbf{0}, 0)$ is the following:

$$\begin{aligned} \dot{x}_1 &= \mu x_1 - x_2 + \sigma x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + \mu x_2 + \sigma x_2(x_1^2 + x_2^2) \end{aligned} \tag{56}$$

  where $\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2$, $\mu \in \mathbb{R}$, and $\sigma = \pm 1$.

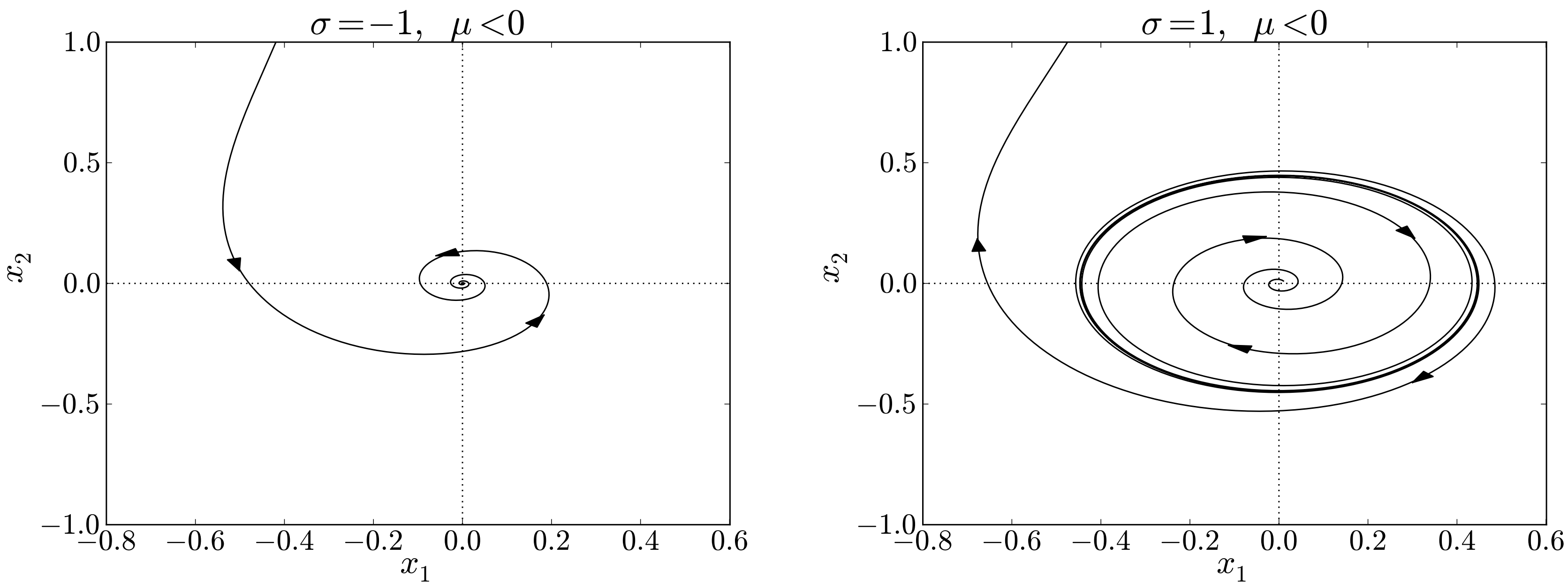
- The sign of σ defines whether the Hopf bifurcation is super- or sub-critical.

Hopf Bifurcation (V)

- *Super-critical Hopf bifurcation*: if $\sigma = -1$, the normal form has an equilibrium at the origin, which is asymptotically stable for $\mu \leq 0$ (weakly at $\mu = 0$) and unstable for $\mu > 0$. Moreover, there is a unique and stable circular limit cycle that exists for $\mu > 0$ and has radius $\sqrt{\mu}$.
- *Sub-critical Hopf bifurcation*: if $\sigma = +1$, the origin in the normal form is asymptotically stable for $\mu < 0$ and unstable for $\mu \geq 0$ (weakly at $\mu = 0$), while a unique and unstable limit cycle exists for $\mu < 0$.

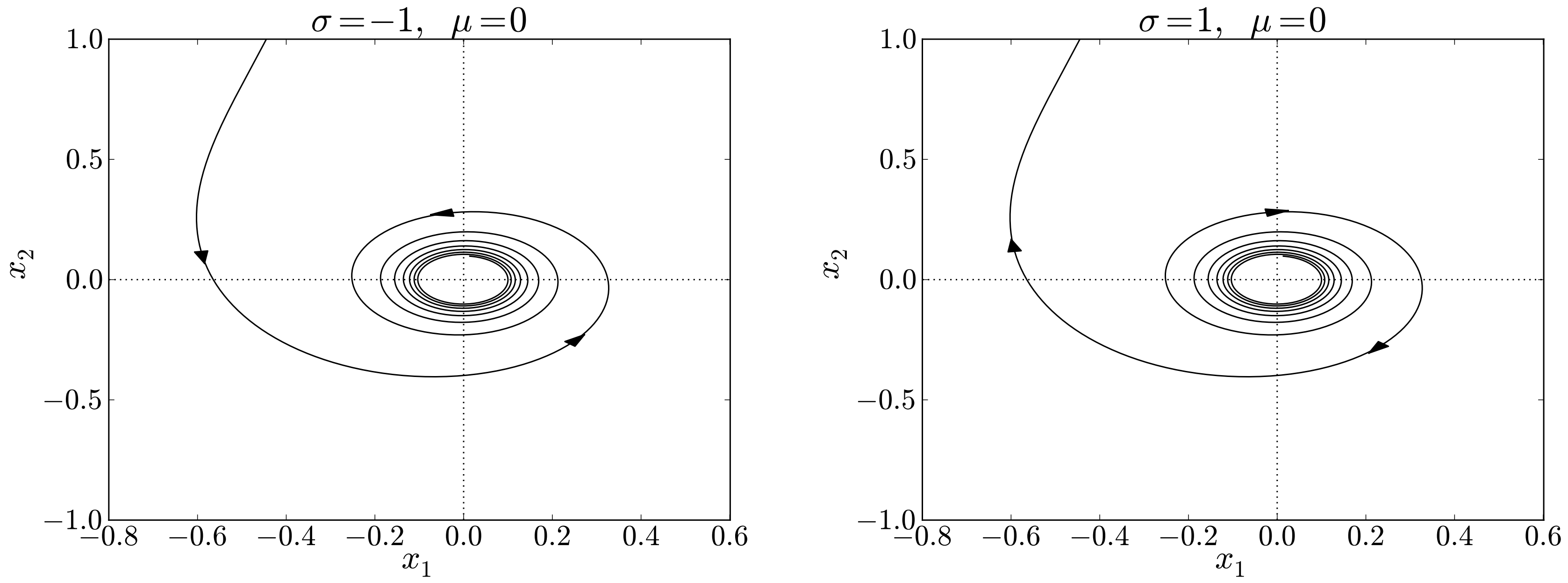
Hopf Bifurcation (VI)

- Illustration of a super-critical Hopf bifurcation (left) and a sub-critical Hopf bifurcation (right) as a function of μ and σ.



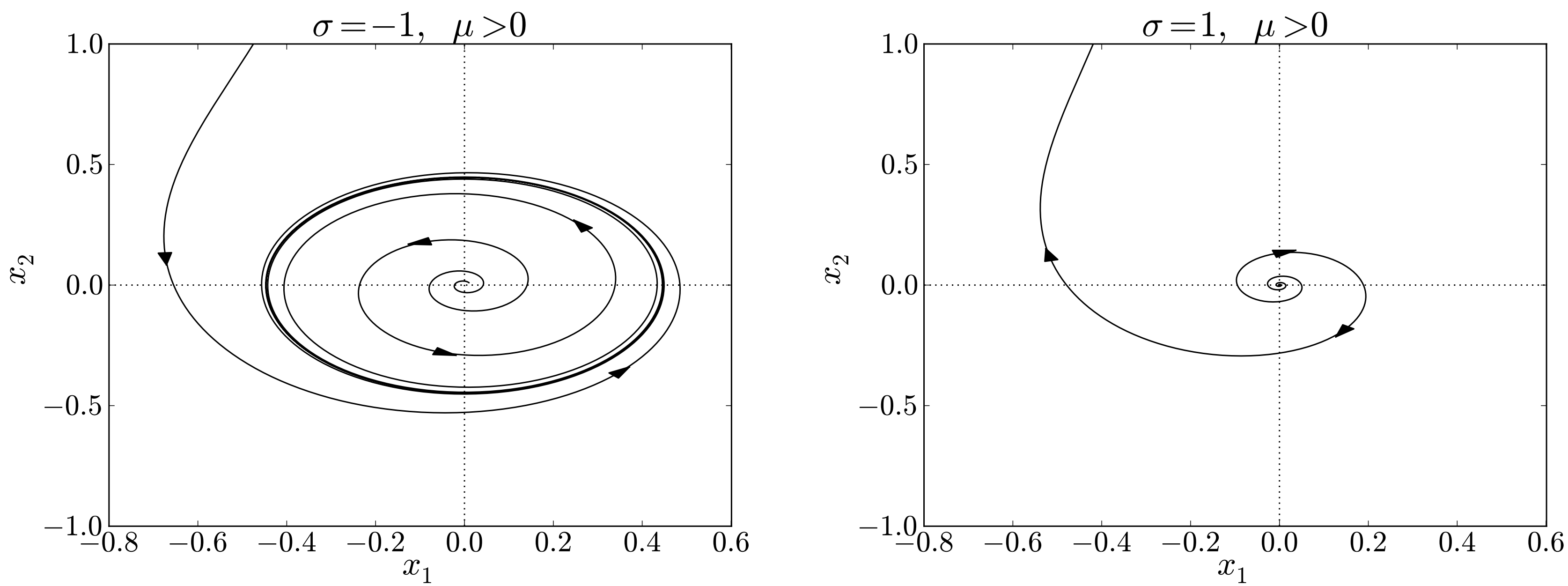
Hopf Bifurcation (VII)

- Illustration of a super-critical Hopf bifurcation (left) and a sub-critical Hopf bifurcation (right) as a function of μ and σ.



Hopf Bifurcation (VIII)

- Illustration of a super-critical Hopf bifurcation (left) and a sub-critical Hopf bifurcation (right) as a function of μ and σ.



Other Bifurcations

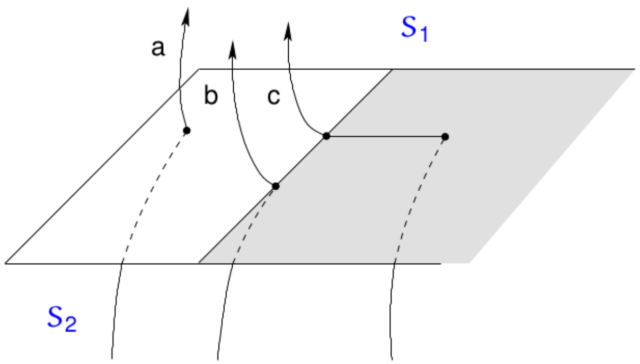
- Hysteresis bifurcation, e.g.:

$$\dot{x} = \mu - x^3$$

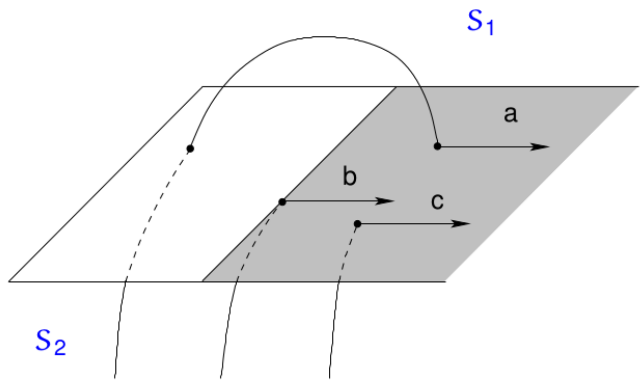
- Sliding bifurcation.
- etc.

Illustration of the Sliding Bifurcations

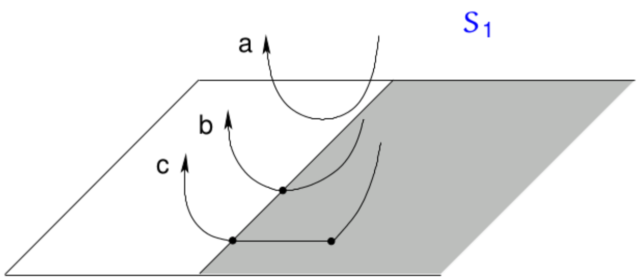
- There exists several different cases of sliding bifurcations:



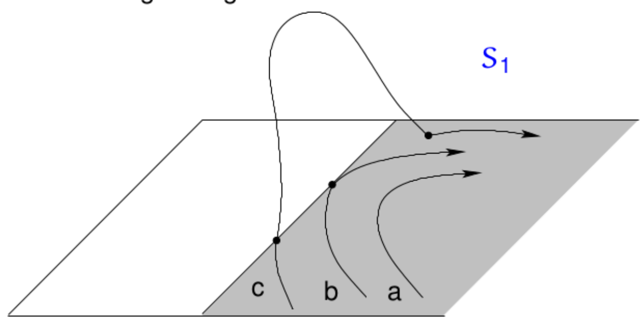
crossing-sliding



switching-sliding



grazing-sliding



adding-sliding

Most Relevant Bifurcations for Power Systems

- Saddle-node bifurcation and limit-induced bifurcation for voltage stability analysis.
- Hopf bifurcation for small signal stability analysis.