AC Electrical Circuits – Phasors & Power

ELECTRICAL ENERGY SYSTEMS (EEEN20090)

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Aims for this Section

- Review basic circuit theory and mathematical concepts
  - Units and notation
  - Complex numbers
  - Ohm's Law
  - Kirchhoff's Laws
  - Capacitors and inductors
  - Impedance $\bar{Z}$ – series and parallel
  - Admittance $\bar{Y}$ – series and parallel
  - Active, reactive and complex powers

Units

- SI base units: m, kg, s, A, K, mol and cd

frequency	$\mathrm{Hz} = \mathrm{s}^{-1}$
radian	$\mathrm{rad} = \mathrm{m}/\mathrm{m}$
joule	$\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \mathrm{W} \cdot \mathrm{s} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$
power	$\mathrm{W} = \mathrm{J}/\mathrm{s} = \mathrm{V} \cdot \mathrm{A} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3}$
electric charge	$\mathrm{C} = \mathrm{As}$
voltage	$\mathrm{V} = \mathrm{W}/\mathrm{A} = \mathrm{J}/\mathrm{C} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3} \cdot \mathrm{A}^{-1}$
electric capacitance	$\mathrm{F} = \mathrm{C}/\mathrm{V} = \mathrm{kg}^{-1} \cdot \mathrm{m}^{-2} \cdot \mathrm{s}^4 \cdot \mathrm{A}^2$
electric resistance	$\Omega = \mathrm{V}/\mathrm{A} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-3} \cdot \mathrm{A}^{-2}$
magnetic flux	$\mathrm{Wb} = \mathrm{J}/\mathrm{A} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{A}^{-1}$
inductance	$\mathrm{H} = \mathrm{V} \cdot \mathrm{s}/\mathrm{A} = \mathrm{Wb}/\mathrm{A} = \mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{A}^{-2}$

Notation (I)

- Notation of most common quantities:

instantaneous quantities	$v(t)$, $i(t)$
	or, simply, v, i
maximum values	V_M, I_M
RMS values (magnitudes)	V, I
phasors	$\bar{V}$, $\bar{I}$

- Examples:

$$v(t) = V_M \sin(\omega t)$$

$$\bar{V} = V e^{j\theta} = V(\cos\theta + j\sin\theta) = V\angle\theta$$

Notation (II)

- Notation of most common quantities:

Per unit (p.u.)	v, i, p, $\bar{s}$
Park vectors	$v_d + jv_q$
Time derivative	$\frac{d}{dt}$ or p operator or $\dot{v}$
Transfer function	$G(s) = K\frac{1+sT_1}{1+sT_2}$
Vectors	$[v]$ or $\boldsymbol{v}$
Matrix	$[A]$ or $\boldsymbol{A}$

- Examples:

$$\dot{x} = \frac{1}{T}(v^{\mathrm{ref}} - x)$$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Frequency

- The frequency of a signal is defined as the number of cycles per second passing a given point.
- The *angular frequency* of a signal is more commonly used and is given by ω, where:

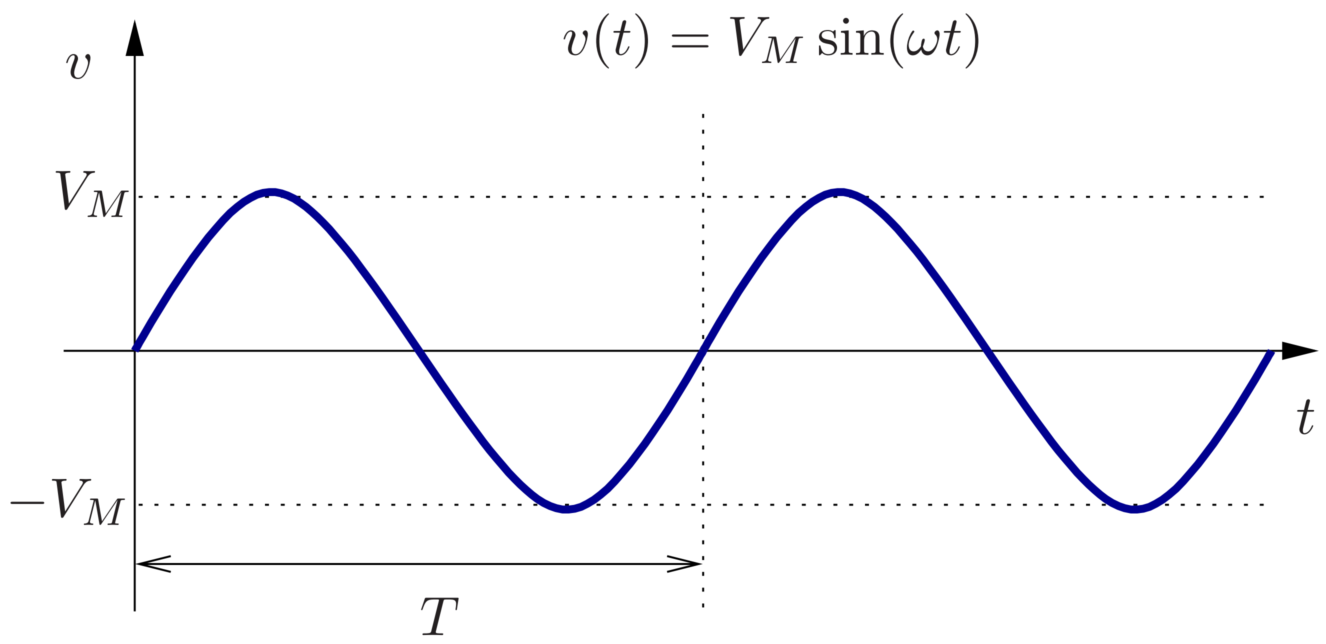
$$\omega = 2\pi f$$

- Example: for a 50 Hz voltage, the value of ω can be easily calculated:

$$\begin{aligned} f &= 50 \text{ Hz} \\ \omega &= 2\pi f \\ &= 2\pi \cdot 50 \text{ Hz} \\ &= 100\pi \text{ rad/s} \\ &= 314.16 \text{ rad/s} \end{aligned}$$

Sinusoidal Voltage

- Let consider a sinusoidal voltage:



- where $T = 1/f$ is the period in s; $f = \frac{\omega}{2\pi}$ is the frequency in Hz; and ω is the angular frequency in rad/s.

Example: Mean Value and RMS (I)

- What are the mean value and RMS of the sinusoidal voltage $v(t)$?
- The mean value is:

$$v_{\text{mean}} = v_m = \frac{1}{T}\int_0^T v(t)dt = 0$$

In fact, the mean value of a sinus over a period is always 0.

- The root-mean-square (RMS) is defined as:

$$V = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt} = \sqrt{\frac{1}{T}\frac{V_M^2}{2}T} = \frac{V_M}{\sqrt{2}}$$

Example: Mean Value and RMS (II)

- Note that the above relation holds for all *sinusoidal* waveforms, irrespective of their frequency.
- Assume:

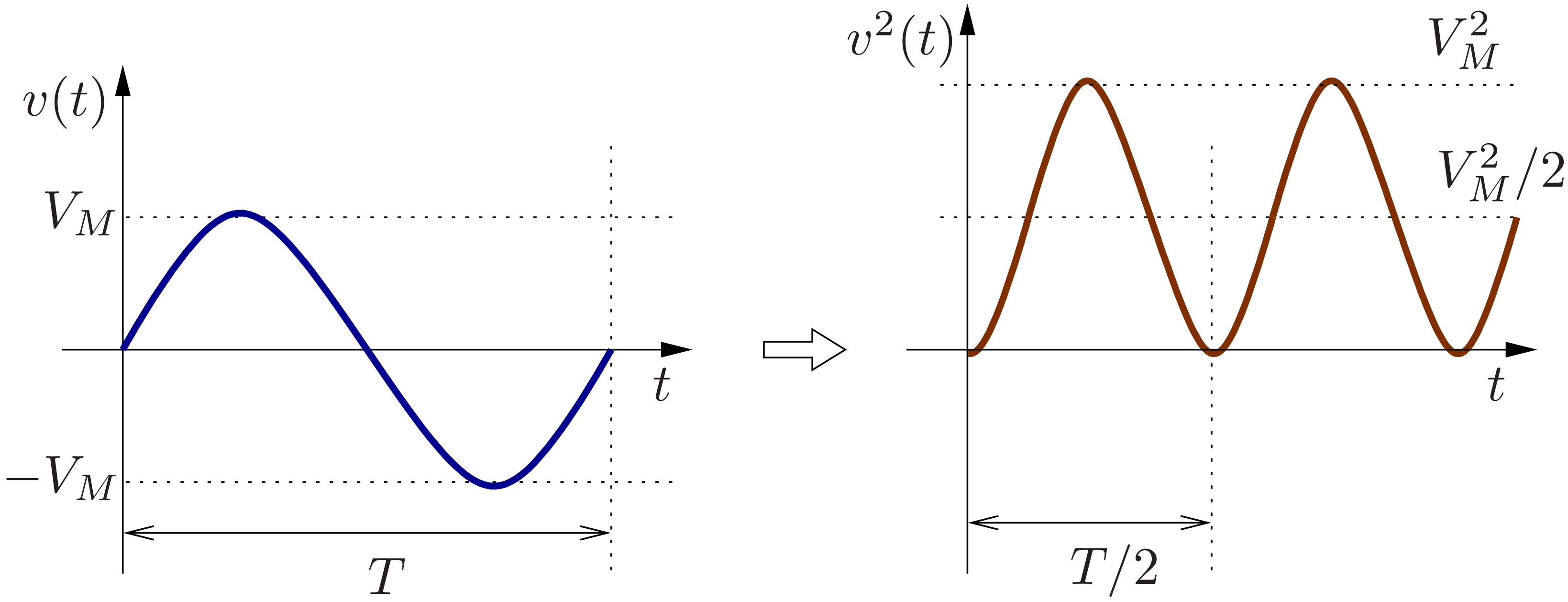
$$V = 220 \text{ V}$$

Then, the maximum amplitude of the signal is:

$$V_M = 220 \cdot \sqrt{2} = 311 \text{ V}$$

Example: Mean Value and RMS (III)

- In this case, the RMS can be evaluated graphically:

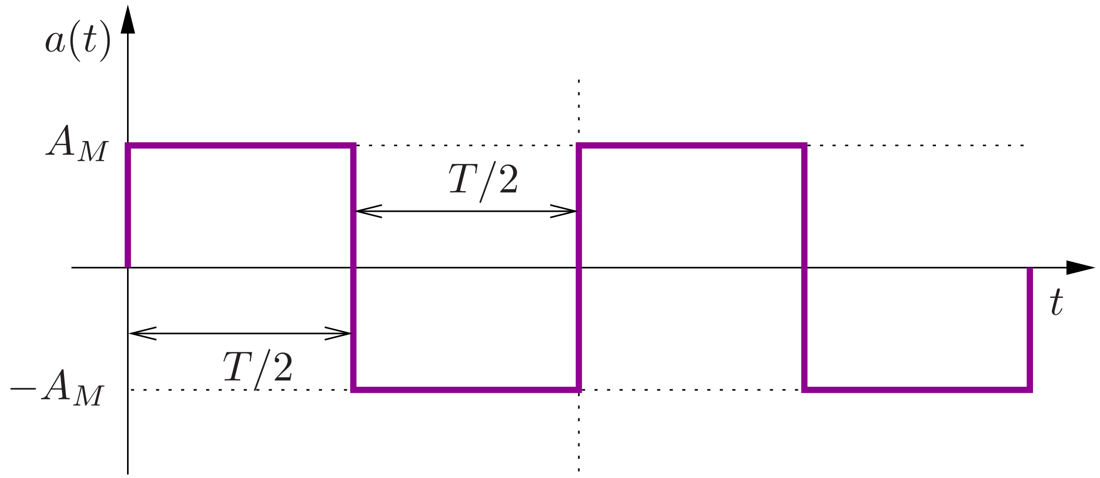


- Observe that:

$$\sin^2 \omega t = \frac{1 - \cos 2\omega t}{2}$$

Example: Square Waveform

- Let consider a square waveform:



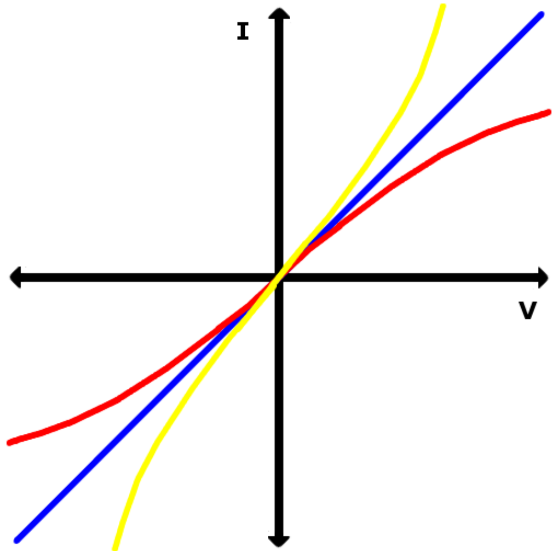
- $a(t)$ can be approximated by a Fourier's series:

$$a(t) = \frac{4}{\pi} A_M \left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \dots \right)$$

- Using the graphical method, the mean and the RMS can be determined easily:

$$a_m = 0, \qquad A = A_M$$

Ohm's Law



$$v(t) = R \cdot i(t)$$

- Fundamental physical law governing electricity:

  Voltage = Resistance $\times$ Current

- The blue line in the I vs. V graph at right represents ohmic devices because current is directly (linearly) proportional to the applied voltage.
- The slope of the blue line is $1/R$. The graph's red line represents a non-ohmic device such as a lamp filament because as more voltage is applied, heating the filament, the filament's resistance rises, forcing the (magnitude of the) slope to decrease.
- The graph's yellow line illustrates the I vs. V characteristics of a non-ohmic two terminal circuit having semi-conductor components (such as paralleled and oppositely oriented diodes).

Motivation of RMS Values

- Let's consider a resistance R:
- The instantneous power disspated in the resistance is:

$$p(t) = R \cdot i(t)^2$$

- If the current is AC:

$$i(t) = I_M \sin(\omega t)$$

- Then:

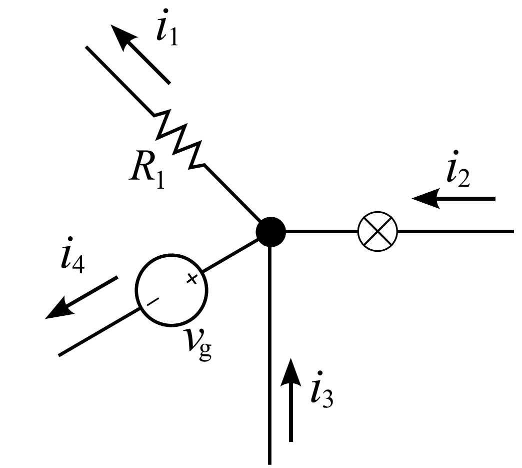
$$p(t) = R \cdot I_M^2 \frac{1 - \cos 2\omega t}{2}$$

- The average power P_m dissipated in the resistance is thus:

$$P_m = \frac{1}{2} R \cdot I_M^2 = R \cdot I^2$$

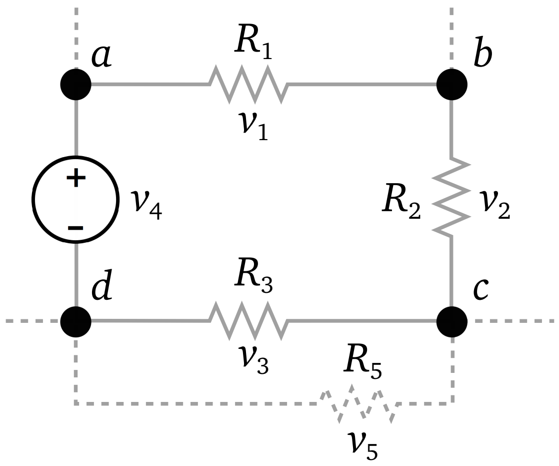
- The RMS value of an AC current is thus *equivalent* to the DC current that dissipates the same energy as the AC current in the resistance R.

Kirchhoff Laws



KCL: At any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node

$$0 = -i_1 + i_2 + i_3 - i_4$$

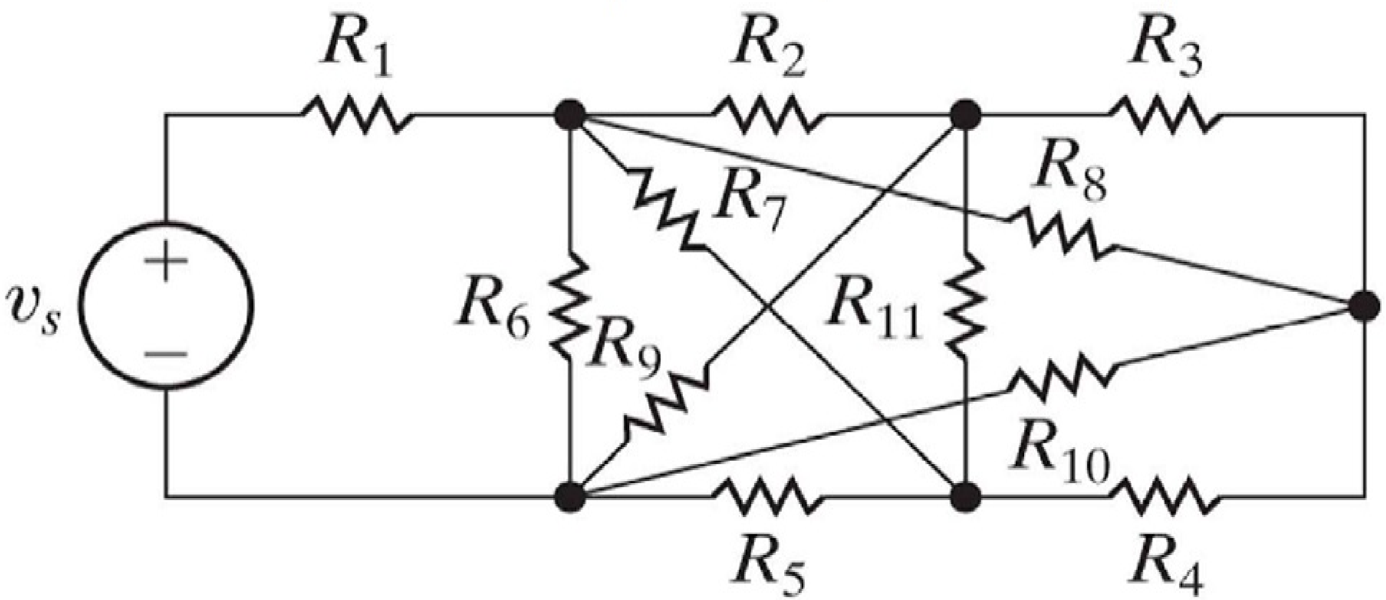


KVL: The directed sum of the electrical potential differences (voltage) around any closed network is zero

$$0 = v_1 + v_2 + v_3 - v_4$$

Node vs. Mesh Methods for Solving Circuits

- Equivalent for planar systems and hand solutions
- Computers implementations are based exclusively on the node (or nodal analysis) method
- The mesh method does not work for non-planar circuits!



Inductance – I

- An inductor in its basic form is a solenoid
- Tightly wound coil of wire
- Energy stored in the magnetic field
- Deal with flux and inductance later in course
- At this stage we will analyse them in terms of ac circuits

Inductance – II

- The voltage across an inductance is given by

$$v(t) = L\frac{di}{dt}$$

- Consider a current $i(t)$ flowing through the device, given by

$$i(t) = I_M \sin(\omega t - \pi/2)$$

- The induced voltage (or back e.m.f.) is

$$\begin{aligned} v(t) &= \omega L I_M \cos(\omega t - \pi/2) \\ &= \omega L I_M \sin(\omega t) \end{aligned}$$

- Note that the current is phase-shifted -90° w.r.t. the voltage.

Capacitance – I

- A capacitor in its basic form is two parallel plates and there is a charging effect between the plates
- Once again there is an energy storage effect
- Can be thought of in terms of ac circuits as the opposite of an inductor

Capacitance – II

- The current flowing in a capacitance is given by

$$i(t) = C\frac{dv}{dt}$$

- Appying a sinusoidal voltage $v(t)$ across a capacitance

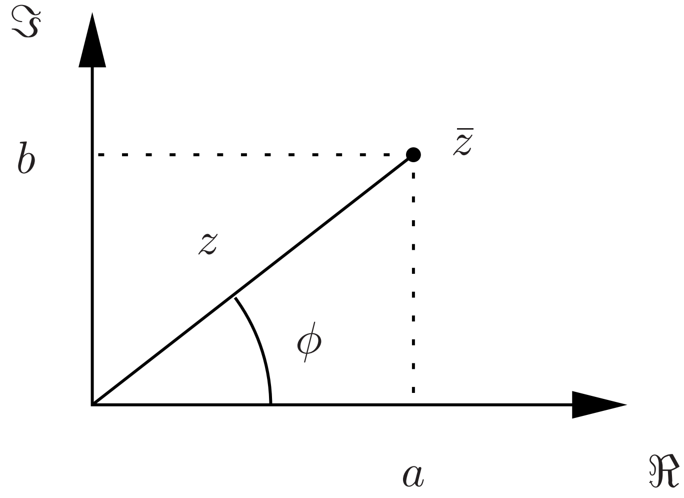
$$v(t) = V_M \sin(\omega t)$$

- The current flowing in it is

$$\begin{aligned} i(t) &= \omega C V_M \cos(\omega t) \\ &= \omega C V_M \sin(\omega t + \pi/2) \end{aligned}$$

- Note that the current is phase-shifted -90° w.r.t. the voltage.

Complex Numbers



- A complex number $\bar{z}$ can be written in:
  - rectangular coordinates: $\bar{z} = a + jb$
  - polar coordinates: $\bar{z} = ze^{j\phi}$

- where:
  - $a = \Re\{\bar{z}\} = z\cos(\phi)$ is the real part of $\bar{z}$
  - $b = \Im\{\bar{z}\} = z\sin(\phi)$ is the imaginary part of $\bar{z}$
  - $z = |\bar{z}| = \sqrt{a^2 + b^2}$ is the amplitude of $\bar{z}$
  - $\phi = \angle\bar{z} = \arctan(b/a)$ is the phase angle of $\bar{z}$

- The complex conjugate of $\bar{z}$ is denoted as $\bar{z}^* = a - jb = ze^{-j\phi}$

Complex Arithmetic

- Let $\bar{z}_1 = a + jb$ and $\bar{z}_2 = c + jd$, then:
  - $\bar{z}_1 + \bar{z}_2 = (a + c) + j(b + d)$
  - $\bar{z}_1 - \bar{z}_2 = (a - c) + j(b - d)$
  - $\bar{z}_1 \bar{z}_2 = (a + jb)(c + jd) = ac - bd + j(ad + bc)$
  - $\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{a + jb}{c + jd} = \dfrac{a + jb}{c + jd} \cdot \dfrac{c - jd}{c - jd} = \dfrac{ac + bd}{c^2 + d^2} + j\dfrac{bc - ad}{c^2 + d^2}$

- Let $\bar{z}_1 = z_1 e^{j\phi_1}$ and $\bar{z}_2 = z_2 e^{j\phi_2}$, then:
  - $\bar{z}_1 \bar{z}_2 = z_1 e^{j\phi_1} \cdot z_2 e^{j\phi_2} = z_1 z_2 e^{j(\phi_1 + \phi_2)}$
  - $\dfrac{\bar{z}_1}{\bar{z}_2} = \dfrac{z_1 e^{j\phi_1}}{z_2 e^{j\phi_2}} = \dfrac{z_1}{z_2} e^{j(\phi_1 - \phi_2)}$

Phasors

- Sinusoidal quantities are identified by their amplitude, phase and frequency:

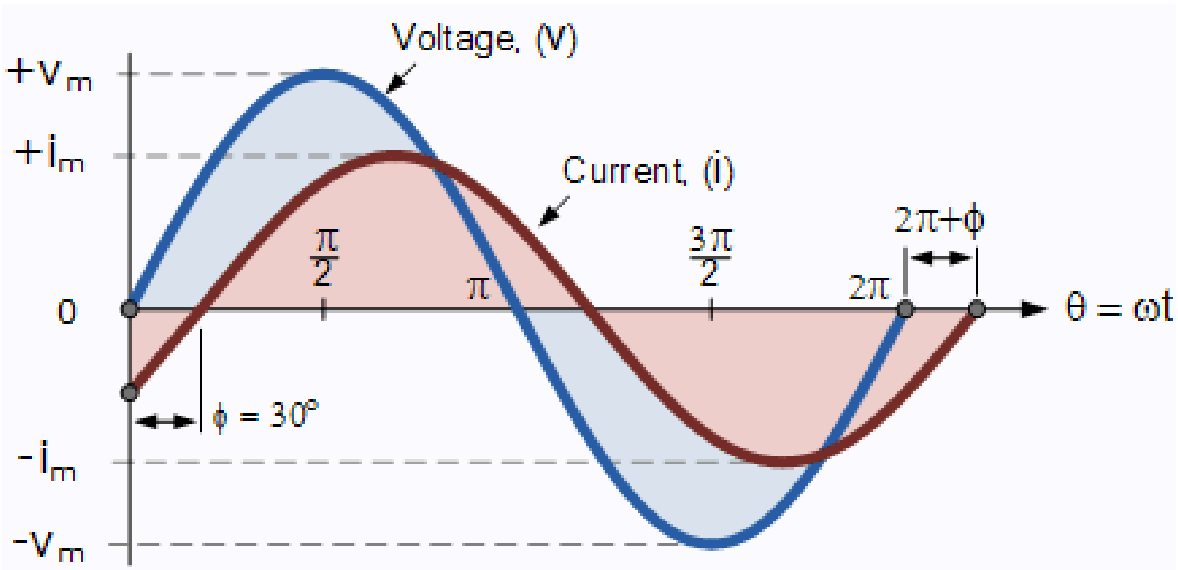
$$v(t) = V_M \sin(\omega t + \phi)$$

- In a system where the frequency is the same everywhere, a convenient way to represent sinusoidal quantities is through *phasors*.
- A convenient way of representing phasor quantities is using complex numbers which are univocally identified by the amplitude and phase (the frequency is assumed to be known and constant).
- The complex number corresponding to $v(t)$ is denoted $\bar{V}$:

$$\bar{V} = \frac{V_M}{\sqrt{2}} e^{j\phi} = V \cos\phi + jV \sin\phi = V_r + jV_i$$

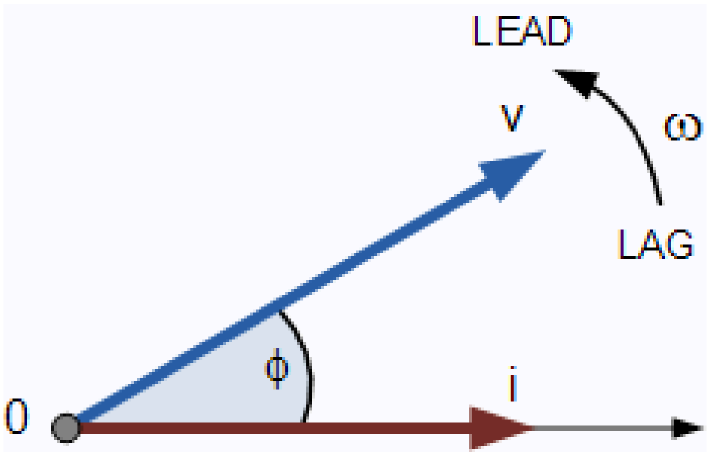
Phase of Phasors

- Two phasors may have identical amplitudes and frequencies, but different in *phase*.
- The phase of a periodic signal is the angular difference between a point on the signal and a corresponding point on a *reference* signal.



Phasor Diagram

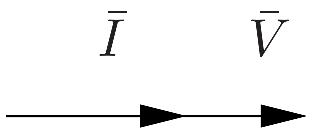
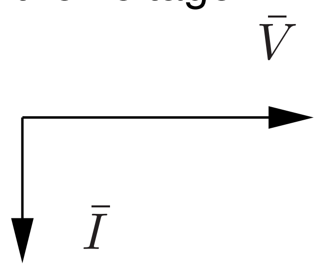
- The Argand diagram depicting phasors is called a *phasor diagram*.



Complex Impedance

- Can simplify phasor analysis by:
  - admitting *complex numbers* into our circuit calculations, and
  - assuming that every quantity in the system varies at *one frequency only*
- To find the impedence of any component, we find the relationship between the voltage across its terminals and the current flowing through it (Ohm's Law)

Time vs. Phasor Domains

	Resistance	Inductance	Capacitance
Time Domain	$v(t) = Ri(t)$	$v(t) = L\dfrac{di(t)}{dt}$	$i(t) = C\dfrac{dv(t)}{dt}$
Phasor Domain	$\bar{V} = R\bar{I}$	$\bar{V} = j\omega L\bar{I} = jX\bar{I}$	$\bar{I} = j\omega C\bar{V} = jB\bar{V}$
		current through inductance is said to *lag* the voltage	current through capacitance is said to *lead* the voltage
Phasor diagram (CIVIL mnemonic)			

Impedance & Admittance

- *Impedance*, $\bar{Z}$, is the phasor domain equivalent of resistance
- The impedance of a capacitance C is $1/j\omega C$, while the impedance of an inductance L is $j\omega L$, where ω is the supply angular frequency (rad/s)

$$\bar{I} = j\omega C\bar{V} \qquad \text{and} \qquad \bar{V} = j\omega L\bar{I}$$

- The reciprocal of impedance, $\bar{Y}$, is called *admittance*

$$\bar{Z} = R + jX \qquad \text{and} \qquad \bar{Y} = G + jB$$

where X is the *reactance* and B is the *susceptance*

Impedance in Series & Parallel

- $\bar{Z}_1$ and $\bar{Z}_2$ connected in series are equivalent to a single impedance, $\bar{Z}_s$, such that:

$$\bar{Z}_s = \bar{Z}_1 + \bar{Z}_2$$

$$R_s + jX_s = (R_1 + R_2) + j(X_1 + X_2)$$

- $\bar{Z}_1$ and $\bar{Z}_2$ connected in parallel are equivalent to a single impedance, $\bar{Z}_p$, such that:

$$\frac{1}{\bar{Z}_p} = \frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2}$$

$$\bar{Z}_p = \frac{\bar{Z}_1 \bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2}$$

$$\bar{Z}_p = \frac{(R_1 R_2 - X_1 X_2) + j(R_1 X_2 + R_2 X_1)}{(R_1 + R_2) + j(X_1 + X_2)}$$

Average Power

- If the voltage and current for a circuit element are

$$v(t) = V_M \sin(\omega t + \phi_v) \quad \text{and} \quad i(t) = I_M \sin(\omega t + \phi_i)$$

where V_M and I_M are the peak values of the sinewaves
$\longrightarrow$ the associated *instantaneous power* at time t is

$$\begin{aligned} p(t) = v(t)i(t) &= V_M I_M \sin(\omega t + \phi_v) \sin(\omega t + \phi_i) \\ &= \frac{V_M I_M}{2} [\cos(\phi_v - \phi_i) - \cos(2\omega t + \phi_v + \phi_i)] \end{aligned}$$

- The *average power* follows as

$$P = \frac{1}{2} V_M I_M \cos(\phi_v - \phi_i) = VI \cos\phi$$

- $\cos(\phi_v - \phi_i) = \cos\phi$ is called the *power factor*

Phasor Representation

- The rms value of the sinusoid

$$V_M \sin(\omega t + \phi_v) \quad \text{is} \quad V = \frac{V_M}{\sqrt{2}}$$

- Then the average power
  - for a resistor, $\bar{V} = R\bar{I}$

$$\phi = 0 \quad \Rightarrow \quad P = VI = \frac{V^2}{R} = I^2 R$$

  - for an inductor, $\bar{V} = j\omega L\bar{I}$

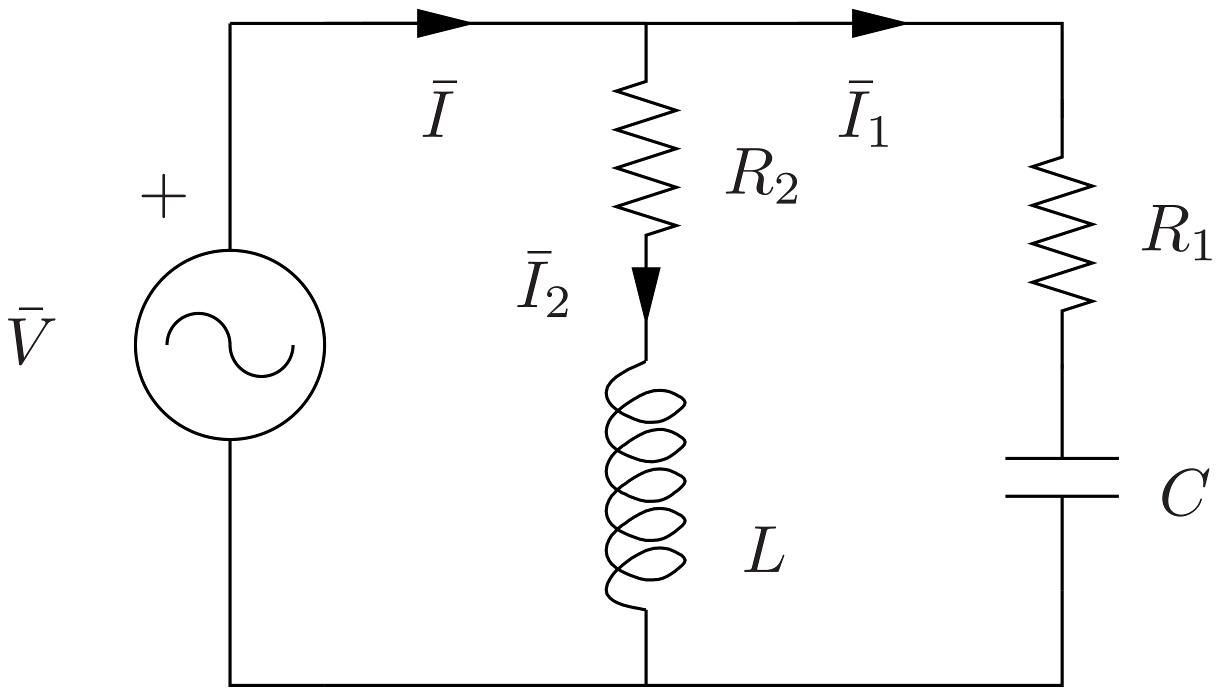
$$\phi = 90^\circ \quad \Rightarrow \quad P = 0$$

  - for a capacitor, $\bar{V} = \bar{I}/j\omega C$

$$\phi = -90^\circ \quad \Rightarrow \quad P = 0$$

Example – I

- Determine the currents flowing in the following circuit



- Data: $V = 220$ V, $f = 50$ Hz, $R_1 = 300\ \Omega$, $R_2 = 100\ \Omega$, $L = 0.795$ mH, and $C = 21.22\ \mu$F.

Example – II

- The current $\bar{I}$ can be obtained as follows.

$$\bar{Z}_1 = 300 - \frac{j}{100 \cdot \pi \cdot 21.22 \cdot 10^{-6}}$$

$$\bar{Z}_2 = 100 + j79.5 \cdot \pi$$

$$\bar{Z} = \frac{\bar{Z}_1 \bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2} = \frac{(300 - j150)(100 + j79.5 \cdot \pi)}{300 - j150 + 100 + j79.5 \cdot \pi}$$

$$\bar{Z} = \frac{67,463 + j59,926.98}{400 - j99.75} = \frac{67,463 + j59,927}{400 - j100}$$

$$\bar{Z} = 193.95 + j101.44\ \Omega$$

$$\bar{I} = \frac{220}{193.95 + j101.44} = 0.89 - j0.466\ \mathrm{A}$$

- Determine $\bar{I}_1$ and $\bar{I}_2$ for exercise.

Active, Reactive, Apparent & Complex Power

- The average (active) power, P, is given as

$$P = V \cdot I \cos\phi \quad [\mathrm{W}]$$

- The reactive power, Q, is given as

$$Q = V \cdot I \sin\phi \quad [\mathrm{VAr}]$$

- The apparent power (measure of the loading), S, is given as

$$S = V \cdot I = \sqrt{P^2 + Q^2} \quad [\mathrm{VA}]$$

- The complex power, $\bar{S}$, is given as

$$\bar{S} = P + jQ$$

- Equipment ratings are expressed in terms of their apparent power (kVA or MVA)

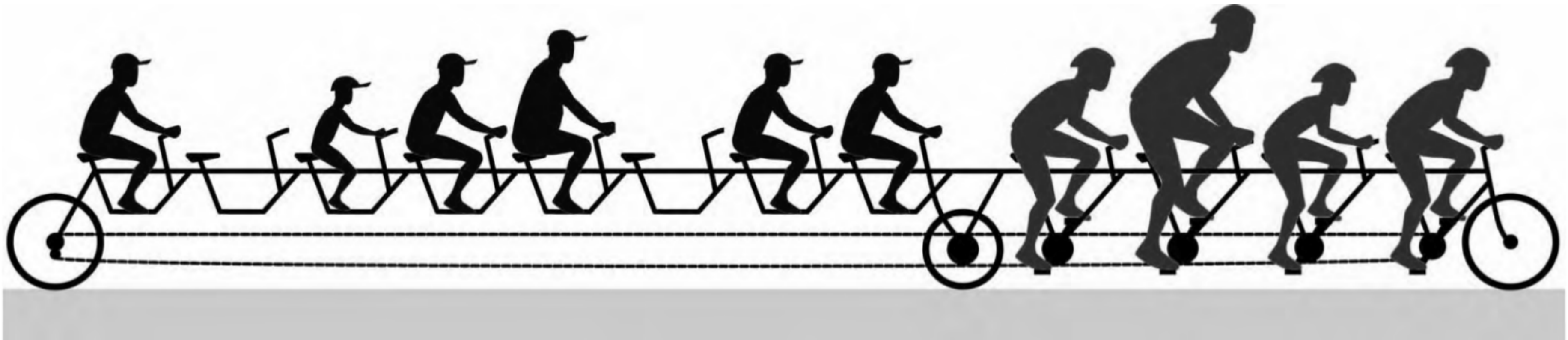
Reactive Power – I

- The reactive power Q (kVAr or MVAr) is sometimes (unfortunately) called the *imaginary power*, but is very real.
- Reactive power is exchanged *back and forth* between inductive and capacitive parts of a *reactive load* (one containing inductors and capacitors) during each ac cycle of the supply, and **does no work**

Reactive Power – II

- Reactive power refers to the energy storage part of a load
- Inductors store energy in their magnetic field while capacitors store energy in their electric field
- There is no net transfer of energy – no work is done
- Loads are generally made up of two components
  - Energy dissipated (active power)
  - Energy stored (reactive power)

Reactive Power Analogy – I



Riders at the back

are *passengers*

Riders at the front

are *drivers*

Loads

Generators

Reactive Power Analogy – II

Effort required to drive bike

is unaffected

. . . BUT bike might fall over

(A reactive load)

Riders at front must compensate

(reactive generation)

Pedalling becomes more difficult

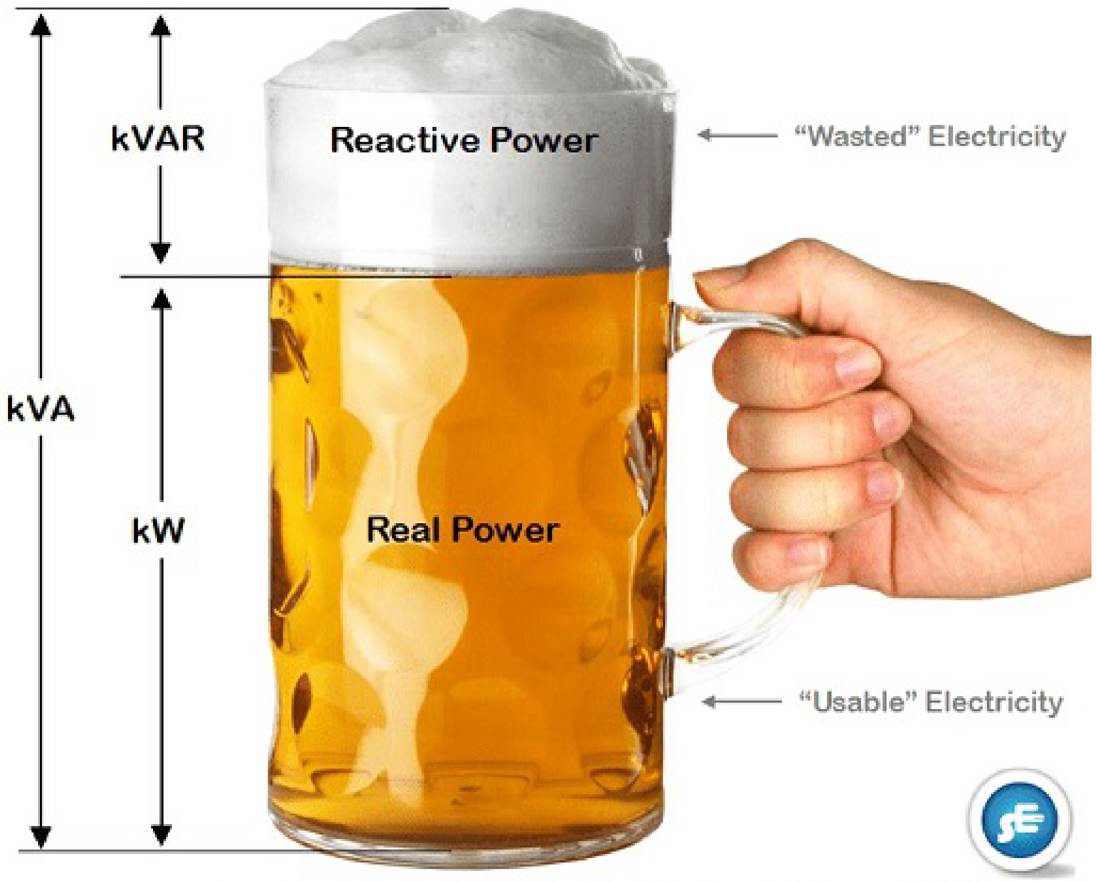
(reduced capability)

Bicycle drag increases

(more losses)

Alternative Reactive Power Analogy

- This is an alternative analogy to explain reactive power. . .



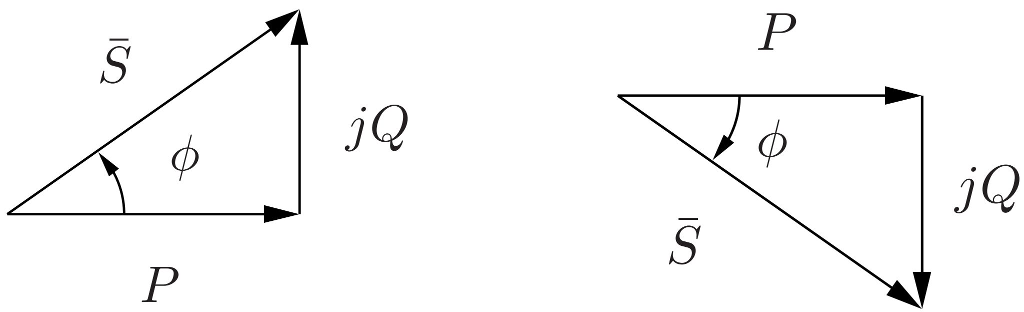
Power Triangles

Lagging (inductive) load Leading (capacitive) load

Circuit element

Phasor diagram

Power triangle



S, P, and Q

- From early equations,

$$P = S\cos\phi$$

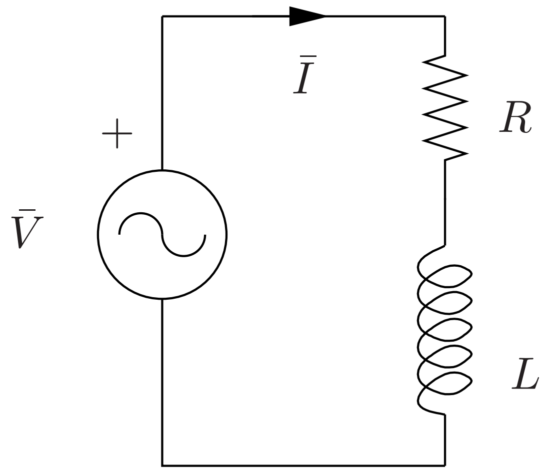
  - P should usually be as close to S as possible, i.e., a power factor near unity $\cos\phi \approx 1$
  - Thus minimising the reactive power, Q, i.e., $\sin\phi \approx 0$

Power Factor (p.f.)

- Inductive loads have lagging power factors
- Capacitive loads have leading power factors
- Most industrial & residential applications are inductive
- Some industries operate a tariff for $\mathrm{p.f.} < 1$
    - Economic sense to correct p.f.
    - done by adding capacitances or synchronous motors in parallel the with rest of load

Example – I

- What is the power factor for this circuit?



- Current drawn:

$$\bar{I} = \frac{\bar{V}}{R + j\omega L} = \frac{VR - jV\omega L}{R^2 + \omega^2 L^2}$$

Example – II

- In polar form:

$$\bar{I} = I \angle \phi$$

- Where the magnitude is:

$$I = \frac{V}{\sqrt{R^2 + \omega^2 L^2}}$$

- And the phase angle:

$$\phi = \tan^{-1} \frac{-\omega L}{R}$$

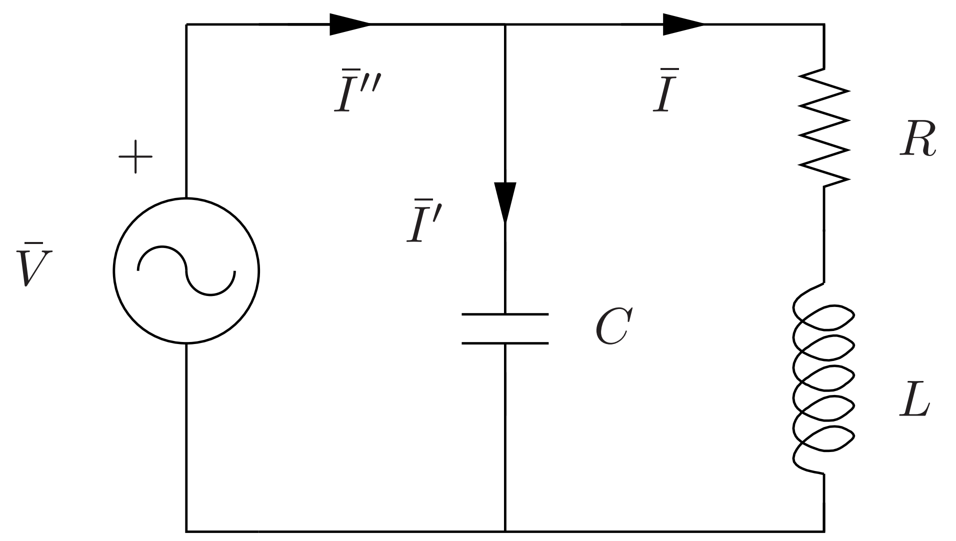
- Power factor:

$$\cos \phi = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}$$

- Is this power factor leading or lagging?
- Draw the phasor diagram

Power Factor Correction – I

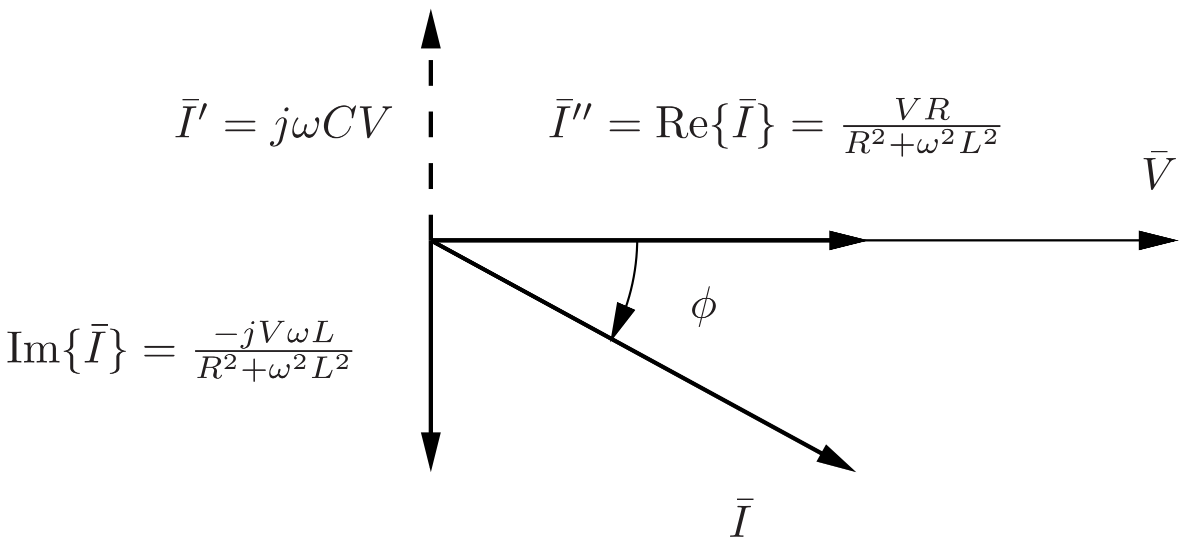
- The current absorbed from the feeder can be made to be in phase with $\bar{V}$.



- This is possible through careful choice of capacitor C
- Then $P \equiv S$, i.e., $\cos\phi = 1$, $\phi = 0$

Power Factor Correction – II

- Unity power factor can be achieved, if $\bar{I}'$ which is purely imaginary is equal in magnitude, but opposite in sign to the imaginary component of $\bar{I}$.



- Hence:

$$C = \frac{L}{R^2 + \omega^2 L^2}$$

Worked Example on PFC

- 10 kW load connected across a 220 V 50 Hz source draws a mean power factor of 0.7 lagging.
- Represent the load by the parallel combination of a resistor and either an inductor or capacitor (as appropriate).
- Calculate the inductance/capacitance of a compensating branch to correct the power factor to unity.

Worked Example on PFC

- Apparent power:

$$S = P/\cos\phi = 10/0.7 = 14.3 \text{ kVA}$$

- Reactive power:

$$Q = S\sin\phi = S\sqrt{1-0.7^2} \approx 0.7 \text{ kVAr}$$

- Reactance (certainly inductive):

$$X = V^2/Q = 4.84\ \Omega$$

- Capacitance:

$$C = \frac{1}{\omega X} = \frac{1}{2\pi f X} = \frac{1}{314.16 \cdot 4.84} = 0.658 \text{ mF}$$

Exercise

- Repeat question with pf of 0.7 leading instead of lagging.