Worked Problems on Energy Conversion

EEEN20090 – Electric Energy Systems

Problem 1

Figure 1 shows an electro-mechanical relay. The current in the coil is 30 mA. The reluctance in the iron core and the fringing flux in the air gap are negligible. Determine:

a. The force that acts on the mobile part of the magnetic circuit and the self-inductance of the relay for an air gap equal to $x = 3.5$ mm.

b. The variation of the stored magnetic energy when the mobile part of the iron core slowly moves from $x_1 = 3.5$ mm to $x_2 = 5$ mm.

Problem 2

The inductances, in H, of the electro-mechanical system shown in Figure 2 are as follows:

$$L_{11} = 5 + 2\cos 2\theta; \quad L_{22} = 3 + \cos 2\theta; \quad L_{12} = L_{21} = 10\cos\theta$$

The currents in the windings are: $i_1 = 1$ A; $i_2 = 0.5$ A. Determine:

a. The magnetic energy stored in the system as a function of the angle θ;

b. The maximum mechanical torque that the system can develop and the corresponding value of θ.

Finally, explain why the period of $L_{12}(\theta)$ is twice the period of $L_{11}(\theta)$ and $L_{22}(\theta)$, and under which hypoteses the condition $L_{12}(\theta) = L_{21}(\theta)$ is satsified.

Problem 3

An electro-mechanical system consists of a coil that can move along adirection x. The relationship between the total flux λ and the current i of the coil is:

$$\lambda = x(i + x)$$

The coil has resistance R and is fed through a sinusoidal voltage with rms V and angular speed ω. Determine the average electro-magnetic force developed

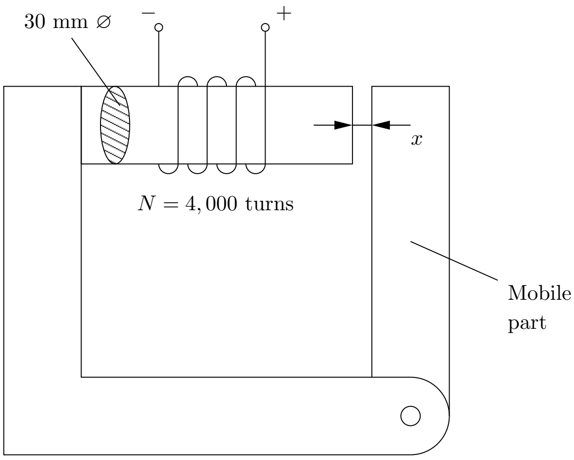

Figure 1

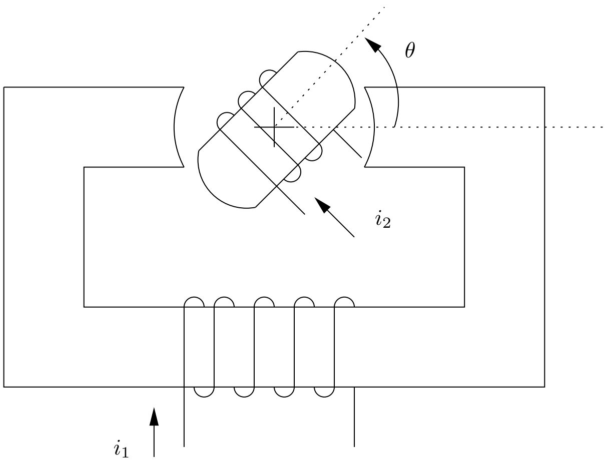

Figure 2

by the system at $x = x_0$ (constant) and in steady-state ac conditions. Assume that the force is null if the current is null.

Note: It is convenient to determine the rms value of the current that flows in the coil as a function of V, ω, R and x_0.

Problem 4

A magnetic circuit is composed of two coils with currents i_1 and i_2, respectively. The total fluxes depend on the currents i_1 and i_2 and on the position x according to the following expressions:

$$\begin{aligned} \lambda_1(i_1, i_2, x) &= x^2 i_1^2 + x i_2 \\ \lambda_2(i_1, i_2, x) &= x^2 i_2^2 + x i_1 \end{aligned}$$

Determine:

a. The magnetic energy $W(i_1, i_2, x)$ of the system.

b. The coenergy.

c. The force developed by the magnetic system.

d. The average force in the following cases:

(a) $i_1(t) = i_2(t) = \sqrt{2} I \cos \omega t$

(b) $i_1(t) = \sqrt{2} I \cos \omega t, \quad i_2(t) = \sqrt{2} I \cos 2\omega t$

Solution of Problem 1

a. Reluctance in the air gap for $x = 3.5$ mm:

$$\mathcal{R}_1 = \frac{x}{\mu_0 S} = \frac{0.0035}{4 \cdot \pi \cdot 10^{-7} \cdot \pi \cdot 0.015^2} = 3940268 \quad \text{A-turn/Wb} \tag{1}$$

Self-inductance for $x = 3.5$ mm:

$$L_1 = \frac{N^2}{\mathcal{R}_1} = \frac{4000^2}{3940268} = 4.06 \quad \text{H} \tag{2}$$

Force on the mobile section for $x = 3.5$ mm:

$$F = \frac{1}{2}\frac{\partial L}{\partial x} i^2 = -\frac{1}{2}\frac{\mu_0 S N^2}{x^2} i^2 = -0.5221 \quad \text{N} \tag{3}$$

b. Reluctance of the air gap for $x = 5$ mm:

$$\mathcal{R}_2 = \frac{x}{\mu_0 S} = \frac{0.005}{4 \cdot \pi \cdot 10^{-7} \cdot \pi \cdot 0.015^2} = 5628955 \quad \text{A-turn/Wb} \tag{4}$$

Self-inductance for $x = 5$ mm:

$$L_2 = \frac{N^2}{\mathcal{R}_2} = \frac{4000^2}{5628955} = 2.84 \quad \text{H} \tag{5}$$

Magnetic energy for $x = 3.5$ mm:

$$W_{m_1} = \frac{1}{2} i^2 L_1 = 0.5 \cdot 0.030^2 \cdot 4.06 = 0.001827 \quad \text{J} \tag{6}$$

Magnetic energy for $x = 5$ mm:

$$W_{m_2} = \frac{1}{2} i^2 L_2 = 0.5 \cdot 0.030^2 \cdot 2.84 = 0.001279 \quad \text{J} \tag{7}$$

Variation of magnetic energy:

$$\Delta W = W_{m_2} - W_{m_1} = -0.000548 \quad \text{J} \tag{8}$$

Solution of Problem 2

a. The general expression of the magnetic energy stored in a magnetic circuit with two coils is:

$$W_m = \frac{1}{2} L_{11} i_1^2 + \frac{1}{2} L_{22} i_2^2 + L_{12} i_1 i_2 \tag{9}$$

and, substitutingy the values of the currents and the inductances, one has:

$$\begin{aligned} W_m &= \frac{1}{2}(5 + 2\cos\theta)1^2 + \frac{1}{2}(3 + \cos 2\theta) \\ &\quad +10(\cos\theta)1 \cdot 0.5 \\ &= 2.5 + \cos 2\theta + 0.375 + 0.125\cos 2\theta + 5\cos\theta \\ &= 2.875 + 1.125\cos 2\theta + 5\cos\theta \end{aligned} \tag{10}$$

b. The mechanical torque developed for constant current is:

$$T = \frac{\partial W_m}{\partial \theta} = -2.25 \sin 2\theta - 5 \sin \theta \tag{11}$$

The maximum torque satisfies the condition $dT/d\theta = 0$:

$$\frac{dT}{d\theta} = 0 = -4.5 \cos 2\theta - 5 \cos \theta \tag{12}$$

which leads to the following expression:

$$9 \cos^2 \theta + 5 \cos \theta - 4.5 = 0 \ . \tag{13}$$

Of the two solutions, only one is < 1, i.e., $\cos \theta = 0.4819$. Two angles satisfy such a condition, namely, 1.0679 rad and 5.2152 rad. The torque is maximum for $\theta = 5.2152$ rad, i.e., $T^{\max} = 6.2813$ Nm (see Figure 3).

The self and mutual inductances are periodical functions of the angle θ due to the salient magnetic poles of the stator and rotor. The self-inductances have period 2θ as the positions θ and $\theta + \pi$ lead to the same reluctance of the system. On the other hand, the positions θ and $\theta + \pi$ have same magnitude but opposite signs when computing the mutual-inductance. Finally, $L_{12} = L_{21}$ holds if the leakage of magnetic flux between stator and rotor is assumed to be null.

Solution of Problem 3

We find first the expression of the current that flows in the coil. The electrical equation of the system is:

$$v(t) = \sqrt{2} V \cos \omega t = R i(t) + \frac{d\lambda}{dt} = R i(t) + \frac{\partial \lambda}{\partial x}\frac{dx}{dt} + \frac{\partial \lambda}{\partial i}\frac{di}{dt} \ . \tag{14}$$

Since x is constant, $dx/dt = 0$. Then, rewriting the equation above using phasors:

$$\bar{V} = (R + j\omega x_0)\bar{I} \ . \tag{15}$$

Hence:

$$\bar{I} = \frac{\bar{V}}{R + j\omega x_0} \ . \tag{16}$$

The rms of the current is:

$$I = \frac{V}{\sqrt{R^2 + \omega^2 x_0^2}} \ , \tag{17}$$

and, in time domain:

$$i(t) = \sqrt{2} I \cos(\omega t - \phi) \tag{18}$$

where

$$\phi = \text{atan}\left(\frac{\omega x_0}{R}\right) \tag{19}$$

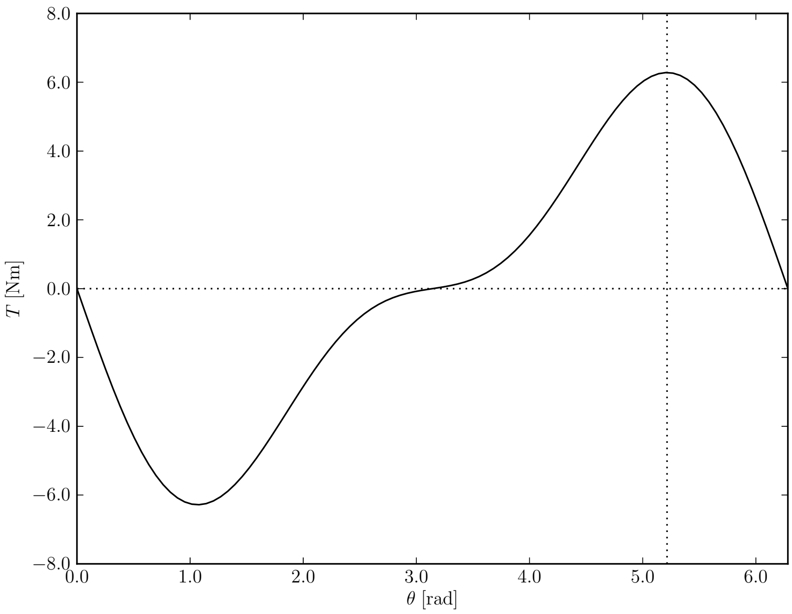

Figure 3

The coenergy W' is:

$$W'(x,i) = \int \lambda di = \int x(i+x)di = \frac{1}{2}xi^2 + x^2 i \tag{20}$$

The force can be obtained as:

$$\begin{aligned} f(x_0,i) &= \frac{\partial W'}{\partial x} = \frac{1}{2}i^2 + 2x_0 i \\ &= I^2\cos^2(\omega t - \phi) + 2x_0\sqrt{2}I\cos(\omega t - \phi) \\ &= \frac{1}{2}I^2(1-\cos(2\omega t - 2\phi)) + 2x_0\sqrt{2}I\cos(\omega t - \phi) \end{aligned} \tag{21}$$

and, finally, the average of the mechanical force is:

$$f_m = \frac{1}{2}I^2 = \frac{1}{2}\frac{V^2}{R^2+\omega^2 x_0^2} \tag{22}$$

Solution of Problem 4

a. The definition of magnetic energy $W(i_1, i_2, x)$ is:

$$W = \int i_1 d\lambda_1 + \int i_2 d\lambda_2 \tag{23}$$

To be able to integrate (23), one has to substitute $d\lambda_1$ and $d\lambda_2$ for their derivatives with respect to di_1 and di_2

$$\begin{aligned} d\lambda_1 &= \frac{\partial \lambda_1}{\partial i_1} di_1 + \frac{\partial \lambda_1}{\partial i_2} di_2 \qquad (24) \\ &= 2x^2 i_1 di_1 + x di_2 \\ d\lambda_2 &= \frac{\partial \lambda_2}{\partial i_1} di_1 + \frac{\partial \lambda_2}{\partial i_2} di_2 \qquad (25) \\ &= 2x^2 i_2 di_2 + x di_1 \ . \end{aligned}$$

In the expressions above, the dependency of the fluxes on x has been neglected since the integral (23) for $x = 0$ and, hence, $dx = 0$.

Let us compute (23) in two parts: (i) $i_2 = 0$, $di_2 = 0$ and variable i_1; and (ii) constant i_1, $di_1 = 0$ and variable i_2. Then, we obtain:

$$\begin{aligned} W &= \int i_1 \frac{\partial \lambda_1}{\partial i_1} di_1 + \int \Big(i_1 \frac{\partial \lambda_1}{\partial i_2} + i_2 \frac{\partial \lambda_2}{\partial i_2} \Big) di_2 \qquad (26) \\ &= \int 2x^2 i_1^2 di_1 + \int (x i_1 + 2x^2 i_2^2) di_2 \\ &= \frac{2}{3} x^2 (i_1^3 + i_2^3) + x i_1 i_2 \end{aligned}$$

We can obtain the same result through the definition of coenergy W':

$$W = \lambda_1 i_1 + \lambda_2 i_2 - W' \tag{27}$$

Refr to the next section for the determination of the expression of the coenergy.

b. The coenergy can be computed in two ways:

$$\begin{aligned} W' &= \int \lambda_1 di_1 + \int \lambda_2 di_2 \qquad (28) \\ &= \lambda_1 i_1 + \lambda_2 i_2 - W \qquad (29) \end{aligned}$$

Using (28) and piece-wise integrating as we did for the magnetic energy above, we obtain:

$$\begin{aligned} W' &= \int x^2 i_1^2 di_1 + \int (x^2 i_2^2 + x i_1) di_2 \qquad (30) \\ &= \frac{1}{3} x^2 (i_1^3 + i_2^3) + x i_1 i_2 \end{aligned}$$

Using (29), we have:

$$W' = x^2(i_1^3 + i_2^3) + 2xi_1i_2 - \frac{2}{3}x^2(i_1^3 + i_2^3) - xi_1i_2 \tag{31}$$
$$= \frac{1}{3}x^2(i_1^3 + i_2^3) + xi_1i_2$$

c. The force can be computed directly from the expression of the coenergy:

$$f(x, i_1, i_2) = \left.\frac{\partial W'}{\partial x}\right|_{i_1,i_2} = \frac{2}{3}x(i_1^3 + i_2^3) + i_1i_2 \tag{32}$$

d. Determination of the mean value of the force:

(a) If $i_1(t) = i_2(t) = \sqrt{2}I \cos \omega t$, one has:

$$f(x, t) = \frac{8\sqrt{2}}{3}xI^3 \cos^3 \omega t + 2I^2 cos^2 \omega t$$
$$\Rightarrow \quad f_m(x) = I^2 \ . \tag{33}$$

Note that the mean value of $\cos^3 \omega t$ is zero and the mean value of $\cos^2 \omega t$ is 0.5.

(b) If $i_1(t) = \sqrt{2}I \cos \omega t$ e $i_2(t) = \sqrt{2}I \cos 2\omega t$, one has:

$$f(x, t) = \frac{4\sqrt{2}}{3}xI^3(\cos^3 \omega t + \cos^3 2\omega t) + 2I^2 cos\omega t \cos 2\omega t$$
$$\Rightarrow \quad f_m(x) = 0 \ . \tag{34}$$