# Complex Numbers

Brendan Hayes

University College Dublin

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# Imaginary Numbers

- What is the square root of $-1$ i.e. $\sqrt{-1}$
- It is clear that no real number can satisfy this since any real number squared is positive
- So we made up a number to satisfy is:

$$i$$

- We say that $i$ is imaginary because there is no real number that satisfies the equation $x^2 = -1$

# Complex Numbers - The Basics

- It has one key property

$$i = \sqrt{-1}$$
$$i^2 = -1$$

- A complex number is just a point on a graph
- Instead of using the $x$-axis and $y$-axis, we use the real and imaginary axis

# The Basics - Examples

- Find $x + y$, $x - y$ and $x.y$:

1. $x = -3 + i4$, $y = 6 - i8$
2. $x = -12 - i16$, $y = 15 + i20$
3. $x = 6 + 3i$,$y = -5 + 2i$

# Complex Conjugate Method

- For any given complex number $z = a + ib$, its complex conjugate, $\bar{z}$, is given by $\bar{z} = a - ib$
- To divide by a complex number, we multiply by the complex conjugate:

$$\frac{a+ib}{c+id} \rightarrow \frac{a+ib}{c+id}.\frac{c-id}{c-id} \rightarrow \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$\frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

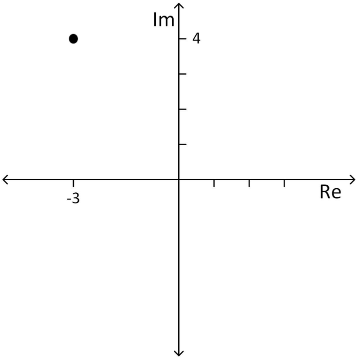
# Complex Conjugate Method - Examples

- Find $x/y$ and express in the form $a + ib$

1. $x = -3 + i4, y = 6 - i8$
2. $x = -12 - i16, y = 15 + 20i$
3. $x = 5 + i\sqrt{5}, y = 5 - i\sqrt{5}$

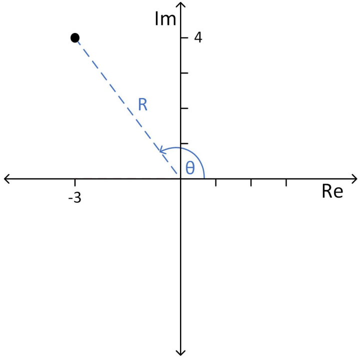
# NB*** Polar to Rectangular ***NB

- Consider a complex number of the form $z = -3 + 4i$

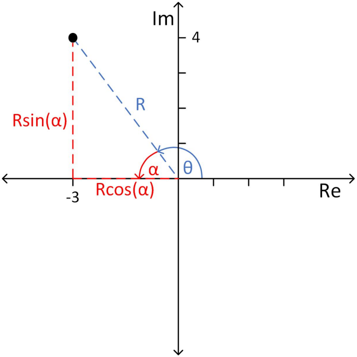


# NB*** Polar to Rectangular ***NB

- Why not represent the complex co-ordinate as the distance from $(0,0)$ and angle to the $x$-axis?



# NB*** The Steps: Polar to Rectangular ***NB




$$R = \sqrt{3^2 + 4^2} = 5$$

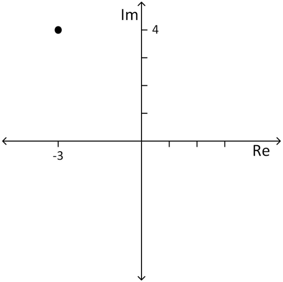
$$\alpha = tan{-}1(4/3) = 0.927$$

$$\theta = \pi - 0.927 = 2.214$$

$$-3 + i4 = 5\angle 2.214$$

# NB*** The Steps: Polar to Rectangular ***NB

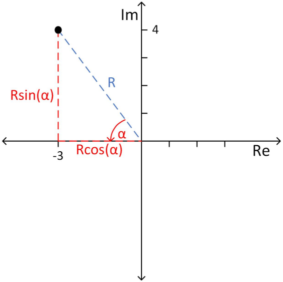
Given $z = a + ib$, write in the form $R\angle\theta$




**Step 1:** Draw the point

# NB*** The Steps: Polar to Rectangular ***NB

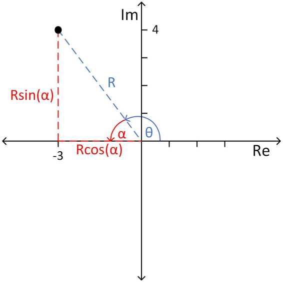
Given $z = a + ib$, write in the form $R\angle\theta$




**Step 1:** Draw the point
**Step 2:** Find $R$ and reference angle $\alpha$
$R = \sqrt{(a^2 + b^2)}$ $\alpha = tan^{-1}(b/a)$

# NB*** The Steps: Polar to Rectangular ***NB

Given $z = a + ib$, write in the form $R\angle\theta$

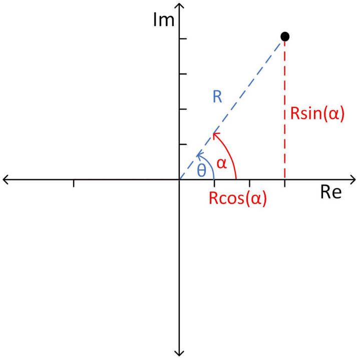



**Step 1:** Draw the point
**Step 2:** Find $R$ and reference angle $\alpha$
$R = \sqrt{(a^2 + b^2)}$ $\qquad \alpha = tan^{-1}(b/a)$
**Step 3:** Find angle $\theta$

# What is $\theta$?

- If you are in the **first quadrant**

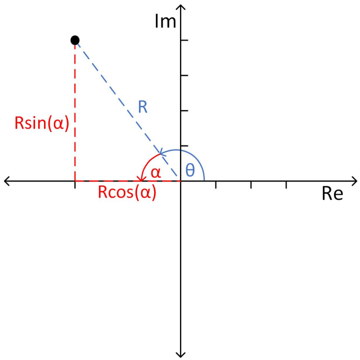
$$\theta = \alpha$$



# What is $\theta$?

- If you are in the **second quadrant**

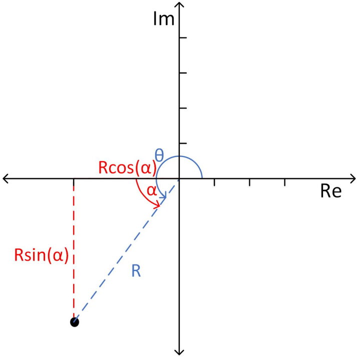
$$\theta = \pi - \alpha$$



# What is $\theta$?

- If you are in the **third quadrant**

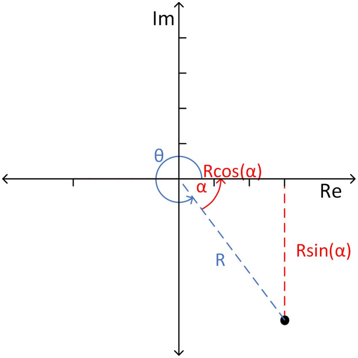
$$\theta = \pi + \alpha$$



# What is $\theta$?

- If you are in the **forth quadrant**

$$\theta = 2\pi - \alpha$$



# NB*** Polar to Rectangular ***NB

- Express the following in the form $R\angle\theta$

1. $6 - i8$

2. $-12 - i16$

3. $15 + i20$

# Rectangular to Polar

- Converting back is much easier:

$$R\angle\theta = R(cos(\theta) + isin(\theta))$$

- To express in rectangular co-ordinates $a + ib$:

$$a = Rcos(\theta) \text{ and } b = Rsin(\theta)$$

# Rectangular to Polar

- Express the following in the form $a + ib$:

1. $25\angle 0.927$
2. $30\angle 4.088$
3. $2(\cos(\pi/6) + i\sin(\pi/6))$

# Phasor Format

- Why is this useful?
- There is a third representation, the exponential representation (also known as Phasors)
- Polar co-ordinates $R\angle\theta$ are written as:

$$R(cos(\theta) + isin(\theta))$$

- Eulers formula states:

$$Re^{j\theta} = R(cos(\theta) + isin(\theta))$$

- This makes multiplication and division **much** easier

# Phasor Multiplication

- Given the two complex numbers $z_1 = 45\angle\frac{3\pi}{4}$ and $z_2 = 5\angle\frac{\pi}{4}$. Find $z_1.z_2$
- We know $z_1 = 45e^{3\pi/4}$ and $z_2 = 5e^{\pi/4}$
- We know: $e^A.e^B = e^{(A+B)}$

$$
\begin{aligned}
z_1.z_2 &= (45)(5)e^{(3\pi/4)+(\pi/4)} \\
&= 225e^{4\pi/4} \\
&= 225e^{\pi}
\end{aligned}
$$

# Phasor Divison

- Given the two complex numbers $z_1 = 45\angle\frac{3\pi}{4}$ and $z_2 = 5\angle\frac{\pi}{4}$. Find $z_1/z_2$
- We know $z_1 = 45e^{3\pi/4}$ and $z_2 = 5e^{\pi/4}$
- We know: $e^A/e^B = e^{(A-B)}$

$$
\begin{aligned}
z_1.z_2 &= (45/5)e^{(3\pi/4)-(\pi/4)} \\
&= 9e^{2\pi/4} \\
&= 9e^{\pi/2}
\end{aligned}
$$

# Phasor Examples 1

- Convert the following complex numbers to polar co-ordinates and find $x.y$

1. $x = -3 + i4$, $y = 6 - i8$

2. $x = -12 - i16$, $y = 15 + i20$

3. $x = 6 + 3i$,$y = -5 + 2i$

# Phasor Examples 2

- Convert the following complex numbers to polar co-ordinates and find $x/y$:

1. $x = -3 + i4$, $y = 6 - i8$
2. $x = -12 - i16$, $y = 15 + i20$
3. $x = 5 + i\sqrt{5}, \quad y = 5 - \sqrt{5}$