Definitions and Formulæ

EEEN20090 – Electric Energy Systems

1 Trigonometry

- $\cos^2\alpha + \sin^2\alpha = 1$
- $\tan\alpha = \sin\alpha / \cos\alpha$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$
- $\sin 2\alpha = 2\sin\alpha\cos\alpha$
- $\sin\alpha\sin\beta = 0.5[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
- $\cos\alpha\cos\beta = 0.5[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
- $\sin\alpha\cos\beta = 0.5[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$
- $\cos\alpha\sin\beta = 0.5[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

2 Complex Numbers

- Imaginary unit:
$$j = \sqrt{-1}$$
- Euler's identity:
$$e^{j\pi} = -1$$
- Complex number notation:
$$\bar{c} = a + jb = c\angle\alpha = c \cdot e^{j\alpha} = c(\cos\alpha + j\sin\alpha)$$
- Magnitude:
$$c = |\bar{c}| = \sqrt{a^2 + b^2}$$

- Phase angle:

$$\alpha = \arctan(b/a)$$

or, equivalently:

$$\cos\alpha = \frac{a}{c}\,, \qquad \sin\alpha = \frac{b}{c}$$

- Complex conjugate:

$$\bar{c}^* = a - jb$$

3 Magnetic Circuits and Energy Conversion

- Ampere's law:

$$\oint_\ell \vec{H} \cdot d\vec{\ell} = Ni\,,$$

where $\vec{H}$ is the magnetic field, ℓ is a closed path, N is the number of turns and i is the current.

- Magnetic induction in a linear magnetic material:

$$\vec{B} = \mu_r \mu_0 \vec{H}\,,$$

where μ_r is the relative permeability and $\mu_0 = 4\pi 10^{-7}$ H/m is the permeability of vacuum.

- Magnetic flux:

$$\phi = \int_A \vec{B} \cdot d\vec{A}\,,$$

- Reluctance $\mathcal{R}$ and permeance $\mathcal{P}$ (only for constant μ_r, i.e., linear magnetic material):

$$\mathcal{R} = \frac{1}{\mathcal{P}} = \frac{\ell}{\mu_r \mu_0 A}$$

- One coil, one mesh of the iron core, linear magnetic material:

$$Ni = \mathcal{R}\phi$$

- Total magnetic flux: $\lambda = N\phi$
- Faraday's law:

$$v = \text{emf} = \frac{d\lambda}{dt}$$

- Inductance:

$$L = \frac{N^2}{\mathcal{R}} = N^2 \mathcal{P}$$

- One coil, one mesh of the iron core, linear magnetic material:

$$\lambda = Li$$

- Faraday's law for a linear magnetic circuit:

$$v = L\frac{di}{dt}$$

- Lorentz force:

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}) = q\vec{E} + i\vec{\ell} \times \vec{B}$$

where q is the electric charge, $\vec{E}$ is the electric field and $\vec{u}$ the speed.

- Magnetic energy W_f:

$$W_f = \int_\lambda i(x, \lambda) d\lambda$$

- Coenergy W'_f:

$$W'_f = \lambda i - W_f = \int_i \lambda(x, i) di$$

- Differentiation of the energy:

$$dW_f = id\lambda - Fdx = \left.\frac{\partial W_f}{\partial \lambda}\right|_x \cdot d\lambda + \left.\frac{\partial W_f}{\partial x}\right|_\lambda \cdot dx$$

- Differentiation of the coenergy:

$$dW'_f = \lambda di + Fdx = \left.\frac{\partial W'_f}{\partial i}\right|_x \cdot di + \left.\frac{\partial W'_f}{\partial x}\right|_i \cdot dx$$

- Expressions of the mechanical force (torque):

$$F = -\left.\frac{\partial W_f}{\partial x}\right|_\lambda = \left.\frac{\partial W'_f}{\partial x}\right|_i$$

- Single coil, linear magnetic material:

$$W_f = W'_f = \frac{1}{2}L(x)i^2$$
$$F = \frac{1}{2}\frac{\partial L(x)}{\partial x}i^2$$

- Mutually coupled pair of coils, linear magnetic material:

$$W_f = \frac{1}{2}L_1(\alpha)i_1^2 + \frac{1}{2}L_2(\beta)i_2^2 + M(\gamma)i_1 i_2$$
$$F = \frac{1}{2}\frac{\partial L_1(\alpha)}{\partial \alpha}i_1^2 + \frac{1}{2}\frac{\partial L_2(\beta)}{\partial \beta}i_2^2 + \frac{\partial M(\gamma)}{\partial \gamma}i_1 i_2$$

4 AC Circuits

- Angular frequency:
$$\omega = 2\pi f \quad [\mathrm{rad/s}]\,,$$
where f is the frequency in Hz.

- AC voltage:
$$v(t) = \sqrt{2}V\sin(\omega t + \theta)\,,$$
where V is the RMS value.

- Phasor:
$$\bar{V} = V\angle\theta = V(\cos\theta + j\sin\theta) = V\exp(j\theta)$$

- Impedance:
$$\bar{Z} = R + jX\,,$$
where R is the resistance and X is the reactance, both measured in Ohms (Ω).

- Amdmittance:
$$\bar{Y} = G + jB\,,$$
where G is the conductance and B is the susceptance, both measured in Siemens or 1/Ohms (Ω^{-1}).

- Relationship between impedance and reactance:
$$\bar{Y} = \frac{1}{\bar{Z}}$$

- Inductive reactance: $X = \omega L$

- Capacitive susceptance: $B = \omega C$, where C is the capacitance.

- Set of symmetrical phase voltages (three-phase system):
$$\begin{aligned}\bar{V}_a &= V_{\mathrm{F}}\angle\theta\\ \bar{V}_b &= V_{\mathrm{F}}\angle\theta - 120^\circ\\ \bar{V}_c &= V_{\mathrm{F}}\angle\theta + 120^\circ\,,\end{aligned}$$
and, hence, $\bar{V}_a + \bar{V}_b + \bar{V}_c = 0$.

- Line-to-line voltages:
$$\begin{aligned}\bar{V}_{ab} &= \bar{V}_a - \bar{V}_b = V_{\mathrm{LL}}\angle\theta + 30^\circ\\ \bar{V}_{bc} &= \bar{V}_b - \bar{V}_c = V_{\mathrm{LL}}\angle\theta - 90^\circ\\ \bar{V}_{ca} &= \bar{V}_c - \bar{V}_a = V_{\mathrm{LL}}\angle\theta + 150^\circ\,,\end{aligned}$$
where $V_{\mathrm{LL}} = \sqrt{3}V_{\mathrm{F}}$.

- Relationship between line current magnitude (I_{L}) and phase current magnitude (I_{F}):

$$I_{\mathrm{F}} = \frac{1}{\sqrt{3}} I_{\mathrm{L}}$$

- Equivalent $Y - \Delta$ connection:

$$\bar{Z}_{\Delta} = 3\bar{Z}_{Y}$$

- Complex power ($\bar{S}$), apparent power (S), power factor ($\cos\phi$), active power (P) and reactive power (Q):

$$\bar{S} = S\cos\phi + jS\sin\phi = P + jQ\,,$$

- Sign of the reactive power:
  - Power factor *lagging*: $Q > 0$ (inductive)
  - Power factor *leading*: $Q < 0$ (capacitive)

- Apparent power:

$$S = \sqrt{P^2 + Q^2}$$

- Complex power in single-phase systems:

$$\bar{S} = \bar{V}\bar{I}^*$$

- Active and reactive power in single-phase systems:

$$P = VI\cos(\phi)$$
$$Q = VI\sin(\phi)\,,$$

  where $\angle\bar{I} = \angle\bar{V} + \phi$.

- Complex power in three-phase systems:

$$\bar{S} = 3\bar{V}_{\mathrm{F}}\bar{I}_{\mathrm{L}}^*$$

- Active and reactive power in three-phase systems:

$$P = 3V_{\mathrm{F}}I_{\mathrm{L}}\cos\phi = \sqrt{3}V_{\mathrm{LL}}I_{\mathrm{L}}\cos\phi = 3V_{\mathrm{LL}}I_{\mathrm{F}}\cos\phi$$
$$Q = 3V_{\mathrm{F}}I_{\mathrm{L}}\sin\phi = \sqrt{3}V_{\mathrm{LL}}I_{\mathrm{L}}\sin\phi = 3V_{\mathrm{LL}}I_{\mathrm{F}}\sin\phi\,,$$

  where $\angle\bar{I}_a = \angle\bar{V}_a + \phi$.

5 Transformers

- Tap ratio:

$$k_T = \frac{V_{1N}}{V_{2N}} \ ,$$

where $V_{!N}$ and V_{2N} are the nominal voltages of the primary and secondary windings, respectively.

- Secondary resistance referred to primary: $R_2' = k_T^2 R_2$
- Secondary reactance referred to primary: $X_2' = k_T^2 X_2$
- Equivalent circuit equations:

$$\begin{aligned}
\bar{V}_1 &= (R_1 + jX_1)\bar{I}_1 + \bar{E}_1 \\
\bar{V}_2' &= -(R_2' + jX_2')\bar{I}_2' + \bar{E}_1 \\
\bar{I}_1 &= \bar{I}_{10} + \bar{I}_2' \\
\bar{I}_{10} &= \frac{\bar{E}_1}{R_{\text{Fe}}} + \frac{\bar{E}_1}{jX_\mu}
\end{aligned}$$

- Short-circuit impedance:

$$\bar{Z}_{\text{sc}}' = R_{\text{sc}}' + jX_{\text{sc}}' = (R_1 + R_2') + j(X_1 + X_2')$$

- Short-circuit voltage:

$$\bar{V}_{\text{sc\%}} = 100\,\frac{\bar{Z}_{\text{sc}}' I_{1N}}{V_{1N}} = 100,\,\frac{\bar{Z}_{\text{sc}}'' I_{2N}}{V_{2N}}$$

- Voltage regulation:

$$\epsilon_V = \frac{V_1 - V_2'}{V_1}$$

- Kapp's formula:

$$V_2' = V_1 - I_2'(R_{sc}' \cos\phi + X_{sc}' \sin\phi)$$

6 Induction Machine

- Angular frequency at the stator:

$$\omega_1 = 2\pi f_1$$

where f_1 is the system frequency, e.g., 50 Hz.

- Number of pair of poles: p
- Synchronous speed: $\omega_{s1} = \omega_1 / p$

- Slip factor:

$$\sigma = \frac{\omega_{s1} - \omega_m}{\omega_{s1}} \,,$$

where ω_m is the mechanical rotor angular speed.

- Rotor current:

$$I_2' = \frac{V_1}{\sqrt{(R_1 + R_2'/\sigma)^2 + (X_{sc}')^2}} \,,$$

where V_1 is the stator phase voltage, and R_1, R_2' and X_{sc}' are the parameters of the equivalent circuit of the machine referred to the stator.

- Torque/slip factor characteristic:

$$T_m = \frac{3p}{\omega_1} {I_2'}^2 \frac{R_2'}{\sigma} = \frac{3p}{\omega_1} \frac{V_1^2}{(R_1 + R_2'/\sigma)^2 + (X_{sc}')^2} \frac{R_2'}{\sigma} \,.$$

- Equivalent resistance of the mechanical load:

$$R_u' = \frac{R_2'}{\sigma} - R_2' = \frac{1-\sigma}{\sigma} R_2'$$

- Mechanical power:

$$P_m = T_m \omega_m = 3 R_u' {I_2'}^2$$

7 Synchronous Generator

- Relationship between internal emf E and excitation current (I_e):

$$E = \psi(I_e)$$

- Rotor angle: δ
- Phasor of the internal emf: $\bar{E} = E\angle\delta$
- Simplified model:

$$\bar{E} = \bar{V} + (R_a + jX_s)\bar{I} \,,$$

where $\bar{E}$ is the internal emf of the machine, $\bar{V}$ is the stator voltage; $\bar{I}$ is the stator current, R_a is the armature resistance, and X_s is the synchronous reactance (assumed to be constant).

- Active power in the simplified model (if $R_a \approx 0$):

$$P = \frac{3EV}{X_s} \sin\delta \,,$$

- Dynamic swing equations:

$$\dot{\delta} = \omega - \omega_{s1}$$
$$M\dot{\omega} = P_m - P(\delta)\,,$$

where ω is the rotor angular speed and ω_{s1} is the synchronous speed. In steady-state, $\omega = \omega_{s1}$ and $P_m = P$. M is the inertia constant.

- Electro-mechanical torque in the simplified model (if $R_a \approx 0$):

$$T = \frac{3p}{\omega_1}\frac{EV}{X_s}\sin\delta$$

- Potier model:

$$\bar{E}_r = \bar{V} + (R_a + jX_d)\bar{I}$$
$$E_r = \psi(I_r) \quad \text{and} \quad \angle\bar{I}_r = \angle\bar{E}_r - 90^\circ$$
$$\bar{I}_e = \bar{I}_r - \frac{\bar{I}}{K_p}$$
$$E = \psi(I_e) \quad \text{and} \quad \angle\bar{I}_e = \angle\bar{E} - 90^\circ$$

where $\bar{E}_r$ is the emf of the machine neto of the armature reaction, and K_p is the Potier coefficient.

- Blondell model:

$$\bar{E} = \bar{V} + jX_d\bar{I}_d + jX_q\bar{I}_q$$
$$\bar{I} = \bar{I}_d + \bar{I}_q$$
$$X_qI_q = V\sin\delta \quad \text{and} \quad X_dI_d = E - V\cos\delta$$
$$P = 3\frac{VE}{X_d}\sin\delta + \frac{3}{2}V^2\left(\frac{1}{X_q} - \frac{1}{X_d}\right)\sin 2\delta$$

or, equivalently:

$$P = 3\frac{E'V}{X_q}\sin\delta$$

where $\bar{E}' = \bar{E} - j(X_d - X_q)\bar{I}_d$.

8 Power Systems

- Power flow equations for a lossless three-phase branch with reactance X_{hk} connecting buses h and k:

$$P_{hk} = 3\frac{V_kV_h}{X_{hk}}\sin(\theta_h - \theta_k)$$
$$Q_{hk} = 3\frac{V_h^2}{}X_{hk} - 3\frac{V_kV_h}{X_{hk}}\cos(\theta_h - \theta_k)$$

- Classification of buses for power flow analysis:
  - Load Bus: knowns P and Q; unknowns V and θ.
  - Generator Bus: knowns P and V; unknowns θ and Q.
  - Slack Bus: knowns V and θ; unknowns P and Q.